HECKE CHARACTERS AND THE K–THEORY OF TOTALLY REAL AND CM NUMBER FIELDS

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Abstract. Let F/K be an abelian extension of number fields with F either CM or totally real and K totally real. If F is CM and the Brumer-Stark conjecture holds for F/K, we construct a family of $G(F/K)$–equivariant Hecke characters for F with infinite type equal to a special value of certain $G(F/K)$–equivariant L–functions. Using results of Greither–Popescu [19] on the Brumer–Stark conjecture we construct l–adic imprimitive versions of these characters, for primes $l > 2$. Further, the special values of these l–adic Hecke characters are used to construct $G(F/K)$–equivariant Stickelberger–splitting maps in the Quillen localization sequence for F, extending the results obtained in [1] for $K = \mathbb{Q}$. We also apply the Stickelberger–splitting maps to construct special elements in $K_{2n}(F)_l$ and analyze the Galois module structure of the group $D(n)_l$ of divisible elements in $K_{2n}(F)_l$. If n is odd, $l \nmid n$, and $F = K$ is a fairly general totally real number field, we study the cyclicity of $D(n)_l$ in relation to the classical conjecture of Iwasawa on class groups of cyclotomic fields and its potential generalization to a wider class of number fields. Finally, if F is CM, special values of our l–adic Hecke characters are used to construct Euler systems in odd K-groups $K_{2n+1}(F, \mathbb{Z}/l^k)$. These are vast generalizations of Kolyvagin's Euler system of Gauss sums [33] and of the K–theoretic Euler systems constructed in [4] when $K = \mathbb{Q}$.

1. Introduction

Notation.

Let L be a number field. For a nontrivial $\mathcal{O}_L$–ideal $\mathbf{a}$, we let, as usual, $N\mathbf{a} := |\mathcal{O}_L/\mathbf{a}|$ denote the norm of $\mathbf{a}$ and $\mathrm{Supp}(\mathbf{a})$ denote the set of distinct prime $\mathcal{O}_L$–ideals which divide $\mathbf{a}$. If M/L is a finite abelian extension and the $\mathcal{O}_L$–ideal $\mathbf{a}$ is prime then $G_{\mathbf{a}}$ denotes the decomposition group associated to $\mathbf{a}$, viewed as a subgroup of $G(M/L)$. Further, if $\mathbf{a}$ is (not necessarily prime) but coprime to the conductor of M/L, then $\sigma_{\mathbf{a}}$ denotes the Frobenius element associated to $\mathbf{a}$ in $G(M/L)$. We will let $S_\infty(L)$ denote the set of archimedean primes of L, I_L the group of fractional $\mathcal{O}_L$–ideals and $I_L(\mathbf{a})$ the group of fractional $\mathcal{O}_L$ ideals which are coprime to $\mathbf{a}$.

For a prime number l the symbol $\omega_{L,l}$ denotes the l–adic cyclotomic character:

$$\omega_{L,l} : G_L \to \mathbb{Z}_l^\times .$$

Recall that $G_L := G(\overline{L}/L)$ acts on the $\mathbb{Z}_l$-modules $\mathbb{Z}_l(n)$, $\mathbb{Q}_l(n)$ and $\mathbb{Q}_l/\mathbb{Z}_l(n) := \mathbb{Z}_l(n) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l = \mathbb{Q}_l(n)/\mathbb{Z}_l(n)$ via the n–th power of $\omega_{L,l}$.

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For $n \geq 1$, let $w_n(L)_l := |(\mathbb{Q}_l/\mathbb{Z}_l)(n)^{G_L}|$ and

$$w_n(L) := \prod_{l \geq 2} w_n(L)_l.$$

Note that $w_1(L)_l = |\mu_L \otimes_{\mathbf{Z}} \mathbb{Z}_l|$, where μ_L is the group of roots of unity in L. For simplicity, we let $w_L := w_1(L) = |\mu_L|$.

If A is an abelian group and $k \in \mathbb{Z}_{\geq 0}$ we let $A[l^k]$ denote the l^k–torsion subgroup of A and $A_l := A \otimes_{\mathbb{Z}} \mathbb{Z}_l$.

For a unital ring R and an integer $m \geq 0$, $K_m(R)$ and $K_m(R, \mathbb{Z}/l^k)$ denote the corresponding Quillen K–group and K–group with coefficients in $\mathbb{Z}/l^k$, respectively.

1.1. **Hecke characters.** In this paper, we consider abelian extensions F/K of number fields, where F is either CM or totally real and K is totally real. We consider two $\mathcal{O}_K$–ideals $\mathbf{f}$ and $\mathbf{b}$, such that $\mathbf{f}$ is divisible by the (finite) conductor of F/K and $\mathbf{b}$ is coprime to $\mathbf{f}$. To the data $(F/K, \mathbf{f}, \mathbf{b})$ one associates the Galois equivariant holomorphic L–function:

$$\Theta_{\mathbf{f},\mathbf{b}} : \mathbb{C} \to \mathbb{C}[G],$$

$$\Theta_{\mathbf{f},\mathbf{b}}(s) := (1 - N\mathbf{b}^{1+s} \cdot \sigma_{\mathbf{b}}^{-1}) \cdot \sum_{\sigma \in G(F/K)} \zeta_{\mathbf{f}}(\sigma, s) \cdot \sigma^{-1}.$$

The special values $\Theta_n(\mathbf{b}, \mathbf{f}) := \Theta_{\mathbf{f},\mathbf{b}}(-n)$, for all $n \in \mathbb{Z}_{\geq 0}$, are what Coates [12] calls higher Stickelberger elements. According to a deep theorem of Deligne–Ribet [14],

$$\Theta_n(\mathbf{b}, \mathbf{f}) \in \mathbb{Z}[G(F/K)], \qquad \text{for all } n \in \mathbb{Z}_{\geq 0},$$

as long as $\mathbf{b}$ is coprime to $w_{n+1}(F)$. In particular, if $\mathbf{b}$ is coprime to w_F, then

$$\Theta_0(\mathbf{b}, \mathbf{f}) \in \mathbb{Z}[G(F/K)].$$

Now, let us assume that F is CM. An equivalent formulation of the Brumer-Stark conjecture (see [29]) is the following.

Conjecture 1.1 (BrSt$(F/K, \mathbf{f})$, Brumer-Stark)**.** *Let F/K and $\mathbf{f}$ be as above. Then, for any prime $\mathcal{O}_K$–ideal $\mathbf{b}$ which is coprime to $w_F \cdot \mathbf{f}$, we have*

$$\Theta_0(\mathbf{b}, \mathbf{f}) \in \mathrm{Ann}_{\mathbb{Z}[G(F/K)]} \mathrm{CH}^1(F)^0_{T_{\mathbf{b}}},$$

where $\mathrm{CH}^1(F)^0_{T_{\mathbf{b}}}$ is the Arakelov class–group associated to $(F, \mathbf{b})$ defined in §3.1.

Under the assumption that the Brumer–Stark conjecture holds for $(F/K, \mathbf{f})$, in §3.1 (see Proposition 3.12) we construct $G(F/K)$–equivariant Hecke characters

$$\lambda_{\mathbf{b},\mathbf{f}} : I_F(\mathbf{b}) \to F^\times,$$

of conductor $\mathbf{b}$ and infinite type $\Theta_0(\mathbf{b}, \mathbf{f})$, for all $\mathcal{O}_K$–ideals $\mathbf{b}$ coprime to $w_F \cdot \mathbf{f}$. For $\mathbf{a} \in I_F(\mathbf{b})$, the value $\lambda_{\mathbf{b},\mathbf{f}}(\mathbf{a})$ is the unique element in $F^\times$ with Arakelov divisor

$$\mathrm{div}_F(\lambda_{\mathbf{b},\mathbf{f}}(\mathbf{a})) = \Theta_0(\mathbf{b}, \mathbf{f}) \cdot \mathbf{a}$$

and with some additional arithmetic properties (see §3.1 for details).

Weil's Jacobi sum Hecke characters [42] can be recovered from our construction for $K = \mathbb{Q}$. The values of our characters $\lambda_{\mathbf{b},\mathbf{f}}$ are generalizations of the classical Gauss sums, which arise in Weil's construction. If viewed in towers of abelian CM extensions of a fixed F/K, these values satisfy norm compatibility relations (see Lemma 3.11) which lead to Euler systems for the algebraic group $\mathbb{G}_m$, generalizing the Euler system of Gauss sums of Kolyvagin and Rubin [33] (see Lemma 3.11 and Remark 7.3). It is also worth remarking that our construction of Hecke characters is somewhat more refined (in that it keeps track of conductors and Galois theoretic and arithmetic properties of special values) than that carried out by Yang in [43], following ideas of Hayes [21].

The Brumer–Stark conjecture is not fully proven yet. However, the results of Greither–Popescu [19] show that if the classical Iwasawa invariant $\mu_{F,l}$ vanishes (as conjectured by Iwasawa), then an l–adic, imprimitive version of $\mathrm{BrSt}(F/K, \mathbf{f})$ holds, for all primes $l > 2$ (see Theorem 3.16 below for the precise result). As a consequence, under the vanishing hypothesis above, in §3.2 (see Lemma 3.20) we construct, for all primes $l > 2$, the l–adic $\mathbb{Z}_l[G(F/K)]$–equivariant versions

$$\lambda_{\mathbf{b},\mathbf{f}} : I_F(\mathbf{b}) \otimes \mathbb{Z}_l \to F^\times \otimes \mathbb{Z}_l$$

of the Hecke characters above, provided that $\mathbf{f}$ is divisible by all l–adic primes in K (an imprimitivity condition.) These l–adic, imprimitive Hecke characters are sufficient for our applications to K–theory in this paper.

1.2. **Euler Systems in odd K–theory with coefficients.** In the case where F is CM, in §7 we push our generalization of Gauss sums farther and use the values of our (l–adic, imprimitive) Hecke characters along with Bott elements to construct Euler systems for the odd K–theory $K_{2n+1}(F, \mathbb{Z}/l^k)$ with coefficients in $\mathbb{Z}/l^k$, for all $n \geq 0$ and all primes $l > 2$. For $n = 0$ and $K = \mathbb{Q}$ one recovers the Euler System of Gauss sums (modulo l^k) of Kolyvagin and Rubin [33]. For $n \geq 1$ and $K = \mathbb{Q}$ one recovers the K–theory Euler systems constructed in [4]. (See Theorem 7.2 and Remark 7.3 for details.)

1.3. **Stickelberger-splitting maps in K–theory.** Assume that F is either CM or totally real, fix a prime $l > 2$, an integer $n \geq 1$ and $\mathcal{O}_K$–ideals $\mathbf{b}$ and $\mathbf{f}$ as above. We consider the l–torsion part of the Quillen localization sequence ([30] and [36])

$$0 \longrightarrow K_{2n}(\mathcal{O}_F)_l \longrightarrow K_{2n}(F)_l \xrightarrow{\partial_F} \bigoplus_v K_{2n-1}(k_v)_l \longrightarrow 0. \tag{1}$$

Above, v runs over all the maximal ideals of $\mathcal{O}_F$ and k_v is the residue field of v. In §4 (see Theorem 4.6), we use special values of the l–adic imprimitive Hecke characters for extensions $F(\mu_{l^k})/K$, with $k \geq 1$, to construct a morphism

$$\Lambda \ : \ \bigoplus_v K_{2n-1}(k_v)_l \longrightarrow K_{2n}(F)_l$$

of $\mathbb{Z}_l[G(F/K)]$–modules, such that for all $x \in \bigoplus_v K_{2n-1}(k_v)_l$

$$(\partial_F \circ \Lambda)(x) = x^{l^{v_l(n)} \cdot \Theta_0(\mathbf{b},\mathbf{f})}.$$

Following [1], we call Λ a Stickelberger–splitting map for exact sequence (1). As shown in [1] and [6] the construction of such maps has far reaching arithmetic applications. The main idea behind constructing Λ is as follows: For each maximal $\mathcal{O}_K$–ideal v_0 we pick an $\mathcal{O}_F$–prime v dividing v_0 and let $l^k := |K_{2n-1}(k_v)_l|$. Then we pick a prime w in $E := F(\mu_{l^k})$ which divides v and consider the special element $\lambda^*_{\mathbf{b},\mathbf{f}}(w) \in E^\times \otimes \mathbb{Z}_l$, where

$$\lambda^*_{\mathbf{b},\mathbf{f}} : I_E \otimes \mathbb{Z}_l \to E^\times \otimes \mathbb{Z}_l$$

is a carefully chosen $\mathbb{Z}_l[G(E/K)]$–linear extension to the group I_E of all fractional ideals in E of the l–adic Hecke character $\lambda_{\mathbf{b},\mathbf{f}}$ associated to the data $(E/K, \mathbf{b}, \mathbf{f})$.

Next, we map $\lambda^*_{\mathbf{b},\mathbf{f}}(w)$ into $E^\times/E^{\times l^k} \simeq K_1(E, \mathbb{Z}/l^k)$. Then we construct the special element:

$$Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}}(w) * b(\beta(\xi_{l^k})^{*n}))^{\gamma_l} \in K_{2n}(F)[l^k],$$

where ξ_{l^k} is a generator of μ_{l^k} in E, $\beta(\xi_{l^k}) \in K_2(E, \mathbb{Z}/l^k)$ is the corresponding Bott element, $\gamma_l \in \mathbb{Z}_l[G(F/K)]$ is an exponent defined in (17) and

$$b : K_{2n+1}(E, \mathbb{Z}/l^k) \to K_{2n}(E)[l^k]$$

is the Bockstein homomorphism. Consequently, for any generator ξ_v of $K_{2n-1}(k_v)_l$ there is a unique group morphism

$$\Lambda_v : K_{2n-1}(k_v)_l \to K_{2n}(F)_l,$$
$$\Lambda_v(\xi_v) = Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}}(w) * b(\beta(\xi_{l^k})^{*n}))^{\gamma_l}.$$

For a carefully chosen generator ξ_v (see Definition 4.1) the map Λ_v satisfies

$$\partial_F \circ \Lambda_v(\xi_v) = \xi_v^{l^{v_l(n)} \cdot \Theta_0(\mathbf{b},\mathbf{f})}$$

and is $\mathbb{Z}_l[G_v]$–equivariant (see Theorem 4.5). The map Λ is the unique $\mathbb{Z}_l[G(F/K)]$–linear morphism which equals Λ_v when restricted to $K_{2n-1}(k_v)_l$, for all the chosen $\mathcal{O}_F$–primes v (see Theorem 4.6).

The above construction generalizes to arbitrary totally real fields K the construction of [1] done in the case $K = \mathbb{Q}$. The above construction is very different from that in [6] and it has the advantage of being $\mathbb{Z}_l[G(F/K)]$–linear unlike the one in loc.cit., a property which leads to new arithmetic applications, as shown below.

1.4. **Divisible elements in K–theory and Iwasawa's conjecture.** For a number field L and an $n > 0$ the group of divisible elements in $K_{2n}(L)_l$ is given by

$$div(K_{2n}(L)_l) := \bigcap_{k>0} K_{2n}(L)_l^{l^k}.$$

It is well known that the groups $div(K_{2n}(L)_l)$ are contained in $K_{2n}(\mathcal{O}_L)_l$ and they are the correct higher K–theoretic analogues of the ideal–class group $Cl(\mathcal{O}_L)_l = (K_0(O_L)_{tor})_l$ (see [2] and [19], for example). The group $div(K_{2n}(L)_l)$ is also one of the main obstructions (see [3, Section 4 and Theorem 6.4]) to the splitting of exact sequence (1) (in the category of $\mathbb{Z}_l$–modules). In particular $div(K_{2n}(\mathbb{Q})_l)$ is the only obstruction [3, Corollary 6.6] to the splitting of (1) for $L = \mathbb{Q}$. Combined with the newly proved Quillen–Lichtenbaum Conjecture [39] and with [3, Theorem 5.10], Theorem 4, p. 299 in [2] can be restated as

$$[K_{2n}(\mathcal{O}_L)_l : div(K_{2n}(L)_l)] = \frac{\prod_{v|l} w_n(L_v)_l}{w_n(L)_l},$$

for all number fields L and all $n \geq 1$, where v runs over all the l–adic primes of L and L_v is the v–adic completion of L. Thus, for $L = \mathbb{Q}$ and all $n \geq 1$ we have

$$K_{2n}(\mathbb{Z})_l = div(K_{2n}(\mathbb{Q})_l).$$

Let $\omega : G(\mathbb{Q}(\mu_l)/\mathbb{Q}) \to \mathbb{Z}_l^\times$ denote the Teichmuller character. Using divisible elements (see [4] and [5]), one of Kurihara's results in [24] can be restated as follows:

$$div(K_{2n}(\mathbb{Q})_l) \text{ is cyclic } \iff Cl(\mathbb{Q}(\mu_l))_l^{\omega^{-n}} \text{ is cyclic,} \tag{2}$$

for all odd $n \geq 1$ and

$$div(K_{2n}(\mathbb{Q})_l) = 0 \iff Cl(\mathbb{Q}(\mu_l))_l^{\omega^{-n}} = 0, \tag{3}$$

for all even $n \geq 1$. The right–hand side of (2) is a deep conjecture of Iwasawa and right–hand side of (3) is an equaly deep conjecture of Kummer-Vandiver.

From the above remarks it is clear that the study of the groups $div(K_{2n}(L)_l)$ is of central importance for understanding the arithmetic of L.

In §5, we use our Stickelberger splitting map to study the $\mathbb{Z}_l[G(F/K)]$–module structure and group structure of the abelian group $div(K_{2n}(F)_l)$. We work in this section under the simplifying hypotheses that $F(\mu_l)/K$ is ramified at all the l–adic primes, $F(\mu_{l^\infty})/F(\mu_l)$ is totally ramified at all these primes and $l \nmid n \cdot |G(F/K)|$. In this context, we show (see Theorem 5.3) that

$$div(K_{2n}(F)_l)^\chi = K_{2n}(O_F)_l^\chi \cap \mathrm{Im}(\Lambda),$$

for all irreducible $\mathbb{C}_l$–valued characters χ of G, such that $\chi(\Theta_n(\mathbf{b}, \mathbf{f})) \neq 0$. In the particular case $F = K$ and $n \geq 1$ odd, this implies that

$$div(K_{2n}(K)_l) = K_{2n}(O_K)_l \cap \mathrm{Im}(\Lambda). \tag{4}$$

These results show that the divisible elements are in fact special values of our maps Λ and can be explicitely constructed, as explained above, out of special values of our l–adic Hecke characters and Bott elements. Further considerations based on (4) lead us to the proof of the following equivalence (see Theorem 5.4):

$$div(K_{2n}(F)_l) \text{ is cyclic } \iff \Lambda_{v_0} \text{ is injective,} \tag{5}$$

for a well chosen ideal $\mathbf{b}$ and $\mathcal{O}_F$–prime v_0, assuming that $\prod_{v|l} w_n(F)_l = 1$.

In particular, in §6 we combine (5) for $F = \mathbb{Q}$ with our explicit construction of Λ to obtain a new proof of Kurihara's result (2) (see Theorem 6.4.) It is hoped that the techniques developed in §6 can be extended to other totally real fields and to a generalization of Iwasawa's conjecture in that context.

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2. The higher Stickelberger elements

Let F/K be a finite, abelian CM or totally real extension of a totally real number field K. Let $\mathbf{f}$ be the (finite) conductor of F/K and let $\mathbf{f}'$ be any nontrivial $\mathcal{O}_K$–ideal divisible by all the primes dividing $\mathbf{f}$, i.e. $\mathrm{Supp}(\mathbf{f}) \subseteq \mathrm{Supp}(\mathbf{f}')$.

For all $\mathbf{f}'$ as above and all $\sigma \in G(F/K)$, let $\zeta_{\mathbf{f}'}(\sigma, s)$ is the $\mathbf{f}'$–imprimitive partial zeta function associated to $\sigma \in G(F/K)$ of complex variable s. For $\Re(s) > 1$, this is defined by the absolutely and compact-uniformly convergent series

$$\zeta_{\mathbf{f}'}(\sigma, s) ;= \sum_{\mathbf{a}} N\mathbf{a}^{-s},$$

where the sum is taken over all the ideals $\mathbf{a}$ of $\mathcal{O}_K$ which are coprime to $\mathbf{f}'$ and such that $\sigma_{\mathbf{a}} = \sigma$. It is well known that $\zeta_{\mathbf{f}'}(\sigma, s)$ has a unique meromorphic continuation to the entire complex plane $\mathbb{C}$ and that it is holomorphic away from $s = 1$.

Definition 2.1 (Coates, [12])**.** For all $n \in \mathbb{Z}_{\geq 0}$, all $\mathbf{f}'$ as above, and all $\mathcal{O}_K$–ideals $\mathbf{b}$ coprime to $\mathbf{f}'$, the Stickelberger elements $\Theta_n(\mathbf{b}, \mathbf{f}') \in \mathbb{C}[G(F/K)]$ are given by

$$\Theta_n(\mathbf{b}, \mathbf{f}') := (1 - N\mathbf{b}^{1+n} \cdot \sigma_{\mathbf{b}}^{-1}) \cdot \sum_{\sigma \in G(F/K)} \zeta_{\mathbf{f}'}(\sigma, -n) \cdot \sigma^{-1}.$$

Remark 2.2. Note that $\zeta_{\mathbf{f}'}(\sigma, s)$ and consequently the elements $\Theta_n(\mathbf{b}, \mathbf{f}')$ only depend on $\mathrm{Supp}(\mathbf{f}')$ and not on $\mathbf{f}'$ per se.

A deep theorem of Siegel (see [35]) implies that $\Theta_n(\mathbf{b}, \mathbf{f}') \in \mathbb{Q}[G(F/K)]$, for all $\mathbf{f}'$, $\mathbf{b}$ and n as above. In [14], Deligne and Ribet proved the following refinement of Siegel's theorem.

Theorem 2.3 (Deligne-Ribet, [14])**.** *Let $\mathbf{f}'$, $\mathbf{b}$ and n be as above and let l be a prime number. Then, we have*

$$\Theta_n(\mathbf{b}, \mathbf{f}') \in \mathbb{Z}_l[G(F/K)],$$

as long as $\mathbf{b}$ is coprime to $w_{n+1}(F)_l$.

Consequently, if $\mathbf{b}$ and $\mathbf{f}'$ are as above and $\mathbf{b}$ is coprime to w_F, then we have

$$\Theta_0(\mathbf{b}, \mathbf{f}') \in \mathbb{Z}[G(F/K)].$$

A fundamental congruence relation between $\Theta_0(\mathbf{b}, \mathbf{f}')$ and $\Theta_n(\mathbf{b}, \mathbf{f}')$, for arbitrary $n \geq 1$, is proved in [14]. In order to state it, let us first note that for every $n \geq 1$ the character $\omega_{K,l}^n$ modulo $w_n(F)_l$ factors through $G(F/K)$. Consequently, we obtain a group morphism

$$\omega_{K,l}^{(n)} : G(F/K) \to (\mathbb{Z}_l/w_n(F)_l)^\times.$$

This extends to a unique $\mathbb{Z}_l/w_n(F)_l$–algebra isomorphism

$$t_n : (\mathbb{Z}_l/w_n(F)_l)[G(F/K)] \simeq (\mathbb{Z}_l/w_n(F)_l)[G(F/K)],$$

which sends $\sigma \to \omega_{K,l}^{(n)}(\sigma)^{-1} \cdot \sigma$, for all $\sigma \in G(F/K)$.

Theorem 2.4 (the Deligne–Ribet congruences, [14])**.** *For all $\mathbf{f}'$, $\mathbf{b}$, l and $n \geq 1$ as above, if $\mathbf{b}$ is coprime to $w_{n+1}(F)_l$, then we have*

$$\widehat{\Theta_n(\mathbf{b}, \mathbf{f}')} = t_n(\widehat{\Theta_0(\mathbf{b}, \mathbf{f}')}),$$

where $\widehat{x}$ is the class of x modulo $w_n(F)_l$, for all $x \in \mathbb{Z}_l[G(F/K)]$.

Remark 2.5. It is easily seen that for all $\mathbf{b}$ and $\mathbf{f}'$ as above we have an equality

$$\Theta_n(\mathbf{b}, \mathbf{f}') := (1 - N\mathbf{b}^{1+n} \cdot \sigma_{\mathbf{b}}^{-1}) \cdot \sum_{\chi \in \widehat{G(F/K)}} L_{\mathbf{f}'}(\chi, -n) \cdot e_{\overline{\chi}},$$

where $L_{\mathbf{f}'}(\chi, s)$ is the $\mathbf{f}'$–imprimitive L–function associated to the complex, irreducible character χ of $G(F/K)$ and e_χ is the usual idempotent associated to χ in the group algebra $\mathbb{C}[G(F/K)]$.

As usual, if F is a CM field we will call a character χ of $G(F/K)$ even if $\chi(j) = 1$ and odd if $\chi(j) = -1$, where j is the unique complex conjugation automorphism of F (contained in $G(F/K)$.) If F is totally real, then all characters χ of $G(F/K)$ are called even. A well known consequence of the functional equation for $L_{\mathbf{f}'}(\chi, s)$ is that its order of vanishing at $s = 0$ is given by the following formula

$$\mathrm{ord}_{s=0} L_{\mathbf{f}'}(\chi, s) = \begin{cases} \mathrm{card}\,\{w \in \mathrm{Supp}(\mathbf{f}') \cup S_\infty(F) \mid \chi(G_w) = 1\}, & \text{if } \chi \neq \mathbf{1}; \\ \mathrm{card}\,\mathrm{Supp}(\mathbf{f}') + \mathrm{card}\, S_\infty(F) - 1, & \text{if } \chi = \mathbf{1}, \end{cases}$$

where $\mathbf{1}$ is the trivial character of $G(F/K)$. The formula above shows that if χ is an even character, then $L_{\mathbf{f}'}(\chi, 0) = 0$. This implies that for all $\mathbf{b}$ and $\mathbf{f}'$ as above, if F is a CM field we have

$$\Theta_0(\mathbf{b}, \mathbf{f}') \in (1 - j) \cdot \mathbb{Z}[G(F/K)]. \tag{6}$$

For a more detailed discussion of the higher Stickelberger elements and their basic properties, the reader can consult for example [6], Section 2.

3. The Brumer–Stark elements and associated Hecke characters

In the 1970s, Brumer formulated a conjecture which generalizes to a great extent Stickelberger's classical theorem. This conjecture was rediscovered in a much more precise form by Stark in [37], as a particular case of what we now call the refined, order of vanishing 1, abelian Stark conjecture. For a very lucid presentation of the Brumer-Stark conjecture, the reader is strongly advised to consult Chpt. IV, §6 of [38]. For a more modern presentation, we refer the reader to §4.3 of [29], where the second author reformulates the Brumer-Stark conjecture in terms of the annihilation of certain generalized Arakelov class–groups.

3.1. **The global theory.** In what follows, we remind the reader the formulation of the Brumer–Stark conjecture stated in [29] in the (slightly more restrictive) context relevant for our current purposes and use it to derive some useful consequences on the existence of certain so–called Brumer–Stark elements and Hecke characters.

Let F/K and $\mathbf{f}'$ be as in the previous section. Throughout this section F is assumed to be a CM number field. Let S_∞ the set of infinite (archimedean) primes in F. Let $S_{\mathbf{f}'}$ be the union of S_∞ with the set consisting of all the primes in F dividing $\mathbf{f}'$. We consider the usual Arakelov divisor group associated to F:

$$\mathrm{Div}_{S_\infty}(F) := \left(\bigoplus_{w\notin S_\infty} \mathbb{Z}\cdot w\right) \bigoplus \left(\bigoplus_{w\in S_\infty} \mathbb{R}\cdot w\right),$$

where the sum is taken over primes w in F. Further, we define a degree map

$$\deg_F : \mathrm{Div}_{S_\infty}(F) \to \mathbb{R}$$

to be the unique map which is $\mathbb{Z}$–linear on the first direct summand above and $\mathbb{R}$–linear on the second and which also satisfies the equalities

$$\deg_F(w) = \begin{cases} 1, & \text{if } w \in S_\infty; \\ \log|\mathrm{N}w|, & \text{if } w \notin S_\infty. \end{cases}$$

Above, as usual, we let $\mathrm{N}w := \mathrm{card}(\mathcal{O}_F/w)$. We let $\mathrm{Div}^0_{S_\infty}(F)$ denote the kernel of the group morphism $\deg_F$. The product formula (for the canonically normalized metrics of F) permits us to define a divisor map (which is a group morphism) by

$$\mathrm{div}_F : F^\times \to \mathrm{Div}^0_{S_\infty}(F), \quad x \to \sum_{w\notin S_\infty} \mathrm{ord}_w(x)\cdot w + \sum_{w\in S_\infty} (-\log|x|_w)\cdot w\,,$$

where $\mathrm{ord}_w(\cdot)$ and $|\cdot|_w$ denote the canonically normalized valuation and metric associated to w, respectively. The Arakelov class group (first Chow group) associated to F is defined as follows.

$$\mathrm{CH}^1(F)^0 := \frac{\mathrm{Div}^0_{S_\infty}(F)}{\mathrm{div}_F(F^\times)}.$$

Next, following [29], we define a generalized version of the above constructions. For that purpose, let $\mathfrak{b}$ be a prime ideal in K, coprime to $\mathbf{f}\cdot w_F$. Let $T_{\mathfrak{b}}$ be the set of primes in F which sit above $\mathfrak{b}$. Further, we let $F^\times_{\mathfrak{b}}$ denote the subgroup of $F^\times$

consisting of those elements which are congruent to 1 modulo $\mathfrak{b}$. We consider the following subgroup of $\mathrm{Div}_{S_\infty}(F)$:

$$\mathrm{Div}_{S_\infty, T_{\mathfrak{b}}}(F) := \left(\bigoplus_{w \notin S_\infty \cup T_{\mathfrak{b}}} \mathbb{Z} \cdot w\right) \bigoplus \left(\bigoplus_{w \in S_\infty} \mathbb{R} \cdot w\right)$$

and let $\mathrm{Div}^0_{S_\infty, T_{\mathfrak{b}}}(F) := \mathrm{Div}_{S_\infty, T_{\mathfrak{b}}}(F) \cap \mathrm{Div}^0_{S_\infty}(F)$. Obviously, we have $\mathrm{div}_F(F^\times_{\mathfrak{b}}) \subseteq \mathrm{Div}^0_{S_\infty, T_{\mathfrak{b}}}(F)$. We define the following generalized Arakelov class group

$$\mathrm{CH}^1(F)^0_{T_{\mathfrak{b}}} := \frac{\mathrm{Div}^0_{S_\infty, T_{\mathfrak{b}}}(F)}{\mathrm{div}_F(F^\times_{\mathfrak{b}})}.$$

For a detailed description of the structure of these (classical or generalized) Arakelov class groups, their links to ideal class groups and special values of zeta functions, the reader may consults [29], Section 4.3. However, the reader should be aware at this point that these Arakelov class groups are by no means finite: if endowed with the obvious topology, their connected component at the origin is compact of volume equal to a certain non-zero (generalized) Dirichlet regulator and their group of connected components is isomorphic to a certain (ray) class group.

Remark 3.1. Obviously, we have $\ker(\mathrm{div}_F) = \mu_F$, so the map div_F factors through $F^\times/\mu_F$. In what follows, we abuse notation and use div_F to denote the factor map as well. However, for a prime ideal $\mathfrak{b}$ coprime to w_F, we have $F^\times_{\mathfrak{b}} \cap \mu_F = \{1\}$. This makes the divisor morphism div_F *injective when restricted to* $F^\times_{\mathfrak{b}}$.

Finally, let us observe that the groups $F^\times$, $F^\times_{\mathfrak{b}}$, $\mathrm{Div}^0_{S_\infty}(F)$ and $\mathrm{Div}^0_{S_\infty, T_{\mathfrak{b}}}(F)$ are endowed with natural $\mathbb{Z}[G(F/K)]$–module structures and that the map div_F is $G(F/K)$–equivariant. Consequently, $\mathrm{CH}^1(F)^0$ and $\mathrm{CH}^1(F)^0_{T_{\mathfrak{b}}}$ are endowed with natural $\mathbb{Z}[G(F/K)]$–module structures.

As we prove in [29] (see Proposition 4.3.5(1)), the classical Brumer-Stark conjecture for the set of data $(F/K, S_{\mathbf{f}'})$ is equivalent to the following statement.

Conjecture 3.2 ($\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$, Brumer-Stark)**.** *For all prime ideals* $\mathfrak{b}$ *in* K *which are coprime to* $\mathbf{f}' \cdot w_F$*, we have*

$$\Theta_0(\mathfrak{b}, \mathbf{f}') \in \mathrm{Ann}_{\mathbb{Z}[G(F/K)]} \mathrm{CH}^1(F)^0_{T_{\mathfrak{b}}}.$$

Remark 3.3. In fact, it is sufficient to prove the conjecture above for all but finitely many prime ideals $\mathfrak{b}$. Moreover, it is sufficient to prove the statement above for any (finite) set $\mathcal{T}$ of prime ideals $\mathfrak{b}$ which are coprime to $\mathbf{f}' \cdot w_F$ and such that the set $\{1 - \mathrm{N}\mathfrak{b} \cdot \sigma_{\mathfrak{b}}^{-1} \mid \mathfrak{b} \in \mathcal{T}\}$ generates the $\mathbb{Z}[G(F/K)]$–ideal $\mathrm{Ann}_{\mathbb{Z}[G(F/K)]}(\mu_F)$. Also, it is important to note that if one proves the statement above for a given $\mathbf{f}'$, then it also follows for any $\mathbf{f}''$, such that $\mathrm{Supp}(\mathbf{f}') \subseteq \mathrm{Supp}(\mathbf{f}'')$. Indeed, this is an immediate consequence of the obvious equality

$$\Theta_0(\mathfrak{b}, \mathbf{f}'') = \prod_w (1 - \sigma_w^{-1}) \cdot \Theta_0(\mathfrak{b}, \mathbf{f}'),$$

where the product runs over all the primes w in K dividing $\mathbf{f}''$ but not dividing $\mathbf{f}'$. All these facts are proved in [29], Section 4.3.

Remark 3.4. Let us note that if $\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$ holds for a prime $\mathfrak{b}$, then

$$\Theta_0(\mathfrak{b}, \mathbf{f}') \in \mathrm{Ann}_{\mathbb{Z}[G(F/K)]} Cl(\mathcal{O}_F)_{\mathfrak{b}} \subseteq \mathrm{Ann}_{\mathbb{Z}[G(F/K)]} Cl(\mathcal{O}_F),$$

where $Cl(\mathcal{O}_F)_{\mathfrak{b}}$ is the ray–class group of conductor $\mathfrak{b}\mathcal{O}_F$ associated to F. Indeed, this is a direct consequence of the existence of (commuting) natural $\mathbb{Z}[G(F/K)]$–linear surjections

$$\begin{tikzcd} \mathrm{CH}^1(F)^0_{T_{\mathfrak{b}}} \arrow[r, two heads] \arrow[d] & Cl(\mathcal{O}_F)_{\mathfrak{b}} \arrow[d] \\ \mathrm{CH}^1(F)^0 \arrow[r, two heads] & Cl(\mathcal{O}_F), \end{tikzcd}$$

explicitly constructed in [29], Section 4.3.

However, for our current purposes we need a more explicit proof and analysis of this consequence of the Brumer–Stark conjecture. So, let us take an ideal class $\widehat{\mathbf{c}}$ in $Cl(\mathcal{O}_F)_{\mathfrak{b}}$ associated to a fractional O_F–ideal $\mathbf{c}$ which is coprime to $\mathfrak{b}$. In the obvious manner, we can associate to $\mathbf{c}$ a divisor $\widetilde{\mathbf{c}}$ in $\mathrm{Div}_{S_\infty, T_{\mathfrak{b}}}(F)$. Now, let us pick an infinite (archimedean) prime ∞ of F and note that $(\widetilde{\mathbf{c}} - \deg_F(\widetilde{\mathbf{c}}) \cdot \infty) \in \mathrm{Div}^0_{S_\infty, T_{\mathfrak{b}}}(F)$. Consequently, by our assumption, there is a unique $\lambda_{\mathfrak{b},\mathbf{f}'}(\mathbf{c}) \in \mathbf{F}_{\mathfrak{b}}^\times$, such that

$$\Theta_0(\mathfrak{b}, \mathbf{f}') \cdot (\widetilde{\mathbf{c}} - \deg_F(\widetilde{\mathbf{c}}) \cdot \infty) = \mathrm{div}_F(\lambda_{\mathfrak{b},\mathbf{f}'}(\mathbf{c})).$$

The uniqueness of $\lambda_{\mathfrak{b},\mathbf{f}'}(\mathbf{c})$ follows from the injectivity of div_F when restricted to $F_{\mathfrak{b}}^\times$. (See the Remark 3.1.) Note that due to (6) and to the obvious fact that $(1-j) \cdot \infty = 0$ in $\mathrm{Div}^0_{S_\infty, T_{\mathfrak{b}}}(F)$, the last equality can be rewritten as

$$\Theta_0(\mathfrak{b}, \mathbf{f}') \cdot \widetilde{\mathbf{c}} = \mathrm{div}_F(\lambda_{\mathfrak{b},\mathbf{f}'}(\mathbf{c})). \tag{7}$$

This shows that the element $\lambda_{\mathfrak{b},f'}(\mathbf{c})$ does not depend on the choice of the infinite prime ∞. Moreover, it shows that $\mathbf{c}^{\Theta_0(\mathfrak{b},\mathbf{f}')}$ is a principal O_F–ideal generated by $\lambda_{\mathfrak{b},\mathbf{f}'}(\mathbf{c})$, which proves the annihilation consequence claimed above.

Remark 3.5. If $\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$ holds, then

$$\Theta_0(\mathbf{b}, \mathbf{f}') \in \mathrm{Ann}_{\mathbb{Z}[G(F/K)]}\mathrm{CH}^1(F)^0 \subseteq \mathrm{Ann}_{\mathbb{Z}[G(F/K)]}Cl(\mathcal{O}_F),$$

for all proper O_K–ideals $\mathbf{b}$ coprime to $\mathbf{f}' \cdot w_F$. Indeed, for a fixed ideal $\mathbf{b}$, this is a direct consequence of the previous remark applied to $\Theta_0(\mathfrak{p}, \mathbf{f}')$, for all primes $\mathfrak{p} \mid \mathbf{b}$ and the obvious equality

$$\Theta_0(\mathbf{ad}, \mathbf{f}') = \mathrm{N}\mathbf{d} \cdot \sigma_{\mathbf{d}}^{-1} \cdot \Theta_0(\mathbf{a}, \mathbf{f}') + \Theta_0(\mathbf{d}, \mathbf{f}'), \tag{8}$$

which holds for all proper O_K–ideals $\mathbf{a}$ and $\mathbf{d}$ coprime to $\mathbf{f}' \cdot w_F$. Moreover, by combining equalities (8) and (7) above, one can easily show that if $\mathbf{b}$ is a proper $\mathcal{O}_K$–ideal, coprime to $\mathbf{f}' \cdot w_F$ and $\mathbf{c}$ is any fractional $\mathcal{O}_F$–ideal, then there exists a unique element $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in F^\times/\mu_F$, such that

$$\mathrm{div}_F(\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c})) = \Theta_0(\mathbf{b}, \mathbf{f}') \cdot \widetilde{\mathbf{c}}. \tag{9}$$

Indeed, if $\mathbf{c}$ is coprime to $\mathbf{b}$, this is obvious. For an arbitrary $\mathbf{c}$, we pick a fractional $\mathcal{O}_F$–ideal $\mathbf{d}$ coprime to $\mathbf{b}$ and an $\widehat{\varepsilon} \in F^\times/\mu_F$, such that we have an equality

$$\widetilde{\mathbf{c}} = \widetilde{\mathbf{d}} + \mathrm{div}_F(\widehat{\varepsilon}) + D_\infty,$$

in $\mathrm{Div}_{S_\infty}(F)$, where $D_\infty \in \bigoplus_{w \in S_\infty} \mathbb{R} \cdot w$. We hit the above equality with $\Theta_0(\mathbf{b}, \mathbf{f}')$ to obtain

$$\Theta_0(\mathbf{b}, \mathbf{f}') \cdot \widetilde{\mathbf{c}} = \mathrm{div}_F(\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{d}) \cdot \widehat{\varepsilon}^{\Theta_0(\mathbf{b},\mathbf{f}')}).$$

Therefore, we can set $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) := \widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{d}) \cdot \widehat{\varepsilon}^{\Theta_0(\mathbf{b},\mathbf{f}')}$.

The reader should notice that if $\mathbf{b}$ *is not prime,* then, in general, it is not true that $\Theta_0(\mathbf{b}, \mathbf{f}')$ annihilates the ray–class group $Cl(\mathcal{O}_F)_{\mathbf{b}}$ of conductor $\mathbf{b}\mathcal{O}_F$ (i.e. the elements $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ cannot be chosen to be classes in $F^\times/\mu_F$ of elements in $F^\times$ which are congruent to $1 \mod \mathbf{b}\mathcal{O}_{\mathbf{F}}$ even if $\mathbf{c}$ is coprime to $\mathbf{b}$, in general.)

Definition 3.6. Let F/K and $\mathbf{f}'$ be as above. Assume that $\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$ holds. Consider a pair $(\mathbf{b}, \mathbf{c})$ consisting of a proper $\mathcal{O}_K$–ideal $\mathbf{b}$ which is coprime to $\mathbf{f}' \cdot w_F$ and a fractional $\mathcal{O}_F$–ideal $\mathbf{c}$. Then, the unique element $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in F^\times/\mu_F$ satisfying (9) is called *the Brumer–Stark element* associated to $(F/K, \mathbf{f}', \mathbf{b}, \mathbf{c})$.

Lemma 3.7. *Under the assumptions and with the notations of the above definition, the following hold.*

(i) *For all pairs $(\mathbf{b}, \mathbf{c})$ as above, we have an equality in $F^\times$*
$$\lambda^{(1+j)} = 1,$$
for any λ in $F^\times$ whose class in $F^\times/\mu_F$ is $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$.

(ii) *For all pairs $(\mathbf{a}, \mathbf{c})$ and $(\mathbf{b}, \mathbf{c})$ as above, we have an equality in $F^\times/\mu_F$*
$$\widehat{\lambda}_{\mathbf{ab},\mathbf{f}'}(\mathbf{c}) = \widehat{\lambda}_{\mathbf{a},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1}} \cdot \widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c}).$$

Proof. In order to prove (i), one combines (9) and (6), to conclude that
$$\mathrm{div}_F(\lambda^{(1+j)}) = (1+j) \cdot \Theta_0(\mathrm{b}, \mathrm{f}') \cdot \widetilde{\mathbf{c}} = 0.$$
Consequently, $\lambda^{(1+j)} \in \mu_F \cap F^+ = \{\pm 1\}$, where $F^+ = F^{j=1}$ is the maximal totally real subfield of F. However, since F is CM, $x^{(1+j)}$ is a totally positive element of F^+, for all $x \in F^\times$. This implies that $\lambda^{(1+j)} = 1$, as stated.

The equality in (ii) is a direct consequence of (8) and (9). $\square$

The considerations in the last two remarks lead naturally to the following.

Proposition 3.8. *Fix F/K and $\mathbf{f}'$ as above. Assume that conjecture* $\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$ *holds true. Fix a fractional $\mathcal{O}_F$–ideal $\mathbf{c}$ and let $\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$ be the set of all proper $\mathcal{O}_K$–ideals $\mathbf{b}$ which are coprime to $\mathbf{c}\mathbf{f}' \cdot w_F$. Then, there exist unique elements $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ in $F^\times$, for all $\mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$, such that the following are satisfied.*

(i) *If $\mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$, then $\mathrm{div}_F(\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})) = \Theta_0(\mathbf{b}, \mathbf{f}') \cdot \widetilde{\mathbf{c}}$.*

(ii) *If $\mathbf{b}$ is a prime ideal in $\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$, then $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in F_{\mathbf{b}}^\times$.*

(iii) *If $\mathbf{a}, \mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$, then we have*
$$\lambda_{\mathbf{ab},\mathbf{f}'}(\mathbf{c}) = \lambda_{\mathbf{a},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1}} \cdot \lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}).$$

Moreover, for all $\sigma \in G(F/K)$ and all $\mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K) (= \mathcal{B}_{\sigma(\mathbf{c}),\mathbf{f}'}(F/K))$, we have
$$\lambda_{\mathbf{b},\mathbf{f}'}(\sigma(\mathbf{c})) = \sigma(\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})).$$

Proof. Let us first note that *if* $\mathbf{b}$ *is prime,* then conditions (i) and (ii) determine $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in F_{\mathbf{b}}^\times$ uniquely. Indeed, Remark 3.4 shows that $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ is the unique element in $F_{\mathbf{b}}^\times$, such that
$$\Theta_0(\mathbf{b}, \mathbf{f}') \cdot \widetilde{\mathbf{c}} = \mathrm{div}_F(\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})).$$
We define elements $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ satisfying (i) and (ii) for all $\mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$ by induction on the number of (not necessarily distinct) prime factors of $\mathbf{b}$ as follows. For ideals $\mathbf{b}$ with one prime factor this has just been achieved. Now, assume that we have achieved this for ideals with $(n-1)$ prime factors, for some $n \geq 2$. Let

$\mathbf{b}' \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$ equal to a product of n primes. Let $\mathfrak{p}$ be a prime dividing $\mathbf{b}'$ and let $\mathbf{b}' = \mathbf{b}\mathfrak{p}$. Obviously, we have $\mathfrak{p}, \mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$. We let

$$\lambda_{\mathbf{b}',\mathbf{f}'}(\mathbf{c}) := \lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathfrak{p}\cdot\sigma_{\mathfrak{p}}^{-1}} \cdot \lambda_{\mathfrak{p},\mathbf{f}'}(\mathbf{c}), \tag{10}$$

which clearly satisfies (i), by Lemma 3.7(ii). In order to check that this definition does not depend on our choice of the prime $\mathfrak{p}$, we need to show that

$$\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathfrak{p}\cdot\sigma_{\mathfrak{p}}^{-1}} \cdot \lambda_{\mathfrak{p},\mathbf{f}'}(\mathbf{c}) = \lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \cdot \lambda_{\mathfrak{p},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1}}. \tag{11}$$

However, equation (8) shows that both sides of the equality to be proved have the same (Arakelov) divisor, namely $\Theta_0(\mathbf{b}\mathfrak{p}, \mathbf{f}') \cdot \widetilde{\mathbf{c}}$. This means that the two sides differ by a root of unity, say $\zeta \in \mu_F$. This implies that

$$\zeta = \lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathfrak{p}\cdot\sigma_{\mathfrak{p}}^{-1}-1} \cdot \lambda_{\mathfrak{p},\mathbf{f}'}(\mathbf{c})^{1-\mathrm{N}\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1}}.$$

Now, since $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ is coprime to $\mathfrak{p}$ and $\lambda_{\mathfrak{p},\mathbf{f}'}(\mathbf{c}) \in F_{\mathfrak{p}}^{\times}$, the last equality implies that $\zeta \in F_{\mathfrak{p}}^{\times}$. However, since $\mathfrak{p}$ is coprime to w_F, this implies that $\zeta = 1$, which concludes the proof of (11).

Now, based on the inductive construction (10) given above, it is easily proved that (iii) is satisfied. Also, the uniqueness of $\{\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \mid \mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)\}$ follows immediately from (iii) and the uniqueness of $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ for $\mathbf{b}$ prime in $\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)\}$.

The last statement in the Proposition follows by checking first that the set $\{\sigma(\lambda_{\mathbf{b}',\mathbf{f}'}(\mathbf{c})) \mid \mathbf{b} \in \mathcal{B}_{\mathbf{c},\mathbf{f}'}(\mathbf{F}/\mathbf{K})\}$ satisfies properties (i)–(iii) with $\mathbf{c}$ replaced with $\sigma(\mathbf{c})$. This follows from the obvious fact that the map div_F is $G(F/K)$–equivariant. Finally, one uses uniqueness in order to prove the last equality in the Proposition. $\square$

Definition 3.9. Let F/K and $\mathbf{f}'$ be as above. Assume that $\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$ holds. Then, for any fractional $\mathcal{O}_F$–ideal $\mathbf{c}$ and any proper $\mathcal{O}_K$–ideal $\mathbf{b}$ corime to $\mathbf{c}\mathbf{f}' \cdot w_F$, the unique element $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ in $F^{\times}$ produced by Proposition 3.8 is called *the strong Brumer–Stark element* associated to the data $(F/K, \mathbf{f}', \mathbf{b}, \mathbf{c})$.

Remark 3.10. The astute reader would have noticed, no doubt, that throughout the current section the elements $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ and $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ depend only on $\mathrm{Supp}(\mathbf{f}')$ and not on $\mathbf{f}'$ per se. This is a direct consequence of Remark 2.2.

Lemma 3.11. *Under the hypotheses of Proposition 3.8, assume in addition that E/K is an abelian CM extension of K, such that $F \subseteq E$, and $\mathbf{e}'$ is a $\mathcal{O}_K$–ideal divisible by the primes dividing the conductor of E/K and those dividing $\mathbf{f}'$. Also, assume that conjectures $\mathrm{BrSt}(E/K, S_{\mathbf{e}'})$ and $\mathrm{BrSt}(F/K, S_{\mathbf{f}'})$ hold. Let $\mathbf{c}$ be a fractional $\mathcal{O}_E$–ideal. Then, for any proper $\mathcal{O}_K$–ideal $\mathbf{b}$ coprime to $\mathbf{c}\mathbf{e}' \cdot w_E$, we have*

$$N_{E/F}(\lambda_{\mathbf{b},\mathbf{e}'}(\mathbf{c})) = \lambda_{\mathbf{b},\mathbf{f}'}(N_{E/F}(\mathbf{c}))^{\prod_{\substack{\mathfrak{p}\mid\mathbf{e}'\\ \mathfrak{p}\nmid\mathbf{f}'}}(1-\sigma_{\mathfrak{p}}^{-1})},$$

where $N_{E/F}$ denotes the usual norm from E down to F at the level of elements and fractional ideals in E, and the product is taken over prime $\mathcal{O}_K$–ideals $\mathfrak{p}$.

Proof. The proof is straightforward. The main idea is to prove that the elements $\{N_{E/F}(\lambda_{\mathbf{b},\mathbf{e}'}(\mathbf{c})) \mid \mathbf{b}\}$ and $\{\lambda_{\mathbf{b},\mathbf{f}'}(N_{E/F}(\mathbf{c}))^{\prod_{\mathfrak{p}}(1-\sigma_{\mathfrak{p}}^{-1})} \mid \mathbf{b}\}$ satisfy properties (i)–(iii) in the statement of Proposition 3.8 with $\mathbf{c}$ and $\mathbf{f}'$ replaced by $N_{E/F}(\mathbf{c})$ and $\mathbf{e}'$, respectively, and then use the uniqueness property. Checking (ii) and (iii) is immediate. In order to check (i), first one uses the inflation property of Artin L–functions to prove that

$$\mathrm{Res}_{E/K}(\Theta_0^E(\mathbf{e}',\mathbf{b}))=\Theta_0(\mathbf{e}',\mathbf{b})=\Theta_0(\mathbf{f}',\mathbf{b})\cdot\prod_{\substack{\mathfrak{p}\mid\mathbf{e}'\\ \mathfrak{p}\nmid\mathbf{f}'}}(1-\sigma_{\mathfrak{p}}^{-1}),$$

for all $\mathbf{b}$ as above, where $\Theta_0^E(\mathbf{e}',\mathbf{b})$ is the Stickelberger element in $\mathbb{Z}[G(E/K)]$ associated to the data $(E/K,\mathbf{e}',\mathbf{b})$ and $\mathrm{Res}_{E/K}:\mathbb{Z}[G(E/K)]\to\mathbb{Z}[G(F/K)]$ is the Galois restriction group ring morphism. Secondly, one uses the easily verified

$$Tr_{E/F}\circ\mathrm{div}_E=\mathrm{div}_F\circ N_{E/F},$$

where $Tr_{E/F}:\mathrm{Div}_{S_\infty}(E)\to\mathrm{Div}_{S_\infty}(F)$ is the usual trace at the level of Arakelov divisors. We leave the details to the interested reader. □

Proposition 3.12 (Hecke characters)**.** *Let F/K and $\mathbf{f}'$ be as above. Assume that* $\mathrm{BrSt}(F/K,\mathbf{f}')$ *holds and fix a proper $\mathcal{O}_K$–ideal $\mathbf{b}$, coprime to $\mathbf{f}'\cdot w_F$. Let $I_F(\mathbf{b})$ denote the group of fractional $\mathcal{O}_F$–ideals coprime to $\mathbf{b}$. Then, the following hold:*

(i) *For all $\mathbf{c},\mathbf{c}'\in I_F(\mathbf{b})$, we have*
$$\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}\cdot\mathbf{c}')=\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})\cdot\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}').$$

(ii) *If $\varepsilon\in F^\times$, such that $\varepsilon\equiv 1 \mod {}^\times\mathbf{b}\mathcal{O}_F$, we have*
$$\lambda_{\mathbf{b},\mathbf{f}'}(\varepsilon O_F)=\varepsilon^{\Theta_0(\mathbf{b},\mathbf{f}')}.$$

(iii) *The group morphism*
$$\lambda_{\mathbf{b},\mathbf{f}'}:I_F(\mathbf{b})\to F^\times,\qquad \mathbf{c}\to\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$$
is a Hecke character for F of conductor $\mathbf{b}\mathcal{O}_F$ and of infinite type $\Theta_0(\mathbf{b},\mathbf{f}')$.

(iv) *The Hecke character $\lambda_{\mathbf{b},\mathbf{f}'}$ is $G(F/K)$–equivariant.*

Proof. In order to prove (i), fix $\mathbf{c}$ and $\mathbf{c}'$ as above. Then observe that both sets

$$\{\lambda_{\mathbf{a},\mathbf{f}'}(\mathbf{cc}')\mid\mathbf{a}\in\mathcal{B}_{\mathbf{cc}',\mathbf{f}'}(\mathbf{F}/\mathbf{K})\},\qquad\{\lambda_{\mathbf{a},\mathbf{f}'}(\mathbf{c})\cdot\lambda_{\mathbf{a},\mathbf{f}'}(\mathbf{c}')\mid\mathbf{a}\in\mathcal{B}_{\mathbf{cc}',\mathbf{f}'}(\mathbf{F}/\mathbf{K})\}$$

satisfy properties (i–iii) in Proposition 3.8 for the fractional $\mathcal{O}_F$–ideal $\mathbf{c}\cdot\mathbf{c}'$. Then, apply the uniqueness property of these elements.

In order to prove (ii), observe that since $\varepsilon^{\Theta_0(\mathfrak{p},\mathbf{f}')}\equiv 1 \mod {}^\times\mathfrak{p}\mathcal{O}_F$ for any prime ideal $\mathfrak{p}\mid\mathbf{b}$ and $\mathrm{div}_F(\varepsilon^{\Theta_0(\mathfrak{p},\mathbf{f}')})=\Theta_0(\mathfrak{p},\mathbf{f}')\cdot\widetilde{\varepsilon\mathcal{O}_F}$ (since div_F is $G(F/K)$–equivariant), we have an equality in $F^\times$

$$\lambda_{\mathfrak{p},\mathbf{f}'}(\varepsilon O_F)=\varepsilon^{\Theta_0(\mathfrak{p},\mathbf{f}')},$$

for any such prime $\mathfrak{p}$. Now, (ii) follows from the above equality and Proposition 3.8(iii), by induction on the number of prime factors of $\mathbf{b}$.

The statement in (iii) is a direct consequence of (i) and (ii) and the definition of a Hecke character of a given conductor $\mathbf{b}$ and given infinite type $\Theta\in\mathbb{Z}[G(F/K)]$.

Finally, (iv) follows from the last statement of Proposition 3.8. □

Remark 3.13. The Hecke characters $\lambda_{\mathbf{b},\mathbf{f}'}$ constructed above are vast generalizations of Weil's Jacobi sum Hecke characters ([42]). Weil's construction can be obtained from the above when setting $K=\mathbb{Q}$. Note that in that case the Brumer–Stark conjecture is known to hold due to Stickelberger's classical theorem (see [29], Section 4.3 for more details.) The effort to "align general Brumer–Stark elements" into Hecke characters was initiated by David Hayes in [21] and achieved with different methods and at a lower level of generality by Yang in [43]. However, the reader should be aware of the fact that the proof of the 2–primary part of Theorem 5 in [43] is incorrect.

Lemma 3.14. *Under the assumptions of Proposition 3.12, there exists a (not necessarily unique) $\mathbb{Z}[G(F/K)]$–module morphism*

$$\lambda^*_{\mathbf{b},\mathbf{f}'} : I_F \to F^\times$$

which extends $\lambda_{\mathbf{b},\mathbf{f}'}$ and satisfies the properties

$$\mathrm{div}_F(\lambda^*_{\mathbf{b},\mathbf{f}'}(\mathbf{c})) = \Theta_0(\mathbf{b},\mathbf{f}') \cdot \widetilde{\mathbf{c}}, \qquad \lambda^*_{\mathbf{b},\mathbf{f}'}(\mathbf{c})^{(1+j)} = 1, \tag{12}$$

for all $\mathbf{c} \in I_F$.

Proof. For ideals $\mathbf{c} \in I_F(\mathbf{b})$, we set $\lambda^*_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) := \lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$. Now, we need to define $\lambda^*_{\mathbf{b},\mathbf{f}'}(w)$ for $\mathcal{O}_F$–primes w which divide $\mathbf{b}$. Since we obviously have

$$(1 - N\mathbf{b} \cdot \sigma_{\mathbf{b}}^{-1}) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_F),$$

Lemme 1.1 in [38, p. 82] implies that we can write (not uniquely)

$$(1 - N\mathbf{b} \cdot \sigma_{\mathbf{b}}^{-1}) = \sum_{i=1}^{n} x_i \cdot (1 - N\mathbf{p}_i \cdot \sigma_{\mathbf{p}_i}^{-1}),$$

for some $n \in \mathbb{N}$, some $x_1, \dots, x_n \in \mathbb{Z}[G]$, and some $\mathcal{O}_K$–primes $\mathbf{p}_1, \dots, \mathbf{p}_n$ which are coprime to $\mathbf{b}\mathbf{f}' \cdot w_F$. Let us fix n, $x_1, \dots, x_n$ and $\mathbf{p}_1, \dots, \mathbf{p}_n$ with the above properties. Note that we have

$$\Theta_0(\mathbf{b},\mathbf{f}') = \sum_{i=1}^{n} x_i \cdot \Theta_0(\mathbf{p}_i,\mathbf{f}').$$

Now, for each $\mathcal{O}_F$–prime w with $w \mid \mathbf{b}$ (therefore $w \in I_F(\mathbf{p}_i)$, for all i), we define

$$\lambda^*_{\mathbf{b},\mathbf{f}'}(w) := \prod_{i=1}^{n} \lambda_{\mathbf{p}_i,\mathbf{f}'}(w)^{x_i}.$$

This extends $\lambda^*_{\mathbf{b},\mathbf{f}'}$ to all $\mathcal{O}_F$–primes dividing $\mathbf{b}$. Finally, for any $\mathbf{c} \in I_F$ we set

$$\lambda^*_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) := \prod_{\mathfrak{p}|\mathbf{c}} \lambda^*_{\mathbf{b},\mathbf{f}'}(\mathfrak{p})^{n_{\mathfrak{p}}}, \qquad \text{if } \mathbf{c} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}.$$

Above, the product is taken over all the $\mathcal{O}_F$–primes $\mathfrak{p}$. The reader can easily check that the map $\lambda^*_{\mathbf{b},\mathbf{f}'}$ satisfies all the desired properties. □

Remark 3.15. If $\mathbf{c} = \epsilon\mathcal{O}_F$ is a principal ideal generated by $\epsilon \in F^\times$, then

$$\lambda^*_{\mathbf{b},\mathbf{f}'}(\epsilon\mathcal{O}_F) = \xi \cdot \epsilon^{\Theta_0(\mathbf{b},\mathbf{f}')},$$

for some root of unity $\xi \in \mu_F$, as the divisor equality (12) easily implies.

Also, with notations as in Lemma 3.11, once we pick extensions $\lambda^*_{\mathbf{b},\mathbf{e}'}$ and $\lambda^*_{\mathbf{b},\mathbf{f}'}$, the norm relations between $\lambda^*_{\mathbf{b},\mathbf{e}'}(\mathbf{c})$ and $\lambda^*_{\mathbf{b},\mathbf{f}'}(N_{E/F}(\mathbf{c}))$ become

$$N_{E/F}(\lambda^*_{\mathbf{b},\mathbf{e}'}(\mathbf{c})) = \xi_{\mathbf{c}} \cdot \lambda^*_{\mathbf{b},\mathbf{f}'}(N_{E/F}(\mathbf{c}))^{\prod_{\substack{\mathfrak{p}|\mathbf{e}' \\ \mathfrak{p}\nmid\mathbf{f}'}}(1-\sigma_{\mathfrak{p}}^{-1})},$$

where $\xi_{\mathbf{c}}$ is a root of unity in μ_F, which depends on $\mathbf{c}$. This follows easily from the divisor equality (12). Of course, if $\mathbf{b}$ and $\mathbf{c}$ are coprime, then $\xi_{\mathbf{c}} = 1$.

3.2. **The imprimitive l–adic theory.** Very recently, Greither and the second author (see [19], section 6.1) proved a strong form of the imprimitive Brumer-Stark conjecture, away from its 2–primary part and under the hypothesis that certain Iwasawa μ–invariants vanish (as conjectured by Iwasawa.) In what follows, we will state a weak form of the main result in loc.cit. This result turns out to imply the existence of an imprimitive l–adic version of (strong) Brumer–Stark elements and Hecke characters, for all odd primes l, which is sufficient for the K–theoretic constructions which follow.

Theorem 3.16 (Greither-Popescu, [19])**.** *Let F/K be as above. Let l be an odd prime and assume that the Iwasawa μ–invariant $\mu_{F,l}$ associated to F and l vanishes. Assume that $\mathbf{f}'$ is a proper $\mathcal{O}_K$–ideal divisible by all the primes dividing $\mathbf{f}l$. Then, for all prime $\mathcal{O}_K$–ideals $\mathbf{b}$ coprime to $\mathbf{f}' \cdot w_F$, we have*

$$\Theta_0(\mathbf{b}, \mathbf{f}') \in \mathrm{Ann}_{\mathbb{Z}_l[G(F/K)]}(\mathrm{CH}^1(F)^0_{T_{\mathbf{b}}} \otimes \mathbb{Z}_l).$$

Remark 3.17. Recall that a major conjecture in number theory due to Iwasawa states that $\mu_{F,l} = 0$, for all primes l. At this point, this conjecture is only known to hold if F is an abelian extension of $\mathbb{Q}$. The general belief is that it holds in general.

For a given odd prime l, the above theorem only settles an l–imprimitive form of l–adic piece of the Brumer-Stark conjecture for F/K. That is so because the ideal $\mathbf{f}'$ is forced to be divisible by all l–adic primes, whether these ramify in F/K or not. Consequently, $\Theta_0(\mathbf{b}, \mathbf{f}')$ is obtained by multiplying $\Theta_0(\mathbf{b}, \mathbf{f})$ with a $\mathbb{Z}_l[G]$–multiple of the element

$$\Pi'_{\mathbf{l}|l}(1 - \sigma_{\mathbf{l}}^{-1}),$$

(product taken over the l–adic primes $\mathbf{l}$ in K which do not divide $\mathbf{f}$) which is not invertible in $\mathbb{Z}_l[G]$, in general, therefore leading to a weaker annihilation result. On the other hand, for any $n \in \mathbb{Z}_{\geq 1}$, $\Theta_n(\mathbf{b}, \mathbf{f}')$ is obtained by multiplying $\Theta_n(\mathbf{b}, \mathbf{f})$ with a $\mathbb{Z}_l[G]$–multiple of the element

$$\Pi'_{\mathbf{l}|l}(1 - \sigma_{\mathbf{l}}^{-1} \cdot N\mathbf{l}^n),$$

which is invertible in $\mathbb{Z}_l[G]$. This explains why imprimitivity is not an issue in our upcoming K–theoretic considerations.

At this point, only very partial results towards the 2–primary piece of the Brumer-Stark conjecture have been proved, which is the reason why we are staying away from $l = 2$ throughout the rest of this paper.

For a given odd prime l, we extend the divisor map div_F by $\mathbb{Z}_l$–linearity to

$$\mathrm{div}_F \otimes \mathbf{1}_{\mathbb{Z}_l} : F^\times \otimes \mathbb{Z}_l \to \mathrm{Div}^0_{S_\infty}(F) \otimes \mathbb{Z}_l.$$

However, for the sake of simplicity, we use div_F to denote this extension as well, whenever the prime l has been chosen and fixed. The following consequences of Theorem 3.16 have identical proofs to those of Lemma 3.7, Proposition 3.8 and Proposition 3.12, respectively.

Corollary 3.18 (imprimitive l–adic Brumer-Stark elements)**.** *Assume that the hypotheses of Theorem 3.16 hold. Let $\mathbf{b}$ be a proper $\mathcal{O}_K$–ideal coprime to $\mathbf{f}' \cdot w_F$ and let $\mathbf{c}$ be a fractional $\mathcal{O}_F$–ideal. Then, the following hold.*

(i) *If $\mathbf{b}$ is a prime not dividing $\mathbf{c}$, then there exists a unique element $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in F_{\mathbf{b}}^\times \otimes \mathbb{Z}_l$, such that*

$$\mathrm{div}_F(\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})) = \Theta_0(\mathbf{b}, \mathbf{f}') \cdot \widetilde{\mathbf{c}}.$$

(ii) *There exists a unique element* $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in (F^\times/\mu_F)\otimes\mathbb{Z}_l$, *such that*

$$\mathrm{div}_F(\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c})) = \Theta_0(\mathbf{b},\mathbf{f}')\cdot\widetilde{\mathbf{c}}.$$

Further, any $\lambda \in F^\times\otimes\mathbb{Z}_l$ *whose class in* $(F^\times/\mu_F)\otimes\mathbb{Z}_l$ *is* $\widehat{\lambda}_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ *satisfies*

$$\lambda^{(1+j)} = 1.$$

Proof. The proofs of (i) and the first equality in (ii) are identical to those of equalities (7) and (9), respectively. The only difference is that instead of assuming the Brumer-Stark conjecture, here one uses Theorem 3.16. Finally, the proof of the last equality in (ii) is identical to that of Lemma 3.7, part (i). □

Corollary 3.19 (imprimitive l–adic strong Brumer–Stark elements)**.** *Assume that the hypotheses of Theorem 3.16 hold. Let* $\mathbf{c}$ *be a fractional* $\mathcal{O}_F$*–ideal. Let* $\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$ *be the set of proper* $\mathcal{O}_K$*–ideals* $\mathbf{b}$ *which are coprime to* $\mathbf{c}\mathbf{f}'\cdot w_F$*. Then, there exist unique elements* $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$ *in* $F^\times\otimes\mathbb{Z}_l$*, for all* $\mathbf{b}\in\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)\otimes\mathbb{Z}_l$*, such that:*

(i) $\mathrm{div}_F(\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})) = \Theta_0(\mathbf{b},\mathbf{f}')\cdot\widetilde{\mathbf{c}}$*, for all* $\mathbf{b}\in\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$*.*

(ii) *If* $\mathbf{b}$ *is a prime ideal in* $\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)$*, then* $\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}) \in F_{\mathbf{b}}^\times\otimes\mathbb{Z}_l$*.*

(iii) *If* $\mathbf{a},\mathbf{b}\in\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)\otimes\mathbb{Z}_l$*, then*

$$\lambda_{\mathbf{ab},\mathbf{f}'}(\mathbf{c}) = \lambda_{\mathbf{a},\mathbf{f}'}(\mathbf{c})^{\mathrm{N}\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1}}\cdot\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}).$$

Moreover, for all $\sigma\in G(F/K)$ *and all* $\mathbf{b}\in\mathcal{B}_{\mathbf{c},\mathbf{f}'}(F/K)\otimes\mathbb{Z}_l(=\mathcal{B}_{\sigma(\mathbf{c}\otimes\mathbb{Z}_l),\mathbf{f}'}(F/K))$*, we have*

$$\lambda_{\mathbf{b},\mathbf{f}'}(\sigma(\mathbf{c})) = \sigma(\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})).$$

Proof. The proof is identical to that of Proposition 3.8. □

Corollary 3.20 (imprimitive l–adic Hecke characters)**.** *Assume that the hypotheses of Theorem 3.16 hold. Fix a proper* $\mathcal{O}_K$*–ideal* $\mathbf{b}$*, coprime to* $\mathbf{f}'\cdot w_F$*. Let* $I_F(\mathbf{b})$ *denote the group of fractional* $\mathcal{O}_F$*–ideals coprime to* $\mathbf{b}$*. Then, we have:*

(i) *For all* $\mathbf{c},\mathbf{c}'\in I_F(\mathbf{b})\otimes\mathbb{Z}_l$*,*

$$\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}\cdot\mathbf{c}') = \lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})\cdot\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c}').$$

(ii) *If* $\varepsilon\in F^\times$*, such that* $\varepsilon\equiv 1 \mod {}^\times\mathbf{b}\mathcal{O}_F$*, we have*

$$\lambda_{\mathbf{b},\mathbf{f}'}(\varepsilon O_F) = \varepsilon^{\Theta_0(\mathbf{b},\mathbf{f}')}.$$

(iii) *The* $\mathbb{Z}_l$*–module morphism*

$$\lambda_{\mathbf{b},\mathbf{f}'} : I_F(\mathbf{b})\otimes\mathbb{Z}_l \to F^\times\otimes\mathbb{Z}_l, \qquad \mathbf{c}\to\lambda_{\mathbf{b},\mathbf{f}'}(\mathbf{c})$$

is $G(F/K)$*–invariant.*

Proof. The proof is identical to that of Proposition 3.12(i), (ii), (iv). □

Remark 3.21. Obvious analogues of Lemmas 3.11 (norm relations) and 3.14 (extension to $I_F\otimes\mathbb{Z}_l$) hold for the imprimitive l–adic maps

$$\lambda_{\mathbf{b},\mathbf{f}'} : I_F(\mathbf{b})\otimes\mathbb{Z}_l \to F^\times\otimes\mathbb{Z}_l$$

as well. We leave the details to the interested reader. An extension of $\lambda_{\mathbf{b},\mathbf{f}'}$ to $I_F\otimes\mathbb{Z}_l$ as in Lemma 3.14 will be denoted by $\lambda^*_{\mathbf{b},\mathbf{f}'}$. Of course, the imprimitive l–adic analogue of Remark 3.15 holds.

4. The Galois equivariant Stickelberger splitting map

In this section, we will construct the l–adic Galois equivariant Stickelberger splitting map in the Quillen localization sequence associated to the top field F in an abelian Galois extension F/K of number fields, with K totally real and F either totally real or CM. The main idea is to use the imprimitive l–adic Brumer–Stark elements for certain cyclotomic extensions of F along with powers of Bott elements to construct special elements in the K–theory of the top field F. Then, one uses these special elements to construct the desired Galois equivariant splitting map.

From now on, we fix an abelian extension F/K as above, denote by G its Galois group, fix an odd prime l and work under the assumption that Iwasawa's conjecture on the vanishing of the μ–invariant associated to F and l holds. Further, we fix nontrivial $\mathcal{O}_K$–ideals $\mathbf{f}$ and $\mathbf{b}$, with $\mathbf{f}$ divisible by the (finite) conductor of F/K and $\mathbf{b}$ coprime to $l\mathbf{f}$.

In the case $K = \mathbb{Q}$, a Stickelberger splitting map was constructed in [1]. The construction in loc. cit. was refined in [4], [5]. However, none of these constructions led to Galois equivariant splitting maps.

In [6], we constructed a Galois equivariant Stickelberger splitting map for arbitrary totally real base fields K. However, that construction was very different from the one we are about to describe in that it relies on a different class of special elements in the K–groups of the top field F.

4.1. K**–theoretic tools.** In what follows, we will use freely K–theory with(out) coefficients as well as the theory of Bockstein morphisms and that of Bott elements at the level of K–theory with coefficients. For the precise definitions and main properties the reader can consult §3 in [6]. However, just to set the notations, we will briefly recall the main objects and facts.

Let R be a unital ring and l be an odd prime number. Then the K–groups with coefficients $K_n(R, \mathbb{Z}/l^k)$, $n \geq 1$, $k \geq 1$, sit inside short exact sequences

$$0 \longrightarrow K_n(R)/l^k \longrightarrow K_n(R, \mathbb{Z}/l^k) \xrightarrow{b=b_R} K_{n-1}(R)[l^k] \longrightarrow 0\,, \tag{13}$$

where b (sometimes denoted b_R, to emphasize dependence on the ring) is the Bockstein morphism associated to R, n and l^k. We remind the reader that once l^k and n are fixed, b and the exact sequences above are functorial in R.

If we assume that the characteristic of R is not l and that R contains the group μ_{l^k} of l^k–roots of unity and fix a generator ξ_{l^k} of μ_{l^k}, then we have canonical special elements $\beta(\xi_{l^k})$ in $K_2(R, \mathbb{Z}/l^k)$ called Bott elements. Consequently, the product structure "$*$" at the level of K–theory with coefficients leads to canonical elements $\beta(\xi_{l^k})^{*n}$ in $K_{2n}(R, \mathbb{Z}/l^k)$, for all $n \geq 1$. For given l^k and n, the elements $\beta(\xi_{l^k})^{*n}$ are functorial in R and the chosen ξ_{l^k} in the obvious sense.

4.2. **Constructing maps** Λ_v**.** In this section, we fix an integer $n \geq 1$, an odd prime l, and a nonzero $\mathcal{O}_F$–prime v. We let $k_v := \mathcal{O}_F/v$ denote the residue field of v. Our main goal is to construct a group morphism

$$\Lambda_v : K_{2n-1}(k_v)_l \to K_{2n}(F)_l$$

satisfying certain properties. Recall that the group $K_{2n-1}(k_v)$ is cyclic of order $q_v^n - 1$, where $q_v = \mid k_v \mid$. The idea behind constructing Λ_v is first to get our hands on an explicit generator ξ_v of $K_{2n-1}(k_v)_l$ and then construct an explicit element in $K_{2n}(F)_l$ annihilated by the order of this generator and declare that to be the image of ξ_v via Λ_v

Obviously, if $\mid K_{2n-1}(k_v)_l \mid = 1$, then $\xi_v = 1$ and Λ_v is the trivial map. So, let us assume that $\mid K_{2n-1}(k_v)_l \mid = l^k$, for some $k > 0$. This implies that $v \nmid l$. Also, it is easily seen (see the proof of Lemma 2 [1, p. 336]) that this also implies that

$$k > v_l(n),$$

where $v_l(n)$ denotes the usual l–adic valuation of n.

Next, we let $E := F(\mu_{l^k})$ and fix an $\mathcal{O}_E$–prime w sitting above v. It is easily seen (for full proofs see the proof of Lemma 2 [1, p. 336]) that $k_w = k_v(\mu_{l^k})$ and consequently that

$$\mid K_{2n-1}(k_w)_l \mid = l^{v_l(n)+k} \tag{14}$$

and that the image of the transfer map $Tr_{w/v} : K_{2n-1}(k_w)_l \to K_{2n-1}(k_v)_l$ satisfies

$$\mathrm{Im}(Tr_{w/v}) = K_{2n-1}(k_v)_l. \tag{15}$$

Fix a generator ξ_{l^k} of μ_{l^k} inside E. By abuse of notation, we will denote by ξ_{l^k} its image in k_w via the reduction modulo w map $\mathcal{O}_E \to \mathcal{O}_E/w = k_w$. This way, we obtain a generator ξ_{l^k} of μ_{l^k} inside k_w. A result of Browder [7] shows that $\beta(\xi_{l^k})^{*n}$ is a generator of $K_{2n}(k_w, \mathbb{Z}/l^k)$. On the other hand, since $K_{2n}(k_w) = 0$, the Bockstein sequence (13) gives a group isomorphism

$$b : K_{2n}(k_w, \mathbb{Z}/l^k) \simeq K_{2n-1}(k_w)[l^k].$$

Therefore, $b(\beta(\xi_{l^k})^{*n})$ is a generator of $K_{2n-1}(k_w)[l^k]$. Consequently, equality (15) allows us to make the following.

Definition 4.1. Let ξ_v be a generator of $K_{2n-1}(k_v)_l$, such that

$$\xi_v^{l^{v_l(n)}} = Tr_{w/v}(b(\beta(\xi_{l^k})^{*n}). \tag{16}$$

Now, we proceed to the construction of a special element in $K_{2n}(F)_l$ annihilated by l^k. Let $\Gamma := G(E/K)$ and let G_v and I_v denote the decomposition and inertia groups of v in $G(F/K)$, respectively. Also, let

$$\mathbf{f}_E^* := \mathbf{f} \cdot l.$$

Note that $\mathbf{f}_E^*$ is divisible by all the primes which ramify in E/K and all the l–adic primes and that $\mathbf{b}$ and $\mathbf{f}_E^*$ are coprime. If we denote by $\{\Theta_m^E(\mathbf{b}, \mathbf{f}_E^*)\}_{m\geq 0}$ the higher Stickelberger elements associated to the data $(E/K, \mathbf{b}, \mathbf{f}_E^*)$, then these are all in $\mathbb{Z}_l[\Gamma]$. Further, if we set

$$\gamma_l := \prod_{\substack{\mathbf{l}\mid l\\ \mathbf{l}\nmid\mathbf{f}}} (1 - \sigma_{\mathbf{l}}^{-1} \cdot N\mathbf{l}^n)^{-1} \in \mathbb{Z}_l[G], \tag{17}$$

then we have the obvious equality

$$\mathrm{Res}_{E/F}(\Theta_n^E(\mathbf{b}, \mathbf{f}_E^*) = \Theta_n(\mathbf{b}, \mathbf{f}) \cdot \gamma_l^{-1}. \tag{18}$$

Also, note that E/K is a CM abelian extension of a totally real number field. Consequently, Lemma 3.20 applies to the data $(E/K, \mathbf{f}_E^*, \mathbf{b})$. In particular, Remark 3.21 allows us to pick a $\mathbb{Z}_l[\Gamma]$–linear morphism

$$\lambda^*_{\mathbf{b},\mathbf{f}_E^*} : I_E \otimes \mathbb{Z}_l \to E^\times \otimes \mathbb{Z}_l,$$

which extends the l–adic imprimitive Hecke character $\lambda_{\mathbf{b},\mathbf{f}_E^*}$ of conductor $\mathbf{b}$ and satisfies the properties

$$\mathrm{div}_E(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(\mathbf{c})) = \Theta_0^E(\mathbf{b},\mathbf{f}_E^*)\cdot\mathbf{c}, \qquad \lambda^*_{\mathbf{b},\mathbf{f}_E^*}(\mathbf{c})^{(1+j)} = 1, \tag{19}$$

for all $\mathbf{c}\in I_E$.

Let us pick an $\mathcal{O}_E$–prime w sitting above v. We let Γ_w and I_w denote its decomposition and inertia groups in E/K. We view $\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w)$ as an element in $K_1(E)_l$ after the obvious identification $E^\times\otimes\mathbb{Z}_l\simeq K_1(E)_l$. Consequently, we obtain an element

$$\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*n}) \in K_1(E)_l * K_{2n-1}(E)[l^k] \subseteq K_{2n}(E)[l^k]$$

which is mapped via the usual transfer morphism $Tr_{E/F}: K_{2n}(E)_l\to K_{2n}(F)_l$ to

$$Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*n})) \in K_{2n}(F)[l^k].$$

Definition 4.2. Since the chosen generator ξ_v of $K_{2n-1}(k_v)_l$ has order l^k, there exists a unique $\mathbb{Z}_l$–linear map $\Lambda_v : K_{2n-1}(k_v)_l\to K_{2n}(F)_l$ which satisfies

$$\Lambda_v(\xi_v) := Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*n}))^{\gamma_l}.$$

Remark 4.3. Note that the map Λ_v depends in an easily described manner on the several choices we have made along the way: that of a prime w sitting above v in E, that of a generator ξ_{l^k} of μ_{l^k} in E, that of a generator ξ_v of $K(k_v)_l$ and, finally, that of a $\mathbb{Z}_l[\Gamma]$–linear extension $\lambda^*_{\mathbf{b},\mathbf{f}_E^*}$ of the l–adic imprimitive Hecke character $\lambda_{\mathbf{b},\mathbf{f}_E^*}$ of conductor $\mathbf{b}$.

The functoriality properties of K–groups imply that we have the following obvious isomorphisms of $\mathbb{Z}_l[G]$– and $\mathbb{Z}_l[\Gamma]$–modules, respectively, for all $m\geq 0$.

$$K_m(k_v)_l\otimes_{\mathbb{Z}_l[G_v]}\mathbb{Z}_l[G]\simeq\bigoplus_{\widehat{\sigma}\in G/G_v}K_m(k_{\sigma(v)})_l, \quad \xi\otimes\sigma\to(1,\dots,1,\sigma(\xi),1,\dots,1) \tag{20}$$

$$K_m(k_w)_l\otimes_{\mathbb{Z}_l[\Gamma_w]}\mathbb{Z}_l[\Gamma]\simeq\bigoplus_{\widehat{\gamma}\in \Gamma/\Gamma_w}K_m(k_{\gamma(w)})_l, \quad \xi\otimes\gamma\to(1,\dots,1,\gamma(\xi),1,\dots,1). \tag{21}$$

Above $\sigma\in G$, $\gamma\in\Gamma$, $\widehat{\sigma}$ and $\widehat{\gamma}$ are their classes in G/G_v and Γ/Γ_w, respectively, and $\sigma(\xi)$ and $\gamma(\xi)$ appear in the $\sigma(v)$ and $\gamma(w)$–components, respectively. In what follows, we will freely identify the left and right hand sides of these isomorphisms. In particular, if $\xi\in K_m(k_w)_l$ (or $\xi\in K_m(k_v)_l$) and $\alpha\in\mathbb{Z}_l[\Gamma]$ (or $\alpha\in\mathbb{Z}_l[G]$), then $\xi\otimes\alpha$ will also be sometimes denoted by ξ^α and thought of as an element in the direct sum on the right hand side of isomorphisms (20) and (21).

Remark 4.4. Let us note that if we set $\mathbf{c}:=w$, then (19) can be rewritten as

$$\partial_E(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w)) = \partial_{E,\Gamma\cdot w}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w)) = 1\otimes\Theta_0^E(\mathbf{b},\mathbf{f}_E^*), \tag{22}$$

where $\partial_{E,\Gamma\cdot w}: K_1(E)_l\to\bigoplus_{\widehat{\gamma}\in\Gamma/\Gamma_w}K_0(k_{\gamma(w)})_l$ is the $\Gamma\cdot w$–supported boundary map and we identify

$$\bigoplus_{\widehat{\gamma}\in\Gamma/\Gamma_w}K_0(k_{\gamma(w)})_l\simeq K_0(k_w)_l\otimes_{\mathbb{Z}_l[\Gamma_w]}\mathbb{Z}_l[\Gamma]\simeq\mathbb{Z}_l\otimes_{\mathbb{Z}_l[\Gamma_w]}\mathbb{Z}_l[\Gamma]$$

as in the last displayed isomorphism above for $m = 0$. Also, let us note that since $K_{2n-1}(k_v)_l$ is a cyclic group (also cyclic $\mathbb{Z}_l[G_v]$–module) generated by ξ_v, we have

$$\bigoplus_{\widehat{\sigma}\in G/G_v} K_{2n-1}(k_{\sigma(v)})_l \simeq K_{2n-1}(k_v)_l \otimes_{\mathbb{Z}_l[G_v]} \mathbb{Z}_l[G] = \mathbb{Z}_l[G]\cdot(\xi_v \otimes 1).$$

Therefore, $\bigoplus_{\widehat{\sigma}\in G/G_v} K_{2n-1}(k_{\sigma(v)})_l$ is a cyclic $\mathbb{Z}_l[G]$–module generated by ξ_v.

Theorem 4.5. *The map Λ_v defined above satisfies the following properties.*

1. *It is $\mathbb{Z}_l[G_v]$–linear.*
2. *If $\partial_{F,G\cdot v} : K_{2n}(F)_l \to \bigoplus_{\widehat{\sigma}\in G/G_v} K_{2n-1}(k_{\sigma(v)})_l$ is the $G\cdot v$–supported boundary map then, after the identification* (20) *for $m := 2n-1$, we have*
$$\partial_F(\Lambda_v(\xi)) = \partial_{F,G\cdot v}(\Lambda_v(\xi)) = \xi^{l^{v_l(n)}\cdot\Theta_n(\mathbf{b},\mathbf{f})},$$
for all $\xi \in K_{2n-1}(k_v)_l$.

Proof. (1) Let $\sigma \in G_v$. Assume that $\sigma = \sigma_v^\alpha \cdot \rho$, where σ_v is a v–Frobenius lift (from G_v/I_v to G_v), $\alpha \in \mathbb{Z}_{\geq 0}$ and $\rho \in I_v$. Since Λ_v is $\mathbb{Z}_l$–linear, we have

$$\Lambda_v(\sigma(\xi_v)) = \Lambda_v(\sigma_v^\alpha(\xi_v)) = \Lambda_v(\xi_v^{q_v^{n\alpha}}) = \Lambda_v(\xi_v)^{q_v^{n\alpha}}.$$

Now, let us consider a lift $\overline{\sigma} \in \Gamma_w$ of σ of the form $\overline{\sigma} = \sigma_w^\alpha \cdot \overline{\rho}$, where σ_w is a w–Frobenius lift in Γ_w which restricts to σ_v and $\overline{\rho} \in I_w$ restricts to ρ. We have

$$\overline{\sigma}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w)) = \lambda^*_{\mathbf{b},\mathbf{f}_E^*}(\overline{\sigma}(w)) = \lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w).$$

Also, since $v \nmid l$, we have $\overline{\sigma}(\xi_{l^k}) = \sigma_w^\alpha(\xi_{l^k}) = \xi_{l^k}^{q_v^\alpha}$. Consequently, the functoriality of the Bockstein and Bott maps b and β, respectively, gives

$$\overline{\sigma}(b(\beta(\xi_{l^k})^{*n})) = b(\beta(\xi_{l^k})^{*n})^{q_v^{n\alpha}}.$$

The last two displayed equalities imply that

$$\begin{aligned} \sigma(\Lambda_v(\xi_v)) &= \sigma \circ Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*n}))^{\gamma_l} = \\ Tr_{E/F}\circ\overline{\sigma}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*n}))^{\gamma_l} &= Tr_{E/F}(\overline{\sigma}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w)) * \overline{\sigma}(b(\beta(\xi_{l^k})^{*n})))^{\gamma_l} = \\ Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*n}))^{q_v^{n\alpha}\gamma_l} &= \Lambda_v(\xi_v)^{q_v^{n\alpha}}. \end{aligned}$$

Consequently, $\Lambda_v(\sigma(\xi_v)) = \sigma(\Lambda_v(\xi_v))$, which concludes the proof of (1).

(2) Note that since $K_{2n-1}(k_v)_l$ is generated by ξ_v and Λ_v is $\mathbb{Z}_l[G_v]$–linear, it suffices to prove (2) for $\xi := \xi_v$. For that purpose, we need the following commutative diagrams in the category of $\mathbb{Z}[\Gamma]$–modules. The first of these is diagram 4.7 in [1]:

$$\begin{tikzcd} K_1(E)\times K_{2n-1}(O_E) \arrow[r, "*"] \arrow[d, "\oplus_{\mathbf{w}}(\partial_{E,\mathbf{w}}\times\mathrm{red}_{\mathbf{w}})"] & K_{2n}(O_E) \arrow[d, "\partial_E"] \\ \bigoplus_{\mathbf{w}}(K_0(k_{\mathbf{w}})\times K_{2n-1}(k_{\mathbf{w}})) \arrow[r, "*"] & \bigoplus_{\mathbf{w}} K_{2n-1}(k_{\mathbf{w}}). \end{tikzcd} \tag{23}$$

Above, $\mathbf{w}$ runs through all nonzero $\mathcal{O}_E$–primes and $\mathrm{red}_{\mathbf{w}} : K_{2n-1}(O_E) \to K_{2n-1}(k_{\mathbf{w}})$ is the map at the level of K–groups induced by the reduction modulo $\mathbf{w}$ morphism $O_E \to O_E/\mathbf{w} = k_{\mathbf{w}}$. In particular, note that the functoriality of the Bockstein and Bott maps b and β implies that we have

$$\mathrm{red}_w(b_E(\beta(\xi_{l^k})^{*n})) = b_w(\beta(\xi_{l^k})^{*n}), \tag{24}$$

where w is the chosen $\mathcal{O}_E$–prime sitting above the chosen $\mathcal{O}_F$–prime v.

The second commutative diagram is diagram 4.1 in [1]:

$$
\begin{tikzcd}
K_{2n}(E) \arrow[r, "\partial_E"] \arrow[d, "Tr_{E/F}"] & \bigoplus_{\mathbf{v}}(\bigoplus_{\mathbf{w}|\mathbf{v}} K_{2n-1}(k_{\mathbf{w}})) \arrow[d, "\bigoplus_{\mathbf{v}}(\prod_{\mathbf{w}|\mathbf{v}} Tr_{\mathbf{w}/\mathbf{v}})=:\overline{Tr}"] \\
K_{2n}(F) \arrow[r, "\partial_F"] & \bigoplus_{\mathbf{v}} K_{2n-1}(k_{\mathbf{v}})
\end{tikzcd}
\tag{25}
$$

where $\mathbf{v}$ runs through all nonzero $\mathcal{O}_F$–primes.

Diagram (25) above yields the equality

$$\partial_F(\Lambda_v(\xi_v)) = \overline{Tr}(\partial_E(\lambda^*_{\mathbf{b},\mathbf{f}^*_E}(w) * b_E(\beta(\xi_{l^k})^{*n})))^{\gamma_l}. \tag{26}$$

Since $\mu_{l^k} \subseteq E^\times$, we have $l^k \mid w_n(E)_l$. This observation combined with the functoriality of the maps b and β lead to the following equality in $\bigoplus_{\widetilde{\gamma}\in\Gamma/\Gamma_w} K_{2n-1}(k_{\gamma(w)})[l^k]$.

$$b_{\gamma(w)}(\beta(\xi_{l^k})^{*n}) = b_w(\beta(\xi_{l^k})^{*n})^{t_n(\gamma)}, \qquad \text{for all } \gamma \in \Gamma,$$

where $t_n : \mathbb{Z}_l/w_n(E)_l[\Gamma] \simeq \mathbb{Z}_l/w_n(E)_l[\Gamma]$ is the map defined right before Theorem 2.4. Consequently, (23) above, combined with equalities (22) and (24) yield

$$\partial_E(\lambda^*_{\mathbf{b},\mathbf{f}^*_E}(w) * b_E(\beta(\xi_{l^k})^{*n})) = \{b_w(\beta(\xi_{l^k})^{*n})\}^{t_n(\widehat{\Theta_0^E(\mathbf{b},\mathbf{f}^*_E)})},$$

where the notations are as in Theorem 2.4. Now, Theorem 2.4 implies that we have

$$t_n(\widehat{\Theta_0^E(\mathbf{b},\mathbf{f}^*_E)}) = \widehat{\Theta_n^E(\mathbf{b},\mathbf{f}^*_E)} \text{ in } \mathbb{Z}_l/w_n(E)_l[\Gamma].$$

Consequently, the last two displayed equalities imply that

$$\partial_E(\lambda^*_{\mathbf{b},\mathbf{f}^*_E}(w) * b_E(\beta(\xi_{l^k})^{*n})) = \{b_w(\beta(\xi_{l^k})^{*n})\}^{\Theta_n^E(\mathbf{b},\mathbf{f}^*_E)}.$$

Now, combine the last equality successively with (26), (16) and (18) to obtain

$$\partial_F(\Lambda_v(\xi_v)) = Tr_{w/v}(b_w(\beta(\xi_{l^k})^{*n}))^{\gamma_l\cdot\Theta_n(\mathbf{b},\mathbf{f})\cdot\gamma_l^{-1}} = \xi_v^{l^{v_l(n)}\cdot\Theta_n(\mathbf{b},\mathbf{f})}.$$

This concludes the proof of the theorem. □

4.3. **Constructing a map Λ.** For every nonzero $\mathcal{O}_K$–prime v_0, we pick an $\mathcal{O}_F$–prime $v \mid v_0$. The isomorphisms (20) for $m := 2n-1$ yield an explicit isomorphism of $\mathbb{Z}_l[G]$–modules

$$\bigoplus_{v'} K_{2n-1}(k_{v'})_l \simeq \bigoplus_{v_0} \left(\mathbb{Z}_l[G] \otimes_{\mathbb{Z}_l[G_v]} K_{2n-1}(k_v)_l\right),$$

where v' runs over all the nonzero $\mathcal{O}_F$–primes and v_0 runs over all the nonzero $\mathcal{O}_K$–primes. For each chosen v, we construct a map $\Lambda_v : K_{2n-1}(k_v)_l \to K_{2n-1}(F)_l$ as in the previous section. Since each of these maps Λ_v is $\mathbb{Z}_l[G_v]$–linear (see Theorem 4.5 (2)), the isomorphism above implies that there exists a unique $\mathbb{Z}_l[G]$–linear map

$$\Lambda : \bigoplus_{v'} K_{2n-1}(k_{v'})_l \to K_{2n}(F)_l$$

which equals Λ_v when restricted to $K_{2n-1}(k_v)_l$ for each of the chosen primes v. Obviously, after identifying the two sides of the isomorphism above, this unique $\mathbb{Z}_l[G]$–linear map is given by

$$\Lambda := \prod_{v_0}(1 \otimes \Lambda_v). \tag{27}$$

Theorem 4.6. *The map Λ defined above satisfies the following.*

(1) *It is* $\mathbb{Z}_l[G]$*–linear.*
(2) *For all* $\xi \in \bigoplus_{v'} K_{2n-1}(k_{v'})_l$*, we have*

$$(\partial_F \circ \Lambda)(\xi) = \xi^{l^{v_l(n)} \cdot \Theta_n(\mathbf{b},\mathbf{f})}.$$

Proof. Part (1) is a direct consequence of (27) and Theorem 4.5(1). Part (2) is a direct consequence of (27) and Theorem 4.5(2). □

5. The Stickelberger splitting map and the divisible elements

In this section, we will use the special elements in the even K–groups of a totally real or CM number field constructed in the previous section to investigate the Galois module structure of the groups of divisible elements in these K–groups. In particular, this will lead to a reformulation and a possible generalization of a classical conjecture of Iwasawa.

The following technical lemma extends the computations in [5, p. 8-9] and will be needed shortly.

Lemma 5.1. *Let* L/F *be a Galois extension of number fields. Let* L^H *be the Hilbert class field of* L*. Let* l *be a prime such that* $L^H \cap L(\mu_{l^\infty}) = L$*. Let* $\mathbf{p} \subset \mathcal{O}_L$ *be a prime. For any positive integer* $m >> 0$ *there exist infinitely many primes* $\mathbf{q} \subset \mathcal{O}_L$*, with* $\mathbf{q} \nmid l$*, such that:*

(1) $[\mathbf{q}] = [\mathbf{p}]$ *in* $Cl(\mathcal{O}_L)$*.*
(2) $l^m \mid\mid N\mathbf{q} - 1$*.*
(3) $\mathbf{q} \cap O_F$ *splits in* L/F*.*

Proof. Let $m \in \mathbb{Z}_{>0}$ sufficiently large so that $L(\mu_{l^m}) \neq L(\mu_{l^{m+1}})$. Since $L(\mu_{l^{m+1}}) \cap L^H = L$, there exists a unique $\sigma \in G(L(\mu_{l^{m+1}})L^H/F)$ such that $\sigma_{|L^H} = Fr_{\mathbf{p}}$, $\sigma_{|L(\mu_{l^{m+1}})}$ is a generator of $G(L(\mu_{l^{m+1}})/L(\mu_{l^m}))$, where $Fr_{\mathbf{p}}$ is the Frobenius automorphism associated to $\mathbf{p}$ in $G(L^H/L)$. Note that, by definition $\sigma_{|L} = \mathrm{Id}$ and therefore $\sigma \in G(G(L(\mu_{l^{m+1}})L^H/L)$, which is an abelian Galois group.

By Chebotarev's density theorem, there are infinitely many primes $q \subset \mathcal{O}_K$ such that $q \nmid l$ and $\sigma = \widetilde{Fr}_q$, where $\widetilde{Fr}_q$ is (any) Frobenius automorphism associated to q in $G(L(\mu_{l^{m+1}})L^H/F)$. Let $\mathbf{q}$ be a prime in O_L sitting above q.

Since $\sigma_{|L} = \mathrm{Id}$, $q = \mathbf{q} \cap O_F$ splits completely in L/F.

Since $\widetilde{Fr}_{\mathbf{q}}\,|_{L(\mu_{l^{m+1}})}$ is a generator of $G(L(\mu_{l^{m+1}})/L(\mu_{l^m}))$, we have

$$\widetilde{Fr_{\mathbf{q}}}(\xi) = \xi^{N\mathbf{q}} = 1, \qquad \text{for all } \xi \in \mu_{l^m},$$

and $\widetilde{Fr_{\mathbf{q}}}(\xi) = \xi^{N\mathbf{q}} \neq 1$ for a generator ξ of $\mu_{l^{m+1}}$. Consequently, $l^m \mid\mid N\mathbf{q} - 1$.

Finally, if we denote by $Fr_{\mathbf{q}}$ the Frobenius morphism associated to $\mathbf{q}$ in $G(L^H/L)$, we have $Fr_{\mathbf{q}} = \widetilde{Fr}_{\mathbf{q}}\,|_{L^H} = Fr_{\mathbf{p}}$. Consequently, Artin's reciprocity isomorphism

$$Cl(\mathcal{O}_L) \to G(L^H/L), \qquad [\mathbf{a}] \to Fr_{\mathbf{a}},$$

implies that $[\mathbf{p}] = [\mathbf{q}]$ in $Cl(\mathcal{O}_L)$. □

We work with the notations and under the assumptions of the previous section. In addition, we will assume from now on that the odd prime l does not divide the order $|G|$ of the Galois group $G := G(F/K)$. We denote by $\widehat{G}(\overline{\mathbb{Q}}_l)$ the set of irreducible $\overline{\mathbb{Q}}_l$–valued characters of G. For $\chi \in \widehat{G}(\overline{\mathbb{Q}}_l)$ we let

$$e_\chi := 1/|G| \sum_{g \in G} \chi(g) \cdot g^{-1}$$

denote its associated idempotent element in $\mathbb{Z}_l[\chi][G]$. Also, we will let

$$\widetilde{e_\chi} := \sum_{\sigma \in G(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} e_{\chi^\sigma}.$$

Note that $\widetilde{e_\chi}$ is the irreducible idempotent in $\mathbb{Z}_l[G]$ associated to the irreducible $\mathbb{Q}_l$–valued character $\widetilde{\chi} = \sum_{\sigma \in G(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)} \chi^\sigma$ of G. Also, note that $\widetilde{e_\chi}$ only depends on the orbit of χ under the natural action of $G(\mathbb{Q}_l/\mathbb{Q}_l)$ on $\widehat{G}(\overline{\mathbb{Q}}_l)$. In what follows, we denote the set of such orbits by $\widehat{G}(\mathbb{Q}_l)$ and think of $\widetilde{\chi}$ as an element of $\widehat{G}(\mathbb{Q}_l)$, for every $\chi \in \widehat{G}(\overline{\mathbb{Q}}_l)$. Obviously, we have

$$\mathbb{Z}_l[G] = \bigoplus_{\widetilde{\chi} \in \widehat{G}(\mathbb{Q}_l)} \widetilde{e_\chi}\mathbb{Z}_l[G], \qquad \widetilde{e_\chi}\mathbb{Z}_l[G] \simeq \mathbb{Z}_l[\chi],$$

where the ring isomorphism above sends $x \to \chi(x)$, for every $x \in \widetilde{e_\chi}\mathbb{Z}_l[G]$ and $\chi \in \widehat{G}(\overline{\mathbb{Q}}_l)$. Also, for every $\mathbb{Z}_l[G]$–module M we have

$$M = \bigoplus_{\widetilde{\chi} \in \widehat{G}(\mathbb{Q}_l)} \widetilde{e_\chi} M, \tag{28}$$

where $\widetilde{e_\chi}M$ is a $\widetilde{e_\chi}\mathbb{Z}_l[G]$–module in the obvious manner. From now on, we denote

$$M^\chi := \widetilde{e_\chi} M$$

and view it as a $\mathbb{Z}_l[\chi]$–module via the ring isomorphism $\widetilde{e_\chi}\mathbb{Z}_l[G] \simeq \mathbb{Z}_l[\chi]$ described above. Obviously, $M \to M^\chi$ are exact functors from the category of $\mathbb{Z}_l[G]$–modules to that of $\mathbb{Z}_l[\chi]$–modules. If $f : M \to N$ is a morphism of $\mathbb{Z}_l[G]$–modules, then $f^\chi : M^\chi \to N^\chi$ denotes its image via the above functor. Also, if M is as above and $x \in M$, then $x^\chi := \widetilde{e_\chi} \cdot x$ will be viewed as an element in M^χ. In particular, if $x \in \mathbb{Z}_l[G]$, then we identify x^χ with $\chi(x) \in \mathbb{Z}_l[\chi]$ via the ring isomorphism $\widetilde{e_\chi}\mathbb{Z}_l[G] \simeq \mathbb{Z}_l[\chi]$ described above.

From now on, we fix an embedding $\mathbb{C} \hookrightarrow \mathbb{C}_l$. As a consequence, the higher Stickelberger elements $\Theta_n(\mathbf{b}, \mathbf{f})$ will be viewed in $\mathbb{Z}_l[G] \subseteq \mathbb{C}_l[G]$. Also, this embedding identifies $\widehat{G}(\mathbb{C})$ and $\widehat{G}(\overline{\mathbb{Q}}_l)$. Therefore, Remark 2.5 and the conventions made above give the following equality in $\mathbb{Z}_l[\chi]$, for all $\chi \in \widehat{G}(\overline{\mathbb{Q}}_l)$:

$$\Theta_n(\mathbf{b}, \mathbf{f})^\chi = (1 - N\mathbf{b}^{n+1} \cdot \chi(\sigma_{\mathbf{b}})^{-1}) \cdot L_{\mathbf{f}}(\chi^{-1}, -n). \tag{29}$$

Let us fix $\chi \in \widehat{G}(\overline{\mathbb{Q}}_l)$ and $n \geq 1$. Consider the χ component of the Quillen localization sequence (1):

$$0 \to K_{2n}(\mathcal{O}_F)_l^\chi \to K_{2n}(F)_l^\chi \xrightarrow{\partial_F^\chi} \bigoplus_{v_0} \Big(\bigoplus_{v|v_0} K_{2n-1}(k_v)_l\Big)^\chi \to 0. \tag{30}$$

Observe that since $\bigoplus_{v|v_0} K_{2n-1}(k_v)_l$ is a cyclic $\mathbb{Z}_l[G]$-module generated by ξ_v (see (20) in the previous section), $\big(\bigoplus_{v|v_0} K_{2n-1}(k_v)_l\big)^\chi$ is a cyclic $\mathbb{Z}_l[\chi]$–module generated by ξ_v^χ. It is easily seen that since $K_{2n-1}(k_v)_l \simeq \mathbb{Z}/l^k$ (with notations as in the previous section), isomorphisms (20) imply that

$$\operatorname{ord}(\xi_v^\chi) = l^k, \text{ whenever } \xi_v^\chi \neq 1. \tag{31}$$

Now, since the map Λ constructed in the previous section is $\mathbb{Z}_l[G]$–equivariant, Theorem 4.6 implies that we have

$$\partial_F^\chi \circ \Lambda^\chi(\xi) = \xi^{l^{v_l(n)}\Theta_n(\mathbf{b},\mathbf{f})^\chi}, \qquad \text{for all } \xi \in \bigoplus_{v_0}\Big(\bigoplus_{v\mid v_0} K_{2n-1}(k_v)_l\Big)^\chi. \tag{32}$$

Definition 5.2. If A is an abelian group, we denote by $div(A)$ its subgroup of divisible elements. In other words, we let

$$div(A) = \bigcap_{r\geq 1} A^r.$$

Let $D(n) := div(K_{2n}(F))$. It is easy to see that we have

$$D(n)_l^\chi = div(K_{2n}(F)_l^\chi), \tag{33}$$

for all χ as above. Also, observe that (30) combined with the finiteness of the Quillen K–groups of finite fields implies right away that for all χ and n we have

$$D(n)_l^\chi \subset K_{2n}(\mathcal{O}_F)_l^\chi. \tag{34}$$

Simplifying Hypothesis: *From now on we assume that all primes above l are ramified in $F(\mu_l)/K$ and totally ramified in $F(\mu_{l^\infty})/F(\mu_l)$.*

Theorem 5.3. *Let $\chi \in \widehat{G}(\mathbb{C})$ and assume that $v_l(n) = 0$ and $\Theta_n(\mathbf{b},\mathbf{f})^\chi \neq 0$. Then*

$$K_{2n}(\mathcal{O}_F)_l^\chi \cap \operatorname{Im}(\Lambda^\chi) = D(n)_l^\chi. \tag{35}$$

In particular, if $v_l(n) = 0$ and $\Theta_n(\mathbf{b},\mathbf{f})^\chi \neq 0$ for all $\chi \in \widehat{G}(\mathbb{C})$, then

$$K_{2n}(\mathcal{O}_F)_l \cap \operatorname{Im}(\Lambda) = D(n)_l. \tag{36}$$

Proof. Note that the exactness of the functors $M \to M^\chi$ combined with (28) shows that equality (36) follows from equalities (35), for all $\chi \in \widehat{G}(\mathbb{C})$. So, we proceed with the proof of (35).

First, take $d \in D(n)_l^\chi$ and take a natural number m such that

$$l^m > l^{v_l(\Theta_n(\mathbf{b},\mathbf{f})^\chi)}|K_{2n}(\mathcal{O}_F)_l^\chi|.$$

Write d as $d = x^{l^m}$, for some $x \in K_{2n}(F)_l^\chi$. By (32), we have

$$\partial_F^\chi \circ \Lambda^\chi \circ \partial_F^\chi(x) = \partial_F^\chi(x)^{\Theta_n(\mathbf{b},\mathbf{f})^\chi} \tag{37}$$

Hence, we have

$$\Lambda^\chi \circ \partial_F^\chi(x) = x^{\Theta_n(\mathbf{b},\mathbf{f})^\chi} y \tag{38}$$

for an element $y \in K_{2n}(\mathcal{O}_F)_l^\chi$. Raising (38) to the power $D_\chi := l^m \cdot |\Theta_n(\mathbf{b},\mathbf{f})^\chi|_l^{-1}$ gives:

$$\Lambda^\chi \circ \partial_F^\chi(x^{D_\chi}) = x^{l^m} = d. \tag{39}$$

Hence, we have $d \in K_{2n}(\mathcal{O}_F)_l^\chi \cap \operatorname{Im}\Lambda^\chi$.

Now, assume that $y \in K_{2n}(\mathcal{O}_F)_l^\chi \cap$ Im Λ^χ. For each prime $v_0 \nmid l$ of $\mathcal{O}_K$ fix a prime $v|v_0$ in $\mathcal{O}_F$. With notations as in the previous section, we can write

$$y = \prod_v \Lambda^\chi(\xi_v^\chi)^{c_v}, \tag{40}$$

where $c_v \in \mathbb{Z}_l[\chi]$. Since $\partial_K^\chi(y) = 1$, (32) implies that

$$(\xi_v^\chi)^{c_v \Theta_n(\mathbf{b},\mathbf{f})^\chi} = 1,$$

for each of the chosen primes v. This, combined with (31) implies that

$$l^k \mid c_v \Theta_n(\mathbf{b},\mathbf{f})^\chi, \text{ whenever } \xi_v^\chi \neq 1. \tag{41}$$

Now, let us fix one of the chosen primes v and assume that $\xi_v^\chi \neq 1$. With notations as in the previous section, (27) and Theorem 4.6(1) imply that

$$\Lambda^\chi(\xi_v^\chi) = \widetilde{e_\chi} \cdot Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E^*}(w) * b(\beta(\xi_{l^k})^{*\,n}))^{\gamma_l}. \tag{42}$$

Let $m \in \mathbb{Z}_{\geq k}$ be any integer such that

$$l^{m-k} \geq | K_{2n}(O_F)_l | \tag{43}$$

Since all the l–adic primes are totally ramified in the extension $F(\mu_{l^\infty})/F(\mu_l)$, Lemma 5.1 applied to the extension E/F allows us to choose a prime w_2 of E which is coprime to $\mathbf{b}l$ such that: $v_2 := w_2 \cap O_F$ splits completely in E/F, and $[w] = [w_2] \in Cl(\mathcal{O}_E)$, and $l^m \mid\mid (Nw_2 - 1)$ for m large. Since $v_l(n) = 0$ and v_2 splits completely in E/F, we have $l^m = |K_{2n-1}(k_{v_2})_l|$. Let $\widetilde{w_2}$ be a prime of $E(\mu_{l^m})$ over w_2. Under our simplifying hypothesis, the projection formula (see [41], Chapter V, §3.3.2) combined with Lemma 3.11, Remark 3.10 and Lemma 5.1(2) gives the following relation:

$$\begin{aligned}\Lambda(\xi_{v_2})^{l^{m-k}} &= Tr_{E(\mu_{l^m})/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E}(\widetilde{w_2}) * b(\beta(\xi_{l^m})^{*\,n}))^{\gamma_l \cdot l^{m-k}} = \\ &= Tr_{E(\mu_{l^m})/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E}(\widetilde{w_2}) * b(\beta(\xi_{l^k})^{*\,n}))^{\gamma_l} = \\ &= Tr_{E/F}(\lambda^*_{\mathbf{b},\mathbf{f}_E}(w_2) * b(\beta(\xi_{l^k})^{*\,n}))^{\gamma_l}\end{aligned} \tag{44}$$

By our choice of w_2 and Corollary 3.18, we have

$$\lambda^*_{\mathbf{b},\mathbf{f}_E}(w_2) = \lambda^*_{\mathbf{b},\mathbf{f}_E}(w)\alpha^{\Theta_0(\mathbf{b},\mathbf{f}_E)}u \tag{45}$$

for some $\alpha \in E^\times \otimes \mathbb{Z}_l$ and $u \in \mu_E \otimes \mathbb{Z}_l$. Since l is odd, $u = N_{E(\mu_{l^m})/E}(u_2)$ for some $u_2 \in \mu_{E(\mu_{l^m})} \otimes \mathbb{Z}_l$. Consequently, by the projection formula and (43),

$$Tr_{E/F}(u * b(\beta(\xi_{l^k})^{*\,n}))^{\gamma_l} = Tr_{E(\mu_{l^m})/F}(u_2 * b(\beta(\xi_{l^m})^{*\,n}))^{\gamma_l})^{l^{m-k}} = 1. \tag{46}$$

Now, the projection formula combined with Theorem 2.4 gives

$$b(Tr_{E/F}(\alpha^{\Theta_0(\mathbf{b},\mathbf{f}_E)} * \beta(\xi_{l^k})^{*\,n})^{\gamma_l})^{c_v} = b(Tr_{E/F}(\alpha * \beta(\xi_{l^k})^{*\,n}))^{c_v\,\Theta_n(\mathbf{b},\mathbf{f})} \tag{47}$$

Hence, if we multiply (47) by $\widetilde{e_\chi}$ and apply (41), we obtain:

$$\widetilde{e_\chi} b(Tr_{E/F}(\alpha^{\Theta_0(\mathbf{b},\mathbf{f}_E)} * \beta(\xi_{l^k})^{*\,n})^{\gamma_l})^{c_v} = \widetilde{e_\chi} b(Tr_{E/F}(\alpha * \beta(\xi_{l^k})^{*\,n}))^{c_v\,\Theta_n(\mathbf{b},\mathbf{f})^\chi} = 1. \tag{48}$$

Now (44), (45), (46) and (48) imply that:

$$\Lambda^\chi(\xi_v^\chi)^{c_v} = \Lambda^\chi(\xi_{v_2}^\chi)^{c_v l^{m-k}}. \tag{49}$$

The product in (40) ranges in fact over a finite set of primes v that depends on y and m is arbitrarily large. Hence (40) and (49) show that $d \in D(n)_l^{\chi}$. □

If we restrict Theorem 5.3 to the situation $K = F$, we obtain the following condition for the cyclicity of the group of divisible elements $D(n)_l$ in $K_{2n}(K)_l$, for an arbitrary totally real number field K, under certain hypotheses.

Theorem 5.4. *Let K be a totally real number field, l an odd prime, and $n \geq 1$ an odd integer such that $v_l(n) = 0$. Let $D(n)_l$ be the group of divisible elements in $K_{2n}(K)_l$. Assume that K satisfies the simplifying hypothesis above and that $|\prod_{v|l} w_n(K_v)|_l = 1$. Then, the following conditions are equivalent:*

1. *The group $D(n)_l$ is cyclic.*
2. *There exists a prime ideal v_0 and an $\mathcal{O}_K$–ideal $\mathbf{b}$ coprime to $w_{n+1}(K)_l$ such that $|K_{2n-1}(k_{v_0})_l|$ is divisible by $|(1 - N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1}$ and the map*
$$\Lambda_{v_0} : K_{2n-1}(k_{v_0})_l \to K_{2n}(K)_l$$
associated to the data $(K/K, n, l, \mathbf{b}, v_0)$ is injective.

Proof. We start with a few preliminary remarks. Let $\mathbf{b}$ be an $\mathcal{O}_K$–ideal coprime to $w_{n+1}(K)$. Note that since K is totally real and n is odd, we have
$$(1 - N\mathbf{b}^{n+1})\zeta_K(-n) \neq 0.$$
(See [23], p. 198.) Consequently, Theorem 5.3 applies to the data $(K/K, n, \chi = 1)$. Construct a map Λ for the data $(K/K, n, l, \mathbf{b})$ as in the previous section. By Theorem 5.3 (see (36)) and (41) we can write every $y \in D(n)_l$ as

$$y = \prod_v \Lambda(\xi_v)^{c_v}, \tag{50}$$

where $c_v \in \mathbb{Z}_l$ are such that

$$\mathrm{ord}(\xi_v) \mid c_v(1 - N\mathbf{b}^{n+1})\zeta_K(-n), \tag{51}$$

for each prime v in $\mathcal{O}_K$. Hence, if we apply ∂_K to (50), we can conclude that $\Lambda(\xi_v)^{c_v} \in K_{2n}(\mathcal{O}_K)_l$ for each v. So, again, by Theorem 5.3 (equality (36)) we have

$$\Lambda(\xi_v)^{c_v} \in D(n)_l, \qquad \text{for all } v. \tag{52}$$

Now, [2, Theorem 3 (ii)] and our assumption that $|\prod_{v|l} w_n(K_v)|_l = 1$ imply that

$$|D(n)_l| = |w_{n+1}(K)\zeta_K(-n)|_l^{-1}. \tag{53}$$

Since $w_{n+1}(K) = \gcd(N\mathbf{b}^{n+1} - 1)$ where the gcd is taken over ideals $\mathbf{b}$ coprime with $w_{n+1}(K)$, there is an ideal $\mathbf{b}$, coprime with $w_{n+1}(K)_l$, such that

$$|w_{n+1}(K)\zeta_K(-n)|_l^{-1} = |(1 - N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1} \tag{54}$$

(1) ⇒ (2). If $D(n)_l$ is trivial, then any l–adic prime v_0 satisfies the conditions in (2). Assume that $D(n)_l$ is cyclic and nontrivial. Let $\mathbf{b}$ be an $\mathcal{O}_K$–ideal satisfying (54). Construct a map Λ for the data $(K/K, n, \mathbf{b})$. Relations (50) and (52) imply that there exists an $\mathcal{O}_K$–prime v_0 such that $\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}$ is a generator of $D(n)_l$. (Take a v_0 such that $\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}$ has maximal order.) By (53) we have
$$|w_{n+1}(K)\zeta_K(-n)|_l^{-1} = |(1 - N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1} = \mathrm{ord}(\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}) \leq \mathrm{ord}(\xi_{v_0}).$$
Since ξ_{v_0} is a generator of $|K_{2n-1}(k_{v_0})_l|$, this implies that
$$|(1 - N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1} \mid |K_{2n-1}(k_{v_0})_l|.$$

Moreover, the map Λ_{v_0} must be injective. Indeed, if $\mathrm{ord}(\xi_{v_0}) = 1$ this is clear. If $\mathrm{ord}(\xi_{v_0}) > 1$ and $\mathrm{ord}(\Lambda_{v_0}(\xi_{v_0})) < \mathrm{ord}(\xi_{v_0})$, then $1 < \mathrm{ord}(\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}) < \mathrm{ord}(\xi_{v_0}^{c_{v_0}})$. But this is impossible, since $\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}$ has order $|w_{n+1}(K)\zeta_K(-n)|_l^{-1}$ and this number also annihilates $\xi_{v_0}^{c_{v_0}}$ by (51).

(2) $\Rightarrow$ (1). Let $r_{v_0} := |K_{2n-1}(k_{v_0})_l|$. Consequently, the number

$$c_{v_0} := \frac{r_{v_0}}{w_{n+1}(K)\zeta_K(-n)}$$

has nonnegative l–adic valuation. Moreover, by Theorem 4.5 we have

$$\partial_K(\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}) = \xi_{v_0}^{r_{v_0}} = 1.$$

Consequently, $\Lambda(\xi_{v_0})^{c_{v_0}} \in K_{2n}(\mathcal{O}_K)_l$. Hence by Theorem 5.3 (equality (36)) we have $\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}} \in D(n)_l$. Since Λ_{v_0} is injective, we have

$$\mathrm{ord}(\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}) = |w_{n+1}(K)\zeta_K(-n)|_l^{-1}.$$

Now, (53) implies that $D(n)_l$ is cyclic generated by $\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}$. □

The following technical lemma refines the implication (1) $\Longrightarrow$ (2) in the above theorem and will be needed in the next section.

Lemma 5.5. *Assume the hypotheses of Theorem 5.4, and assume that $D(n)_l$ is cyclic. Then for any $\mathcal{O}_K$–ideal $\mathbf{b}$ which is coprime to $w_{n+1}(K)_l$ and such that*

$$(1 - N\mathbf{b}^{n+1})\mathbb{Z}_l = w_{n+1}(K)_l\mathbb{Z}_l \tag{55}$$

and any integer $m >> 0$, there exist infinitely many $\mathcal{O}_K$–primes v such that Λ_v is injective and $l^m \mid (Nv - 1)$.

Proof. Let us fix a $\mathbf{b}$ which satisfies (55). The proof of (1) $\Longrightarrow$ (2) in Theorem 5.4 shows that if one has an $\mathcal{O}_K$–ideal v_0 such that $|(1-N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1} \mid (Nv_0^n - 1)$ and $\Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}$ generates $D(n)_l$ for some $c_{v_0} \in \mathbb{Z}$, then the map Λ_{v_0} is injective. Let us fix a v_0 and a c_{v_0} satisfying all these properties.

Let $m \in \mathbb{Z}_{\geq k}$ such that

$$l^{m-k} \geq \max\left\{|(1 - N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1}, \quad |K_{2n}(\mathcal{O}_K)_l|\right\}.$$

The technique developed in the proof of Theorem 5.3 allows us to construct infinitely many $\mathcal{O}_K$–primes v which are coprime to $\mathbf{b}l$ and which satisfy

$$l^m \mid (Nv - 1), \qquad \Lambda_v(\xi_v)^{c_{v_0}\cdot l^{m-k}} = \Lambda_{v_0}(\xi_{v_0})^{c_{v_0}}.$$

(See (49) and the arguments preceding it.) For any such prime v, we have

$$|(1 - N\mathbf{b}^{n+1})\zeta_K(-n)|_l^{-1} \mid (Nv^n - 1)$$

and $\Lambda_v(\xi_v)^{c_{v_0}\cdot l^{m-k}}$ generates $D(n)_l$. Consequently, the map Λ_v is injective. □

In the particular case $K = \mathbb{Q}$, the above theorem is closely related to a classical conjecture of Iwasawa. We make this link explicit in what follows. For that purpose, let A be the l-torsion part of the class group of $\mathbb{Z}[\mu_l]$. Let $\omega : G(\mathbb{Q}(\mu_l)/\mathbb{Q}) \to \mathbb{Z}_l^\times$ be the Teichmüller character. For every $i \in \mathbb{Z}_{\geq 0}$, let $e_{\omega^i} \in \mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]$ be the idempotent associated to ω^i. View A as a $\mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]$–module and let

$$A^{[i]} := A^{\omega^i} = e_{\omega^i}A,$$

in the notations used at the beginning of this section.

Conjecture 5.6. (Iwasawa) $A^{[l-1-n]}$ *is cyclic for all n odd,* $1 \leq n < l-1$.

Recall the following result (see [4], [5]):

Proposition 5.7. *Let n be odd, $1 \leq n < l-1$. Then $A^{[l-1-n]}$ is cyclic if and only if $D(n)_l$ is cyclic, where $D(n)_l$ is the group of divisible elements in $K_{2n}(\mathbb{Q})_l$.*

In the next section, we will give a new proof of this result (see Theorem 6.4), based on our construction of the Galois equivariant Stickelberger splitting map Λ. For the moment, let us simply observe that if combined with Proposition 5.7, Theorem 5.4 gives the following equivalent formulation of Iwasawa's conjecture in terms of our map Λ.

Corollary 5.8. *Let n be an odd integer such that $1 \leq n < l-1$. The following conditions are equivalent:*

(1) *The group $A^{[l-1-n]}$ is cyclic.*
(2) *The group $D(n)_l$ is cyclic.*
(3) *There exists a prime number p and an integer $b \in \mathbb{Z}_{\geq 1}$ coprime to $w_{n+1}(\mathbb{Q})_l$ such that $|(1-b^{n+1})\zeta_{\mathbb{Q}}(-n)|_l^{-1}$ divides $|K_{2n-1}(\mathbb{F}_p)_l|$ and the map*
$$\Lambda_p \,:\, K_{2n-1}(\mathbb{F}_p)_l \to K_{2n}(\mathbb{Q})_l$$
associated to the data $(\mathbb{Q}/\mathbb{Q}, n, l, b\mathbb{Z}, p)$ is injective.

Proof. This is a direct consequence of Theorem 5.4 upon observing that $v_l(n) = 0$ and $|w_n(\mathbb{Q}_l)|_l = 1$, for each n odd and $1 \leq n < l-1$ and also that $\mathbb{Q}$ satisfies our simplifying hypothesis for all odd primes l. $\square$

Remark 5.9. In light of Theorem 5.4 and Corollary 5.8, the question regarding the cyclicity of $D(n)_l \in K_{2n}(K)_l$ may be viewed as an extension of Iwasawa's conjecture to arbitrary totally real number fields K, under the obvious hypotheses on l and n. At this point we do not have sufficient data to conjecture that $D(n)_l$ is cyclic at this level of generality. In the next section we will do a close analysis of the injectivity of Λ_{v_0} in the case $F = K = \mathbb{Q}$. In the process, we will give a new proof of Proposition 5.7 (see Theorem 6.4.) In future work, we hope to extend the techniques developed in the next section to more general totally real number fields K and study a generalization of Iwasawa's cyclicity conjecture in that setting.

6. Conditions for the injectivity of Λ_{v_0} in the case $K = F = \mathbb{Q}$

In this section we assume that $K = F = \mathbb{Q}$ and fix an odd prime l. Next we pick a natural number $\mathbf{b}$ which is coprime to l and satisfies the two conditions in the following elementary Lemma.

Lemma 6.1. *Let $\mathbf{b}$ be a natural number coprime to l. Then $(1 - \mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1})$ is a generator of the ideal $\mathrm{Ann}_{\mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]}(\mu_l)$ if and only if the following conditions are simultaneously satisfied.*

(1) *$\omega(\overline{\mathbf{b}})$ is a generator of μ_{l-1}, where $\omega : (\mathbb{Z}/l\mathbb{Z})^\times \to \mu_{l-1}$ is the Teichmüller character and $\overline{\mathbf{b}} := \mathbf{b} \mod l$.*
(2) *$\mathbf{b} \not\equiv \omega(\overline{\mathbf{b}}) \mod l^2\mathbb{Z}_l$.*

Proof (sketch.) Note that there is a $\mathbb{Z}_l$–algebra isomorphism
$$\mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})] \simeq \bigoplus_{i=0}^{l-2} \mathbb{Z}_l,$$

sending $\sigma \to (\omega^i(\sigma))_i$, for all $\sigma \in \mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]$. Under this isomorphism the ideal $\mathrm{Ann}_{\mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]}(\mu_l)$ is sent into $\mathbb{Z}_l \oplus l\mathbb{Z}_l \oplus \mathbb{Z}_l \oplus \cdots \oplus \mathbb{Z}_l$. Consequently, the group ring element $(1 - \mathbf{b} \cdot \sigma_{\mathbf{b}}^{-1})$ generates $\mathrm{Ann}_{\mathbb{Z}_l[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]}(\mu_l)$ if and only if

$$l \,||\, (1 - \mathbf{b} \cdot \omega(\overline{\mathbf{b}})^{-1}), \qquad l \nmid (1 - \mathbf{b} \cdot \omega(\overline{\mathbf{b}})^{-i}), \text{ for all } i \not\equiv 1 \mod (l-1) \tag{56}$$

These are exactly conditions (2) and (1) in the Lemma, respectively. □

Remark 6.2. If $\mathbf{b}$ is chosen as above, then we also have

$$w_{n+1}(\mathbb{Q})\mathbb{Z}_l = (1 - \mathbf{b}^{n+1})\mathbb{Z}_l,$$

for all $n \in \mathbb{Z}_{\geq 0}$, as one can easily prove based on relations (56).

Let n be an odd natural number, with $l \nmid n$. Let v_0 be a rational prime such that

$$v_0 \equiv 1 \mod l, \quad v_0 \nmid \mathbf{b}, \quad v_l((1 - \mathbf{b}^{n+1})\zeta_K(-n)) \leq v_l(|K_{2n-1}(k_{v_0})_l|).$$

Note that since $|K_{2n-1}(k_{v_0})_l| = v_0^n - 1$, once n and $\mathbf{b}$ are fixed the set of such primes v_0 has positive density, as a consequence of Chebotarev's density theorem.

In this context, our goal is to analyze the injectivity of the map

$$\Lambda_{v_0} : K_{2n-1}(k_{v_0})_l \to K_{2n}(\mathbb{Q})_l.$$

We resume the notations of §4. In particular, if $v_l({v_0}^n - 1) = k$, then $E := \mathbb{Q}(\mu_{l^k})$ and w is a prime sitting above v_0 in E. Since in this case the exponent $\gamma_l = (1 - l^n)^{-1}$ lies in $\mathbb{Z}_l^{\times}$ and it does not affect injectivity, in order to simplify notations we work with the following slightly modified definition of Λ_{v_0}.

$$\Lambda_{v_0}(\xi_{v_0}) = b(Tr_{E/\mathbb{Q}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w) * \beta(\xi_{l^k})^{*\,n})) = Tr_{E/\mathbb{Q}}(b(\lambda_{\mathbf{b},\mathbf{f}_E}(w) * \beta(\xi_{l^k})^{*\,n})), \tag{57}$$

where ξ_{v_0} is the distinguished generator of $K_{2n-1}(k_{v_0})_l$ picked in §4. Note that since $w \nmid \mathbf{b}$, $\lambda^*_{\mathbf{b},\mathbf{f}_E}(w) = \lambda_{\mathbf{b},\mathbf{f}_E}(w)$ in this case.

Now, let us note that since $v_l(v_0 - 1) > 0$, $v_l({v_0}^n - 1) = k$ and $v_l(n) = 0$, we have $v_l(v_0 - 1) = k$. Consequently, v_0 splits completely in E and $q_w = v_0$. Therefore, the arguments at the beginning of §4.2 (see equality (15)) imply that all arrows in the following commutative diagram are isomorphisms.

$$\begin{array}{ccc} K_{2n}(k_w, \mathbb{Z}/l^k) & \xrightarrow[\cong]{b_w} & K_{2n-1}(k_w)_l \\ {\scriptstyle Tr_{w/v_0}}\downarrow\cong & & {\scriptstyle Tr_{w/v_0}}\downarrow\cong \\ K_{2n}(k_{v_0}, \mathbb{Z}/l^k) & \xrightarrow[\cong]{b_{v_0}} & K_{2n-1}(k_{v_0})_l \end{array}$$

Consider the following commutative diagram:

$$\begin{array}{ccc} K_{2n+1}(E, \mathbb{Z}/l^k) & \xleftarrow{\Lambda_{w,\, l^k}} & K_{2n}(k_w, \mathbb{Z}/l^k) \\ {\scriptstyle Tr_{E/K}}\downarrow & & {\scriptstyle Tr_{w/v_0}}\downarrow\cong \\ K_{2n+1}(K, \mathbb{Z}/l^k) & \xleftarrow{\Lambda_{v_0,\, l^k}} & K_{2n}(k_{v_0}, \mathbb{Z}/l^k) \end{array}$$

where the horizontal arrows are defined as follows:

$$\Lambda_{w,\, l^k}(\beta(\xi_{l^k})^{*\,n}) := \lambda_{\mathbf{b},\mathbf{f}_E}(w) * \beta(\xi_{l^k})^{*\,n}, \tag{58}$$

$$\Lambda_{v_0,\, l^k}(Tr(\beta(\xi_{l^k})^{*\,n})) := Tr_{E/\mathbb{Q}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w) * \beta(\xi_{l^k})^{*\,n}) \tag{59}$$

So, we can rewrite

$$\Lambda_{v_0}(\xi_{v_0}) = b\circ Tr_{E/K}\circ\Lambda_{w,l^k}(\beta(\xi_{l^k})^{*n}) = Tr_{E/\mathbb{Q}}\circ b\circ\Lambda_{w,l^k}(\beta(\xi_{l^k})^{*n}). \tag{60}$$

Remark 6.3. Note that in the case under consideration, we have

$$\Theta_0(\mathbf{b},\mathbf{f}_E) = (1-\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1})\cdot\sum_{a\in(\mathbb{Z}/l^k)^\times}\zeta_{l^k}(\sigma_a,0)\cdot\sigma_a^{-1} =$$

$$= (1-\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1})\cdot\sum_{a\in(\mathbb{Z}/l^k)^\times}(\frac{1}{2}-\frac{a}{l^k})\cdot\sigma_a^{-1}, \tag{61}$$

where the second sum above is taken with respect to all $1\leq a<l^k$, with $l\nmid a$. The projection of $\Theta_0(\mathbf{b},\mathbf{f}_E)$ to $\mathbb{Z}[G(E_1/\mathbb{Q})]$, where $E_1:=\mathbb{Q}(\mu_l)$, is given by

$$\Theta_0(\mathbf{b},\mathbf{f}_{E_1}) = (1-\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1})\cdot\sum_{a\in(\mathbb{Z}/l)^\times}(\frac{1}{2}-\frac{a}{l})\cdot\sigma_a^{-1}.$$

Via the first condition in (56), it is clear that $\omega(\Theta_0(\mathbf{b},\mathbf{f}_{E_1}))\not\equiv 0 \mod l$. Thus,

$$\Theta_0(\mathbf{b},\mathbf{f}_{E_1})\notin l\mathbb{Z}_l[G(E_1/\mathbb{Q})].$$

This implies in particular that

$$\Theta_0(\mathbf{b},\mathbf{f}_E)\notin l\mathbb{Z}_l[G(E/\mathbb{Q})]. \tag{62}$$

Now, we begin our study of the injectivity of Λ_{v_0}, for the chosen $l,n,\mathbf{b},v_0$. We will use expression (60) for Λ_{v_0}. As Steps I and II below will show, the map $b\circ\Lambda_{w,l^k}$ turns out to be injective unconditionally. The point where Iwasawa's cyclicity conjecture (Conjecture 5.6) comes into play is when one analyzes the injectivity of $Tr_{E/\mathbb{Q}}$ restricted to $\mathrm{Im}(b\circ\Lambda_{w,l^k})$, as it will be made clear in Step III below.

Step I. The unconditional injectivity of $\Lambda_{w,\,l^k}$.

By the definition of $\lambda_{\mathbf{b},\mathbf{f}_E}(w)$, its Arakelov divisor is

$$\mathrm{div}_E(\lambda_{\mathbf{b},\mathbf{f}_E}(w)) = (1-\mathbf{b}\cdot\sigma_{\mathbf{b}}^{-1}))\sum_{a\in(\mathbb{Z}/l^k)^\times}(\frac{1}{2}-\frac{a}{l^k})\cdot\sigma_a^{-1}(w). \tag{63}$$

Since v_0 splits completely in $E/\mathbb{Q}$, (62) implies that

$$\mathrm{div}_E(\lambda_{\mathbf{b},\mathbf{f}_E}(w))\notin\ l\cdot\mathrm{Div}^0_{S_\infty}(E). \tag{64}$$

Consequently, we have

$$\lambda_{\mathbf{b},\mathbf{f}_E}(w)\notin\mu_E\cdot(E^\times)^l\,. \tag{65}$$

Consider the following commutative diagram.

$$\begin{tikzcd} K_{2n+1}(E,\,\mathbb{Z}/l^k) \arrow[d] & K_{2n}(k_w,\,\mathbb{Z}/l^k) \arrow[l, "\Lambda_{w,\,l^k}"'] \arrow[d, "\cong"] \\ H^1(E,\,\mathbb{Z}/l^k(n+1)) & H^0(k_w,\,\mathbb{Z}/l^k(n)) \arrow[l, "\Lambda^{et}_{w,\,l^k}"'] \end{tikzcd} \tag{66}$$

The vertical arrows are the Dwyer-Friedlander maps [16] and the right vertical arrow is an isomorphism [16, Corollary 8.6]. The bottom horizontal arrow is defined by

$$\Lambda^{et}_{w,\,l^k}(\xi_{l^k}^{\otimes\,n})\ :=\ \lambda_{\mathbf{b},\mathbf{f}_E}(w)\cup\xi_{l^k}^{\otimes\,n}. \tag{67}$$

Since $\xi_{l^k} \in \mu_E$ it is clear, by (65), that the map $\Lambda^{et}_{w,\,l^k}$ is injective. Hence the map $\Lambda_{w,\,l^k}$ is also injective.

Step II. The unconditional injectivity of the map $(b \circ \Lambda_{w,\,l^k})$.

Consider the following commutative diagram with exact rows and surjective vertical arrows. (Surjectivity follows from [16, Theorem 8.5]. See also [36, Thm. 4, p. 278].)
(68)

$$\begin{tikzcd}
0 \arrow[r] & K_{2n+1}(E)/l^k \arrow[r] \arrow[d] & K_{2n+1}(E,\mathbb{Z}/l^k) \arrow[r, "b"] \arrow[d] & K_{2n}(E)[l^k] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(E,\,\mathbb{Z}_l(n{+}1)/l^k \arrow[r] & H^1(E,\,\mathbb{Z}/l^k(n{+}1)) \arrow[r, "b^{et}"] & H^2(E,\,\mathbb{Z}_l(n{+}1))[l^k] \arrow[r] & 0
\end{tikzcd}$$

By Step I the middle vertical arrow in (68) induces an isomorphism

$$\operatorname{Im}\Lambda_{w,\,l^k} \cong \operatorname{Im}\Lambda^{et}_{w,\,l^k}. \tag{69}$$

Hence (69) and (68) give the isomorphism

$$\operatorname{Im}\Lambda_{w,\,l^k} \cap \operatorname{Ker} b \quad \cong \quad \operatorname{Im}\Lambda^{et}_{w,\,l^k} \cap \operatorname{Ker} b^{et}. \tag{70}$$

Let S denote the set of l–adic primes in E. Since $H^1(\mathcal{O}_{E,S},\,\mathbb{Z}_l(n+1)) = H^1(E,\,\mathbb{Z}_l(n+1))$ then $\operatorname{Ker} b^{et} \simeq H^1(\mathcal{O}_{E,S},\,\mathbb{Z}_l(n+1))/l^k$. Hence, isomorphism (70) and (93) in the Appendix imply that for an element of the form $\lambda_{\mathbf{b},\mathbf{f}_E}(w)^r * \beta(\xi_{l^k})^{*\,n}$ we have an equivalence

$$\lambda_{\mathbf{b},\mathbf{f}_E}(w)^r * \beta(\xi_{l^k})^{*\,n} \in \operatorname{Ker}(b) \Longleftrightarrow \lambda_{\mathbf{b},\mathbf{f}_E}(w)^r \cup \xi_{l^k}^{\otimes\, n} = (\xi_{l^k}{}^{\otimes (n+1)})^s, \tag{71}$$

for some $s \in \mathbb{N}$. (When applying (93) it is important to note that since n is odd and $\lambda_{\mathbf{b},\mathbf{f_E}}(w)^{1+j} = 1$, we have $\lambda_{\mathbf{b},\mathbf{f}_E}(w)^r \cup \xi_{l^k}^{\otimes\, n} \in H^1(E,\mathbb{Z}_l/l^k(n+1))^+$.)

Now, the right hand side of (71) is equivalent to $\lambda_{\mathbf{b},\mathbf{f}_E}(w)^r \in \langle \xi_{l^k}\rangle \cdot E^{\times\, l^k}$. Combined with (64), this implies that l^k divides r. Consequently, $\lambda_{\mathbf{b},\mathbf{f}_E}(w)^r * \beta(\xi_{l^k})^{*\,n} = 0$. Therefore $\operatorname{Ker}(b \circ \Lambda_{w,l^k}) = 0$.

Step III. The question of injectivity for the map $Tr_{E/\mathbb{Q}}$ restricted to $\operatorname{Im}(b \circ \Lambda_{w,\,l^k})$.

At this point we use a trick which allows us to assume that $k = 1$. First, let us note that Λ_{v_0} is injective if and only if Λ_{v_0} restricted to the unique subgroup $K_{2n-1}(k_{v_0})[l]$ of order l in $K_{2n-1}(k_{v_0})_l$ is injective. Let

$$\eta_{v_0} := \xi_{v_0}^{l^{k-1}}.$$

Observe that η_{v_0} is a generator of $K_{2n-1}(k_{v_0})[l]$. Observe that in our case $\mathbf{f}_{\mathbb{Q}(\mu_l)} = l\mathbb{Z}$ and all the l–adic primes in $\mathbb{Q}(\mu_l)$ are totally ramified in $\mathbb{Q}(\mu_{l^\infty})/\mathbb{Q}(\mu_l)$. Also, recall that $v_0 \nmid \mathbf{b}$. Hence if we apply Lemma 3.11, Remark 3.10 and Lemma 5.1(3) (with $K = F = \mathbb{Q}$ and $E_1 = \mathbb{Q}(\mu_l)$), we obtain:

$$N_{E/E_1}\lambda_{\mathbf{b},\mathbf{f}_E}(w) = \lambda_{\mathbf{b},\mathbf{f}_{E_1}}(w_1)$$

where w_1 is a prime of $\mathcal{O}_{E_1}$ below w. Hence

$$\begin{aligned}
\Lambda_{v_0}(\eta_{v_0}) = \Lambda_{v_0}(\xi_{v_0})^{l^{k-1}} = {} & b(Tr_{E/\mathbb{Q}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w) * \beta(\xi_{l^k})^{*\,n}))^{l^{k-1}} = \\
= {} & b(Tr_{E_1/F}(N_{E/E_1}\lambda_{\mathbf{b},\mathbf{f}_E}(w) * \beta(\xi_l)^{*\,n})) = \\
= {} & b(Tr_{E_1/\mathbb{Q}}(\lambda_{\mathbf{b},\mathbf{f}_{E_1}}(w_1) * \beta(\xi_l)^{*\,n})).
\end{aligned}$$

So the injectivity of Λ_{v_0} is equivalent to the injectivity of the following map:

$$\Lambda_{v_0}\,:\,K_{2n-1}(k_{v_0})[l]\to K_{2n}(\mathbb{Q})[l] \tag{72}$$

$$\Lambda_{v_0}(\eta_{v_0}):=b(Tr_{E_1/\mathbb{Q}}(\lambda_{\mathbf{b},\mathbf{f}_{E_1}}(w_1)*\beta(\xi_l)^{*\,n})). \tag{73}$$

If one compares (73) and (57), it becomes clear that it is sufficient to set $k=1$ and $E=\mathbb{Q}(\mu_l)$ in all of the above considerations. We will use this notation for the rest of the section. Since in this case the natural map $K_{2n}(\mathbb{Q})_l\to K_{2n}(E)_l$ is injective (as $l\nmid |G(E/Q)|$), the injectivity of Λ_{v_0} is equivalent to

$$\operatorname{ord}\,(N_l\circ b\circ\Lambda_{w,\,l})(\beta(\xi_l)^{*\,n})=l,$$

where $N_l:=\sum_{c=1}^{l-1}\sigma_c$. The reader should note that we have already showed (in Steps I and II above for $k=1$) that

$$\operatorname{ord}\,(b\circ\Lambda_{w,\,l})(\beta(\xi_l)^{*\,n})=l.$$

Let $\omega\,:\,G(E/\mathbb{Q})\to\mathbb{Z}_l^\times$ denotes the Teichmüller character and let $e_{\omega^i}\in\mathbb{Z}_l[G]$ denote the group ring idempotent corresponding to ω^i, for all $i\in\mathbb{Z}$. Note that

$$e_{\omega^i}\equiv\sum_{c=1}^{l-1}-c^i\,\sigma_c^{-1}\mod l\mathbb{Z}_l[G]. \tag{74}$$

Also, note that $E^\times/E^{\times l}$ has a natural $\mathbb{Z}_l[G]$–module structure. For simplicity, if $x\in E^\times$, we will use the notation

$$e_{\omega^i}\cdot x:=e_{\omega^i}\cdot(x\mod E^{\times l})$$

and view this as an element in $E^\times/E^{\times l}$, for all $i\in\mathbb{Z}$. In light of the above notation, we have the following equalities:

$$N_l(\Lambda_{w,\,l}(\beta(\xi_l)^{*\,n}))=\sum_{c=1}^{l-1}\sigma_c(\lambda_{\mathbf{b},\mathbf{f}_E}(w)*\beta(\xi_l)^{*\,n})= \tag{75}$$

$$=(\prod_{c=1}^{l-1}\lambda_{\mathbf{b},\mathbf{f}_E}(\sigma_c(w))^{c^n})*\beta(\xi_l)^{*\,n}=-e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w))*\beta(\xi_l)^{*\,n}.$$

$$N_l(\Lambda_{w,\,l}^{et}(\xi_l^{\otimes\,n}))=\sum_{c=1}^{l-1}\sigma_c(\lambda_{\mathbf{b},\mathbf{f}_E}(w)\cup\xi_l^{\otimes\,n})= \tag{76}$$

$$=(\prod_{c=1}^{l-1}\lambda_{\mathbf{b},\mathbf{f}_E}(\sigma_c(w))^{c^n})\cup\xi_l^{\otimes\,n}=-e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w))\cup\xi_l^{\otimes\,n}.$$

Now, (75) and (76) combined with the arguments at the end of Step II (applied to $k:=1$) lead to the following equivalence.

$$N_l(\Lambda_{w,\,l}(\beta(\xi_l)^{*\,n}))\in\operatorname{Ker}(b)\Longleftrightarrow e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w))\in(\mu_l\cdot E^{\times l})/E^{\times l}. \tag{77}$$

Now, we are ready to prove the main result of this section.

Theorem 6.4. *Let $l>2$ be a prime number. Let $n\geq 1$ be an odd integer, such that $l\nmid n$. Let $\mathbf{b}\geq 1$ be an integer satisfying the two conditions in Lemma 6.1. As above, let $E:=\mathbb{Q}(\mu_l)$. Then, the following are equivalent.*

(1) *$D(n)_l:=div\,(K_{2n}(\mathbb{Q})_l)$ is cyclic.*

(2) *There exists an O_E–prime w for which*

$$e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w)) \notin (\mu_l \cdot E^{\times l})/E^{\times l}.$$

and which satisfies the following additional hypotheses: it is coprime to $\mathbf{b}$, *it sits above a rational prime p which splits completely in $E/\mathbb{Q}$, and*

$$|(1-\mathbf{b}^{n+1}) \cdot \zeta_{\mathbb{Q}}(-n)|_l^{-1} \mid p^n - 1.$$

(3) $A^{[l-1-n]} := e_{\omega^{-n}} \cdot (Cl(\mathcal{O}_E) \otimes \mathbb{Z}_l)$ *is cyclic.*

Proof. **(1) $\Longleftrightarrow$ (2).** Remark 6.2 and Lemma 5.5 applied to $(\mathbb{Q}/\mathbb{Q}, n, l, \mathbf{b})$ show that $D(n)_l$ is cyclic if and only if there exists a rational prime satisfying the additional hypotheses in (2) for which the map Λ_p associated to $(\mathbb{Q}/\mathbb{Q}, n, l, \mathbf{b}, p)$ is injective.

Now, since for an $\mathcal{O}_E$–prime $w|p$, we have an equality

$$(N_l \circ b \circ \Lambda_{w,\,l})(\beta(\xi_l)^{*\,n}) = b(N_l(\Lambda_{w,\,l}(\beta(\xi_l)^{*\,n}))),$$

(75) and (77) imply that the injectivity of Λ_p holds if and only if

$$e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w)) \notin (\mu_l \cdot E^{\times l})/E^{\times l},$$

for some $\mathcal{O}_E$–prime w sitting above p.

(2) $\Longleftrightarrow$ (3). Our choice of $\mathbf{b}$ (see Lemma 6.1) combined with a theorem of Mazur–Wiles (see Theorem 8.8 and the Remark which follows in the Appendix of [25]) gives us the following equalities.

$$| A^{[l-1-i]} | = [\mathbb{Z}_l : \omega^{-i}(\Theta_0(\mathbf{b}, \mathbf{f}_E))\mathbb{Z}_l], \qquad \text{for all odd } i. \tag{78}$$

The reader should note that our choice of $\mathbf{b}$ implies that $\Theta_0(\mathbf{b}, \mathbf{f}_E)$ is a generator of the Stickelberger ideal in $\mathbb{Z}_l[G(E/\mathbb{Q})]$, denoted by $\mathcal{R}_0$ in the Remark following Theorem 8.8 in loc.cit. Also, the equality in the Remark in question can be easily extended to the case $\chi := \omega$ as both sides are equal to 1 in that case.

We consider the standard exact sequence of $\mathbb{Z}_l[G(E/\mathbb{Q})]$–modules

$$0 \longrightarrow (E^\times/\mathcal{O}_E^\times) \otimes \mathbb{Z}_l \xrightarrow{\text{div}} \text{Div}(\mathcal{O}_E) \otimes \mathbb{Z}_l \longrightarrow Cl(\mathcal{O}_E) \otimes \mathbb{Z}_l \longrightarrow 0,$$

where $\text{Div}(\mathcal{O}_E)$ is the non-archimedean part of the group $\text{Div}_{S_\infty}(E)$ of Arakelov divisors of E, div is the divisor map extended by $\mathbb{Z}_l$–linearity to $(E^\times \otimes \mathbb{Z}_l)$ and the projection onto $Cl(O_E) \otimes \mathbb{Z}_l$ is the usual divisor–class map (taking $d \in \text{Div}(\mathcal{O}_E)$ to its class $\widehat{d} \in Cl(\mathcal{O}_E)$) extended by $\mathbb{Z}_l$–linearity. Since n is odd, we have an equality $e_{\omega^{-n}} \cdot (\mathcal{O}_E^\times \otimes \mathbb{Z}_l) = e_{\omega^{-n}} \cdot (\mu_l \otimes \mathbb{Z}_l)$. Consequently, by taking ω^{-n}–components in the exact sequence above we obtain the following exact sequence of $\mathbb{Z}_l$–modules

$$0 \longrightarrow e_{\omega^{-n}}(E^\times/\mu_l \otimes \mathbb{Z}_l) \xrightarrow{\text{div}} e_{\omega^{-n}}(\text{Div}(\mathcal{O}_E) \otimes \mathbb{Z}_l) \longrightarrow A^{[l-1-n]} \longrightarrow 0. \tag{79}$$

Now, according to (78), $A^{[l-1-n]}$ is cyclic if and only if it contains an ideal class of order $|\omega^{-n}(\Theta_0(\mathbf{b}, \mathbf{f}_E))|_l^{-1}$. Consequently, exact sequence (79) and Lemma 5.1 applied to $E/\mathbb{Q}$ and a sufficiently large m, imply that $A^{[l-1-n]}$ is cyclic if and only if there exists an O_E–prime w satisfying the additional hypotheses in (2), such that

$$\omega^{-n}(\Theta_0(\mathbf{b}, \mathbf{f}_E)) \cdot e_{\omega^{-n}}(w \otimes 1) \in \text{div}(e_{\omega^{-n}}(E^\times \otimes \mathbb{Z}_l)) \setminus l \cdot \text{div}(e_{\omega^{-n}}(E^\times \otimes \mathbb{Z}_l)). \tag{80}$$

In the argument above, it is important to note that since w sits over a rational prime p which splits completely in $E/\mathbb{Q}$, we have $e_{\omega^{-n}}(w\otimes 1)\notin l\cdot(\mathrm{Div}(\mathcal{O}_E)\otimes\mathbb{Z}_l)$. However, by the definition of $\lambda_{\mathbf{b},\mathbf{f}_E}(w)$, we have

$$\mathrm{div}(e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w)\otimes 1)) = \omega^{-n}(\Theta_0(\mathbf{b},\mathbf{f}_E))\cdot e_{\omega^{-n}}(w\otimes 1). \tag{81}$$

Equality (81) combined with exact sequence (79) shows that (80) is equivalent to

$$e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w)\otimes 1)\notin e_{\omega^{-n}}(\mu_l\cdot E^{\times l}\otimes\mathbb{Z}_l).$$

This is clearly equivalent to $e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w))\notin(\mu_l\cdot E^{\times l})/E^{\times l}$. □

Remark 6.5. Note that for $n\not\equiv -1 \mod (l-1)$, we have an equivalence

$$e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w))\notin(\mu_l\cdot E^{\times l})/E^{\times l}\Longleftrightarrow e_{\omega^{-n}}(\lambda_{\mathbf{b},\mathbf{f}_E}(w))\neq 0,$$

since we have an obvious inclusion $(\mu_l\cdot E^{\times l})/E^{\times l}\subseteq e_{\omega^1}\cdot(E^\times/E^{\times l})$.

7. An Euler system in the higher odd K–theory with coefficients

In this section we construct an Euler System for the odd K-theory (with coefficients) of a CM abelian extension of a totally real number field. Our construction generalizes those of [33] and [4] to arbitrary totally real number fields and it is quite different from that in [6].

As above, we fix a finite abelian CM extension F/K of a totally real number field of conductor $\mathbf{f}$, a rational prime $l>2$ and a natural number $n\geq 1$. Next, we fix an $\mathcal{O}_K$–ideal $\mathbf{b}$ which is coprime to $\mathbf{f}l$. We let $\mathbf{L}=\mathbf{l}_1\cdot\dots\cdot\mathbf{l}_t$ run through all the products of mutually distinct prime ideals $\mathbf{l}_1,\dots,\mathbf{l}_t$ of $\mathcal{O}_K$, coprime to $l\cdot\mathbf{bf}$. Let $F_{\mathbf{L}}:=FK_{\mathbf{L}}$, where $K_{\mathbf{L}}$ is the ray class field of K of conductor $\mathbf{L}$. Obviously, the conductor of the CM-extension $F_{\mathbf{L}}/K$ divides $\mathbf{Lf}$. We let $F_{\mathbf{L}l^k}:=F_{\mathbf{L}}(\mu_{l^k})$, for every $k\geq 0$.

For each CM extension $F_{\mathbf{L}l^k}/K$ we fix an $\mathcal{O}_K$–ideal ideal $\mathbf{f}'_{\mathbf{L}l^k}$ such that

$$\mathrm{Supp}(\mathbf{f}'_{\mathbf{L}l^k}) \;=\; \mathrm{Supp}(\mathbf{f})\cup\mathrm{Supp}(\mathbf{L})\cup\mathrm{Supp}(l\mathcal{O}_K).$$

Also, we fix roots of unity $\xi_{l^k}\in F(\mu_{l^k})$ of order l^k, such that

$$\xi_{l^{k+1}}^l=\xi_{l^k},$$

for all $k\geq 0$. We let

$$\beta_{\mathbf{L},k}:=\beta_{F_{\mathbf{L}l^k}}(\xi_{l^k})$$

be the corresponding Bott elements in $K_2(F_{\mathbf{L}l^k},\mathbb{Z}/l^k)$, for all $\mathbf{L}$ and k as above. Next, we fix a prime v in $\mathcal{O}_F$, such that $v\nmid l\mathbf{b}$. For each $\mathbf{L}$ as above and each $k\geq 0$, we fix a prime $w_k(\mathbf{L})$ of $O_{F_{\mathbf{L}l^k}}$ sitting above v, such that $w_{k'}(\mathbf{L}')$ sits above $w_k(\mathbf{L})$ whenever $l^k\mathbf{L}\mid l^{k'}\mathbf{L}'$. We let $v(\mathbf{L}):=w_0(\mathbf{L})$, for all $\mathbf{L}$ as above.

Let $\mathbf{f}^*_{\mathbf{L}l^k}$ denote the conductor of $F_{\mathbf{L}l^k}/K$ multiplied by all the l–adic primes in K. We will use the l–adic imprimitive Brumer–Stark elements $\{\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}l^k}}(w_k(\mathbf{L}))\}_{\mathbf{L}}$ viewed as special elements in $\{K_1(F_{\mathbf{L}l^k})_l\}_{\mathbf{L}}$ to construct special elements $\{\boldsymbol{\lambda}_{v(\mathbf{L}),k}\}_{\mathbf{L}}$ in the K–theory with coefficients $\{K_{2n+1}(F_{\mathbf{L}},\mathbb{Z}/l^k)\}_{\mathbf{L}}$, for all $k>0$ as follows.

Definition 7.1. For all $\mathbf{L}$ and k as above and $n\geq 0$ define $\boldsymbol{\lambda}_{v(\mathbf{L}),k}\in K_{2n+1}(F_{\mathbf{L}},\,\mathbb{Z}/l^k)$ as follows.

$$\boldsymbol{\lambda}_{v(\mathbf{L}),k} \;:=\; Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}l^k}}(w_k(\mathbf{L}))*\beta_{\mathbf{L},k}^{*\,n})^{\gamma_l},$$

where the operator $\gamma_l\in\mathbb{Z}_l[G(F_{\mathbf{L}}/K)]$ is given by (17).

The reader should note that in the definition above, for simplicity, our notation does not capture the dependence of γ_l on $\mathbf{L}$.

In what follows, we will apply the following three natural maps in K-theory: the transfer map (82), the reduction modulo $w_k(\mathbf{L})$ map (83), the boundary map in the Quillen localization sequence for K–theory with coefficients (84) and the reduction modulo l^k map (85):

$$Tr_{w_k(\mathbf{L})/w_0(\mathbf{L})} \,:\, K_{2n}(k_{w_k(\mathbf{L})},\, \mathbb{Z}/l^k) \to K_{2n}(k_{w_0(\mathbf{L})},\, \mathbb{Z}/l^k) \tag{82}$$

$$\pi_{w_k(\mathbf{L})} \,:\, K_{2n}(\mathcal{O}_{F_{\mathbf{L}l^k},S},\, \mathbb{Z}/l^k) \to K_{2n}(k_{w_k(\mathbf{L})},\, \mathbb{Z}/l^k) \tag{83}$$

$$\partial_{F_{\mathbf{L}}} \,:\, K_{2n+1}(F_{\mathbf{L}},\, \mathbb{Z}/l^k) \longrightarrow \bigoplus_{v} K_{2n}(k_v,\, \mathbb{Z}/l^k) \tag{84}$$

$$r_{k'/k} : K_{2n+1}(F_{\mathbf{L}},\, \mathbb{Z}/l^{k'}) \to K_{2n+1}(F_{\mathbf{L}},\, \mathbb{Z}/l^k), \qquad \text{for all } k' \geq k. \tag{85}$$

The following theorem captures the main properties of the special elements defined above. In particular, part (3) of the theorem below simply states the fact that these special elements form an Euler system in $\{K_{2n+1}(F_{\mathbf{L}}, \mathbb{Z}/l^k)\}_{\mathbf{L}}$, for all $k > 0$.

Theorem 7.2. *Assume that $\mathbf{L}' = \mathbf{L}\mathbf{l}'$, for an $\mathcal{O}_K$–prime $\mathbf{l}' \nmid \mathbf{L}$. Let us denote $N_{\mathbf{L}} := Tr_{k_{w_k(\mathbf{L})}/k_{w_0(\mathbf{L})}}$.*

(1) *If $k' \geq k$ and $w_k(\mathbf{L})$ splits completely in $F_{\mathbf{L}l^{k'}}$, then:*

$$r_{k'/k}(\boldsymbol{\lambda}_{v(\mathbf{L}),k'}) = \boldsymbol{\lambda}_{v(\mathbf{L}),k}.$$

(2) $\partial_{F_{\mathbf{L}}}(\boldsymbol{\lambda}_{v(\mathbf{L}),k}) = N_{\mathbf{L}}(\pi_{w_k(\mathbf{L})}(\beta^{*\,n}_{\mathbf{L},k}))^{l^{v_l(n)}\Theta_n(\mathbf{b},\,\mathbf{Lf})}$.

(3) *If $\sigma_{\mathbf{l}'}$ denotes the Frobenius morphism associated to $\mathbf{l}'$ in $G(F_{\mathbf{L}}/K)$, then*

$$Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}}(\boldsymbol{\lambda}_{v(\mathbf{L}'),k}) = (\boldsymbol{\lambda}_{v(\mathbf{L}),k})^{1-N(\mathbf{l}')^n\cdot\sigma_{\mathbf{l}'}^{-1}}.$$

Proof. Part (1) follows by Lemma 3.11 and the projection formula. The proof part (2) is similar to the proof of Theorem 4.5(2). Let us prove the third formula above (the Euler system property). We apply Definition 7.1 and Lemma 3.11 combined with Remark 3.21 to conclude that we have the following equalities:

$$\begin{gathered}
Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}}(\boldsymbol{\lambda}_{v(\mathbf{L}'),k}) = Tr_{F_{\mathbf{L}'}/F_{\mathbf{L}}}Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}'}}(\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}'\mathbf{l}^{\mathbf{k}}}}(w_k(\mathbf{L}')) * \beta^{*\,n}_{\mathbf{L}',k})^{\gamma_l} = \\
Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}Tr_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}}(\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}'l^k}}(w_k(\mathbf{L}')) * \beta^{*\,n}_{\mathbf{L}',k})^{\gamma_l} = \\
Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(N_{F_{\mathbf{L}'l^k}/F_{\mathbf{L}l^k}}\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}'l^k}}(w_k(\mathbf{L}')) * \beta^{*\,n}_{\mathbf{L},k})^{\gamma_l} = \\
Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}l^k}}(w_k(\mathbf{L}))^{1-\sigma_{\mathbf{l}'}{}^{-1}} * \beta^{*\,n}_{\mathbf{L},k})^{\gamma_l} = \\
(Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}l^k}}(w_k(\mathbf{L})) * \beta^{*\,n}_{\mathbf{L},k})^{\gamma_l})^{1-N(\mathbf{l}')^n\cdot\sigma_{\mathbf{l}'}{}^{-1}} = \\
(\boldsymbol{\lambda}_{v(\mathbf{L}),k})^{1-N(\mathbf{l}')^n\cdot\sigma_{\mathbf{l}'}^{-1}}.
\end{gathered}$$

□

Remark 7.3. Note that if $K = \mathbb{Q}$, then our Euler system construction recaptures that of [4]. Also, if $n = 0$, our Euler system lives in $\{F_{\mathbf{L}}^{\times}/F_{\mathbf{L}}^{\times l^k}\}_{\mathbf{L}}$ and it is given by

$$\boldsymbol{\lambda}_{v(\mathbf{L}),k} \;:=\; Tr_{F_{\mathbf{L}l^k}/F_{\mathbf{L}}}(\lambda_{\mathbf{b},\mathbf{f}^*_{\mathbf{L}l^k}}(w_k(\mathbf{L})))^{\gamma_l} \quad \bmod F_{\mathbf{L}}^{\times l^k}.$$

This is a vast generalization of Kolyvagin's Euler system of Gauss sums (mod l^k) (see [33]) which can be obtained from our construction by setting $K = \mathbb{Q}$.

8. Appendix: The class groups and étale cohomology groups of rings of S-integers

All the cohomology groups in what follows are étale cohomology groups. The following two lemmas are very useful in section 6.

Lemma 8.1. *Let l be an odd prime. Let L be a number field such that there is only one prime v_l over l in $\mathcal{O}_L$. Let $S := S_l = \{v_l\}$. Then for any $m \in \mathbb{N}$ we have the following natural isomorphisms:*

$$Cl(\mathcal{O}_{L,S})/l^m \xrightarrow{\cong} H^2(\mathcal{O}_{L,S},\, \mathbb{Z}/l^m(1)) \tag{86}$$

$$Cl(\mathcal{O}_{L,S})_l \xrightarrow{\cong} H^2(\mathcal{O}_{L,S},\, \mathbb{Z}_l(1)) \tag{87}$$

$$Cl(\mathcal{O}_{L,S})[l^m] \xrightarrow{\cong} H^2(\mathcal{O}_{L,S},\, \mathbb{Z}_l(1))[l^m] \tag{88}$$

Proof. It is well known (see [23, §2] and the references therein) that we have the following canonical isomorphisms

$$H^0(\mathcal{O}_{L,S}, \mathbb{G}_m) \simeq \mathcal{O}_{L,S}^{\times} \tag{89}$$

$$H^1(\mathcal{O}_{L,S}, \mathbb{G}_m) \simeq Cl(\mathcal{O}_{L,S}) \tag{90}$$

$$H^2(\mathcal{O}_{L,S}, \mathbb{G}_m) \simeq Br(\mathcal{O}_{L,S}) \simeq (\mathbb{Q}/\mathbb{Z})^{|S|-1} \tag{91}$$

where $\mathbb{G}_m$ is the étale sheaf associated to the multiplicative group scheme $\mathbb{G}_m$ and Br stands for Brauer group. Since in our case $|S| = 1$, we have $H^2(\mathcal{O}_{L,S}, \mathbb{G}_m) = 0$. Therefore, the long exact sequence in étale cohomology associated to the short exact sequence of étale sheaves on $\mathrm{Spec}(\mathcal{O}_{L,S})$

$$1 \to \mu_{l^m} \to \mathbb{G}_m \xrightarrow{l^m} \mathbb{G}_m \to 1 \tag{92}$$

combined with (90) gives the isomorphism

$$Cl(\mathcal{O}_{L,S})/l^m \simeq H^2(\mathcal{O}_{L,S},\, \mu_{l^m}) = H^2(\mathcal{O}_{L,S},\, \mathbb{Z}/l^m(1)),$$

which is precisely (86). Isomorphism (87) follows upon taking an inverse limit with respect to m. Isomorphism (88) follows upon applying the multiplication by l^m map to both sides of (87). $\square$

Lemma 8.2. *Let $E := \mathbb{Q}(\mu_{l^k})$, for some $k \geq 0$. Let v_l be the unique prime of $\mathcal{O}_E$ over l and let $S := \{v_l\}$. Assume that n is odd. Then, there are natural isomorphisms of $\mathbb{Z}/l^k[G(E/\mathbb{Q})]$ modules:*

$$(H^1(\mathcal{O}_{E,S},\, \mathbb{Z}_l(n+1))/l^k)^+ \;\cong\; \mathbb{Z}/l^k(n+1) \tag{93}$$

$$Cl(\mathcal{O}_{E,S})\,[l^k]^- \otimes \mathbb{Z}/l^k(n) \;\cong\; H^2(\mathcal{O}_{E,S}, \mathbb{Z}_l(n+1))^+\,[l^k] \tag{94}$$

Above, the upper scripts $\pm$ stand for the corresponding eigenspaces with respect to the action of the unique complex conjugation automorphism of E.

Proof. Since $n+1 \geq 2$ and $(n+1)$ is even, if we combine [23, Prop. 2.9] with [23, p. 239] we obtain the following natural isomorphisms of $\mathbb{Z}_l[G(E/\mathbb{Q})]$–modules

$$H^1(\mathcal{O}_{E,S},\, \mathbb{Z}_l(n+1))^+ \simeq H^1(\mathcal{O}_{E^+,S},\, \mathbb{Z}_l(n+1)) \simeq [\mathbb{Q}_l/\mathbb{Z}_l(n+1)]^{G_{E^+}},$$

where E^+ is the maximal real subfield of E and G_{E^+} is its absolute Galois group. Since $2 = [E^+(\mu_{l^k}) : E^+]$ divides $n+1$, we have $[\mathbb{Q}_l/\mathbb{Z}_l(n+1)]^{G_{E^+}} \simeq \mathbb{Z}/l^{\alpha}(n+1)$, for some $\alpha \geq k$. Consequently, we have

$$(H^1(\mathcal{O}_{E,S},\, \mathbb{Z}_l(n+1))/l^k)^+ \simeq H^1(\mathcal{O}_{E,S},\, \mathbb{Z}_l(n+1))^+/l^k \simeq \mathbb{Z}/l^k(n+1),$$

which proves isomorphism (93).

Now, let $E_\infty := E(\mu_{l^\infty})$ and $\Gamma := G(E_\infty/E)$. Obviously, $\Gamma \simeq G(E_\infty^+/E^+)$. Since the l–adic primes in $E, E^+, E_\infty, E_\infty^+$ are principal, we have $Cl(O_{E,S}) = Cl(O_E)$ and also $Cl(O_{E_\infty,S}) = Cl(O_{E_\infty})$. Similar equalities hold for E^+ and E_∞^+. Moreover, since $l \neq 2$, the natural map at the level of ideal classes induces an injection

$$Cl(O_E)_l^- \subseteq Cl(O_{E_\infty})_l^-.$$

(See [40, Prop. 13.26].) It is also well known (as a direct consequence of the cohomological triviality of $Cl(O_{E(\mu_{l^m})})_l^-$ as a $G(E(\mu_{l^m})/E)$–module, for all m, see [18]) that the inclusion above induces an equality

$$Cl(O_E)_l^- = [Cl(O_{E_\infty})_l^-]^\Gamma. \tag{95}$$

Now, if $\mathfrak{X}_\infty^+$ denotes the Galois group of the maximal pro-l abelian extension of E_∞^+ which is unramified outside of l, [23, Prop. 2.20] and [23, p. 238] give natural isomorphisms of $\mathbb{Z}_l[G(E/\mathbb{Q})]$–modules

$$H^2(\mathcal{O}_{E,S}, \mathbb{Z}_l(n+1))^+ \simeq H^2(\mathcal{O}_{E^+,S}, \mathbb{Z}_l(n+1)) \simeq (\mathfrak{X}_\infty^+(-(n+1))_\Gamma)^\vee,$$

where $*_\Gamma$ stands for Γ–coinvariants and $*^\vee$ stands for Pontrjagin dual. However, we have a natural perfect duality pairing (see [40, Prop. 13.32])

$$\mathfrak{X}_\infty^+ \times Cl(O_{E_\infty})_l^- \to \mathbb{Q}_l/\mathbb{Z}_l(1).$$

When combined with the last displayed isomorphisms this pairing leads to

$$H^2(\mathcal{O}_{E,S}, \mathbb{Z}_l(n+1))^+ \simeq [Cl(O_{E_\infty})_l^-(n)]^\Gamma.$$

When combined with (95) this isomorphism leads to

$$\begin{aligned} H^2(\mathcal{O}_{E,S}, \mathbb{Z}_l(n+1))^+[l^k] &\simeq [Cl(O_{E_\infty})_l[l^k]^-(n)]^\Gamma \\ &\simeq [Cl(O_{E_\infty})_l[l^k]^- \otimes \mathbb{Z}/l^k(n)]^\Gamma \\ &= Cl(O_E)_l[l^k]^- \otimes \mathbb{Z}/l^k(n). \end{aligned}$$

The last equality above is a consequence of the fact that Γ acts trivially on $Z/l^k(n)$ (as $\mu_{l^k} \subseteq E^\times$.) This concludes the proof of isomorphism (94). $\square$

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