FITTING IDEALS OF ℓ-ADIC REALIZATIONS OF PICARD 1-MOTIVES AND CLASS GROUPS OF GLOBAL FUNCTION FIELDS

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Abstract. In previous work, we proved a result on the equivariant Fitting ideal of the ℓ-adic realization of the Picard 1-motive attached to an abelian covering of curves defined over a finite field. In this paper, we build upon this work to deduce results on the equivariant Fitting ideal of the Tate modules of the Jacobian of the top curve, and on the equivariant Fitting ideal of the (dual of the) degree zero class group of the top curve. At the end, we discuss refinements of Brumer's conjecture and consider examples.

Introduction

Let $X \to Y$ be an abelian G-Galois covering of smooth projective curves over a finite field $\mathbb{F}_q$ of characteristic p. We may equivalently speak about G-Galois extensions K/k, where K and k are the function fields $\mathbb{F}_q(X)$ and $\mathbb{F}_q(Y)$, respectively, and we always assume that $\mathbb{F}_q$ is the field of constants of k (which is no loss of generality).

One main goal of this paper is to determine the $\mathbb{Z}_\ell[G]$-Fitting ideal of the Pontryagin dual of the degree zero class group $cl^0(K)\{\ell\}$, where ℓ is an arbitrary prime. This is achieved via some intermediate steps. The starting point is afforded by the main result of [GP], as we will explain shortly. We would like to stress that important parts of the results in [GP] do not require G to be abelian; but we need G abelian in the present paper, in order to work freely with Fitting ideals.

We will have to work with the base-changed curves $\bar{X} \to \bar{Y}$, where $\bar{X} = \bar{\mathbb{F}} \times_{\mathbb{F}_q} X$ (and similarly for Y), and $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}$. Note that $\bar{X}$ may not be connected. The principal intermediate step is to calculate the Λ-Fitting ideal of $T_\ell(J_X(\bar{\mathbb{F}}))$, the ℓ-adic Tate module of the Jacobian of X. Here $\Lambda = \mathbb{Z}_\ell[G][[\Gamma]]$ is the group ring of G over the completed group ring of $\Gamma = \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F}_q)$ (equivalently: the completed group ring of $\Gamma \times G$) over $\mathbb{Z}_\ell$. This calculation is made possible by using the following input:

Deligne defined a 1-motive $\mathcal{M} = M^X_{\mathcal{S},\mathcal{T}}$ attached to $\bar{X}$ and two finite disjoint sets $\mathcal{S}$ and $\mathcal{T}$ of closed points of $\bar{X}$; it is required that both are nonempty and G-invariant, and that $\mathcal{S}$ contains all ramified points. (For more details we refer to [Ta] and [De], and also to [GP].) We will take $\mathcal{S} = \bar{S}$ (the set of points on $\bar{X}$ above points in S), and $\mathcal{T} = \bar{\Sigma}$, where S and Σ are sets of closed points "below" (that is, on X) satisfying analogous requirements. To this motive one may then attach its ℓ-adic realization, or Tate module $T_\ell(M^X_{\bar{S},\bar{\Sigma}})$, for all prime numbers ℓ. This is a free $\mathbb{Z}_\ell$-module of finite rank, and, most importantly, it is a G-module.

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It contains a copy of $T_\ell(J_X(\bar{\mathbb{F}}))$ (details are given in §1). From [GP] we have the crucial information that $T_\ell(M^X_{\bar{S},\bar{\Sigma}})$ is cohomologically trivial, and we also know its Fitting ideal over Λ, which turns out to be principal, generated by an equivariant L-function. In §2 we work our way back from $T_\ell(M^X_{\bar{S},\bar{\Sigma}})$ to $T_\ell(J_X(\bar{\mathbb{F}}))$, and in §3 we descend from $\bar{\mathbb{F}}$ back to $\mathbb{F}_q$, arriving at Fitting ideals of (duals of) degree zero class groups of curves over $\mathbb{F}_q$. Actually we are over-simplifying here: in §2 and §3 we only do part of the job. We have to exclude the ε-part, where $\varepsilon \in \mathbb{Z}_\ell[G]$ is the idempotent corresponding to the trivial character of the non-ℓ-part of G (unless $\ell = p$, where there are other slight restrictions). This unfortunately means that we have done nothing in the case of ℓ-groups unless $\ell = p$. This deficiency is partly mended in §4. A ray-class group version of the results in §§2-3 is presented in §5; this version is actually simpler, allowing us to be very brief.

The resulting formulas for Fitting ideals of class groups are very reminiscent of Sinnott's description of Stickelberger ideals. In particular they involve a lot of generators, which are attached to subextensions of K/k. The Brumer- and Brumer-Stark conjectures only "see" the top-level generator among all these. Note here that every module is annihilated by its Fitting ideal.

Whilst the validity of the Brumer-Stark conjecture in this context is certainly a well-known fact, it still seems worthwhile to discuss the Brumer conjecture, and we do so briefly in the final section §6. There is no new general result since the Brumer-Stark conjecture is known to imply the Brumer conjecture. However, we use the latter as an illustration of, and a checkup on our results, and it turns out that we often obtain slightly better annihilating elements at top level. In any case we obtain improved overall annihilation, just by finding so many new elements in the Fitting ideal.

The detour via cohomologically trivial modules used in our approach fits in with the philosophy of the Equivariant Tamagawa Number Conjectures (as dealt with in many important papers of Burns and others), but we manage to keep everything fairly explicit here. Moreover there is certainly some connection with constructions of Ritter and Weiss. There seems to be a connection to local and global canonical classes and their interplay. It is hoped that this can be explored further, and that the ideas exploited here in the function field case can also be brought to bear on number fields. Apart from this, the results in [GP] include applications to the (étale) Coates-Sinnott Conjecture; we hope to deal with the conjectures of Rubin–Stark and Gross in further work.

Notation: For any finite group G, the sum $\sum_{\sigma\in G}\sigma$ in the group ring will be written $s(G)$. Other notation will be introduced as we go along.

§1. *The setting, and previous results on Fitting ideals and 1-motives*

We will fix a finite field $\mathbb{F}_q$ of characteristic p and cardinality q, and a finite group G throughout (which will be assumed abelian later), and we will consider a fixed G-Galois covering $X \to Y$ of smooth complete curves over $\mathbb{F}$. We will assume that the base curve Y is connected, but not necessarily the top curve X. This corresponds to a G-Galois extension K/k of function algebras in one variable over $\mathbb{F}_q$. Here k is a function field, and the Galois algebra K is a product of copies of one function field K_0, the so-called core field, which is G_0-Galois over k with a subgroup $G_0 \subset G$ unique up to conjugacy. On the geometric side, X is the disjoint union of $[G : G_0]$ mutually isomorphic copies of one curve X_0, which is a G_0-Galois cover of Y. The following conventions are almost self-evident: A point y of Y is said to ramify in X iff it ramifies in X_0, and said to split totally in X iff it does so in X_0. The multidegree of a divisor on X is the tuple whose entries are the degrees of that divisor restricted to each connected component in turn.

Moreover we fix a prime ℓ. The interesting case $\ell = p$ is *not* excluded, but often needs separate treatment.

We now consider two finite disjoint nonempty sets S and Σ of closed points on X. We further assume that both are G-invariant and S contains all ramified points in the covering X/Y. *These assumptions are in force for the whole paper.* For the image of S in Y we write somewhat abusively S/G (so as not to introduce new letters or dashes etc.), and likewise Σ/G. The set of all points of $\bar{X}$ (see introduction) above S (or Σ) will be written $\bar{S}$ and $\bar{\Sigma}$ respectively. The notation $\bar{S}/G$ of course means the image of $\bar{S}$ in $\bar{Y}$. The data $\bar{X}/\bar{Y}, \bar{S}, \bar{T}$ now serve as input for Deligne's construction of the 1-motive. We will sketch this briefly and refer to [De] and [GP] for further details. For background information on Fitting ideals (which will not be repeated here) the reader should consult [GP] or the standard references given there.

Let $\Gamma = \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F}_q)$ and $\Lambda = \mathbb{Z}_\ell[G][[\Gamma]]$. Let J_X and $J_{X,\Sigma}$ denote the Jacobian (resp. Σ-Jacobian) of X. Let τ_Σ denote the torus such that $\tau_\Sigma(\mathbb{F}) = (\oplus_{v\in\Sigma_{\mathbb{F}}}\mathbb{F}(v)^\times)/\Gamma(X_{\mathbb{F}}, O^\times_{X_{\mathbb{F}}})$, for each $\mathbb{F}_q \subset \mathbb{F} \subset \bar{\mathbb{F}}$. The $\mathbb{F}$-points of $J_{X,\Sigma}$ are given as the quotient of divisors of multidegree zero on $X_{\mathbb{F}}$ supported outside $\Sigma_{\mathbb{F}}$ modulo the principal divisors $\mathrm{div}(f)$ where $f(s) = 1$ at all $s \in \Sigma_{\mathbb{F}}$. There is the fundamental exact sequence

$$1 \to \tau_\Sigma \to J_{X,\Sigma} \to J_X \to 1.$$

One may think of the Σ-Jacobian as a close analog of a ray class group.

Let L_0 be the kernel of the multidegree map on $\mathrm{Div}(\bar{S})$ (the group of divisors supported on $\bar{S} \subset X$), and let $L = \mathbb{Z}_\ell \otimes_{\mathbb{Z}} L_0$. The module L is called the lattice associated with S. The Picard 1-motive $\mathcal{M} = M_{\bar{S},\bar{\Sigma}}$ is, formally, just the following arrow:

$$L_0 \to J_{X,\Sigma}(\bar{\mathbb{F}}).$$

(This map is completely canonical: to a divisor it associates its class.)

The general machinery of 1-motives associates to this kind of data an ℓ-adic realization $T_\ell(\mathcal{M})$, also called the (ℓ-adic) Tate module of $\mathcal{M}$. (One can also construct torsion modules $\mathcal{M}[n]$, but we will not need them in this paper.) We will not review the construction (see [Ta]), and just mention the most important property: there is a short exact sequence

$$0 \to T_\ell(J_{X,\Sigma}(\bar{\mathbb{F}})) \to T_\ell(\mathcal{M}) \to L \to 0.$$

Since there is another s.e.s.

$$0 \to T_\ell(\tau_\Sigma(\bar{\mathbb{F}})) \to T_\ell(J_{X,\Sigma}(\bar{\mathbb{F}})) \to T_\ell(J_X(\bar{\mathbb{F}})) =: T(X) \to 0,$$

the module $T_\ell(\mathcal{M})$ comes with a three-step filtration

$$0 \to T'(\mathcal{M}) \to T'(\mathcal{M}) \to T_\ell(\mathcal{M}),$$

where $T'(\mathcal{M} = T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))$ (the toric part), $T_\ell(\mathcal{M})/T''(\mathcal{M}) = L$ (the lattice part), and $T''(\mathcal{M})/T'(\mathcal{M}) = T(X)$ (the Jacobian part, which is the object of primary arithmetic interest). All objects constructed so far are Λ-modules.

The main results of [GP] (Thm. 3.10 and Thm. 4.3) say under these assumptions and with all this notation:

Theorem 1.1. *(a) The module $T_\ell(M_{\bar{S},\bar{\Sigma}})$ is cohomologically trivial over G and hence (since it is free of finite rank over $\mathbb{Z}_\ell$) projective over $\mathbb{Z}_\ell[G]$.*

(b) Suppose G is abelian. Then the Fitting ideal of $T_\ell(M_{\bar{S},\bar{\Sigma}})$ over Λ is principal, generated by the equivariant L-function $\Theta_S^\Sigma(\gamma^{-1})$. The element $\Theta_S^\Sigma(u)$ (written $\Theta_{S,\Sigma}(u)$ in [GP]*) is constructed in* [Ta]*, and γ is the q–power arithmetic Frobenius (a distinguished generator of the free pro-cyclic group Γ).*

For emphasis we repeat that the prime ℓ is arbitrary ($\ell = p$ is allowed!), and that X is not assumed to be connected. Even if X is connected, $\bar{X}$ need not be. Theorem 1.1 will be the point of departure for all that follows.

§2 *The Fitting ideal of the Tate module of the Jacobian, I*

We keep our notation: $X \to Y$ is a G-covering of curves as before, $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}_q$, and overbar denotes basechange from $\mathbb{F}_q$ to $\bar{\mathbb{F}}$.

Our task is to pass from $T_\ell(M_{\bar{S},\bar{\Sigma}}) = T_\ell(\mathcal{M})$ to the "standard Iwasawa module" $T(X) = T_\ell(J_X(\bar{\mathbb{F}}))$.

Remark: The epithet "Iwasawa module" is justified, since one can prove that $T(X)$ is isomorphic to the projective limit of the ℓ-parts of the class groups $cl^0(\mathbb{F}_{q^m} \times_{\mathbb{F}_q} X)$. As this will not be used, we will not give the argument, cf. Lemma 5.6 (2) in [GP].

We recall from §1 that $T_\ell(\mathcal{M})$ has, by construction, a three-step filtration $0 \subset T'(\mathcal{M}) \subset T''(\mathcal{M}) \subset T_\ell(\mathcal{M})$ in which $T(X)$ is the middle quotient. The top quotient is L, and the bottom quotient (in other words, the lowest nontrivial module of the filtration) is $T'(\mathcal{M}) = T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))$. This is also the kernel of the canonical surjection from $T''(\mathcal{M}) = T_\ell(J_{X,\Sigma}(\bar{\mathbb{F}}))$ to $T(X) = T_\ell(J_X(\bar{\mathbb{F}}))$. The element $\Theta_S^\Sigma(\gamma^{-1})$ will be simply written Θ_S^Σ in this section.

We assume that G is abelian and that k has $\mathbb{F}_q$ as field of constants.

We work, for now, in a simplified setting, which systematically neglects the trivial character of the non-ℓ-part of G. For the case that the ℓ-part of G is cyclic, this restriction will be eliminated partly in §4. Also, in the case $p = \ell$, we can do without this simplification, to a certain extent.

Since we do not want to impose further restrictions on S and Σ, we have to put up with some technicalities, concerning the G-modules $T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))$ and L. The $\mathbb{Z}_\ell$-structure of these modules is clear, but perhaps not the G-structure.

For every $v \in Y$ we let $\bar{X}(v)$ denote the set of points of $\bar{X}$ above v, and we are interested in two modules over $\mathbb{Z}_\ell[G][[\Gamma]]$:

$$H_v = \mathbb{Z}_\ell[\bar{X}(v)],$$

$$\tilde{H}_v = T_\ell(\bigoplus_{w \in \bar{X}(v)} \kappa(w)^\times).$$

Here $\kappa(w)$ denotes the residue class field at w, which is actually just $\bar{\mathbb{F}}$. This field contains all ℓ-power roots of unity, so $T_\ell(\bar{\mathbb{F}}^\times)$ is canonically identified with $\mathbb{Z}_\ell(1)$ if $\ell \neq p$, and $T_p(\bar{\mathbb{F}}^\times)$ is the zero module. However, we prefer a more functorial notation.

If v is unramified, both modules H_v and $\tilde{H}_v$ are $\mathbb{Z}_\ell[G]$-free of rank d_v, where d_v is the degree of the point v on Y, with the exception that $\tilde{H}_v$ is again zero for $\ell = p$.

Let us also look at "Euler factors". Let σ_v denote a Frobenius of v in G (if v is unramified, then it is the unique Frobenius automorphism attached to v; in the ramified case, any lift of a Frobenius of v in G_v/I_v will do). Let

$$e_v = 1 - \sigma_v^{-1} u^{d_v}; \quad \tilde{e}_v = 1 - q^{d_v} \sigma_v^{-1} u^{d_v} \in \mathbb{Z}[G][u].$$

Note that for $v \in \Sigma$, $\tilde{e}_v$ is just the "modifying Euler factor" used in the definition of Θ_S^{Σ} (see [Ta] p.121). Similarly, for $v \in S$ unramified, e_v is the Euler factor for v used in loc.cit.

We intend to link these factors to Fitting ideals; let us do the unramified case first. We begin by noting that for $\ell = p$, the element $\tilde{e}_v(\gamma^{-1})$ is a unit of $\mathbb{Z}_\ell[G][[\Gamma]]$, for the simple reason that it is congruent to 1 modulo $\ell = p$.

Lemma 2.1. *If $v \in Y$ is unramified in X, then we have:*

(a) $\mathrm{Fitt}_{\mathbb{Z}_\ell[G][[\Gamma]]}(H_v) = (e_v(\gamma^{-1}))$;

(b) $\mathrm{Fitt}_{\mathbb{Z}_\ell[G][[\Gamma]]}(\tilde{H}_v) = (\tilde{e}_v(\gamma^{-1}))$.

(Note: H_v and $\tilde{H}_v$ are cyclic over $\mathbb{Z}_\ell[G][[\Gamma]]$, and hence isomorphic to the quotient of that ring by the respective Fitting ideal.)

PROOF: We already remarked that H_v and $\tilde{H}_v$ are $\mathbb{Z}_\ell[G]$-free. By Lemma V.2.1 in [Ta], the determinant of $1 - F_* u$ on H_v is $e_v(u)$, and therefore by Prop. 4.1 in [GP], part (a) follows. Part (b) for $\ell \neq p$ is a twisted variant of this, and for $\ell = p$ the equality holds trivially, since $\tilde{H}_v$ is zero and $\tilde{e}_v(\gamma^{-1})$ is a unit, as just pointed out. Remarks on notation: Tate's F_v agrees with our σ_v^{-1}; our H_v is his $\mathrm{H}_0(X_v, \mathbb{Z}_\ell)$ and our $\tilde{H}_v$ is his $\mathrm{H}_2(X_v, \mathbb{Z}_\ell)$. QED

We need a version of 2.1(a) which covers the ramified case. Let $I_v \subset G \times \Gamma$ be the inertia group of v in the extension $\bar{X}/Y$. Then I_v actually lies in G, acts trivially on H_v, and v is unramified in the subcovering X^{I_v}/Y. Furthermore, H_v is a cyclic module over $\mathbb{Z}_\ell[G][[\Gamma]]$ (or over $\mathbb{Z}_\ell[G/I_v][[\Gamma]]$). Using Lemma 2.1 for X^{I_v}/Y (including the note at the end), we can deduce:

Lemma 2.2. *The module H_v is isomorphic to $\mathbb{Z}_\ell[G][[\Gamma]]/A_v$, where the ideal A_v is generated by $\tau - 1$ for all $\tau \in I_v$ and $e_v(\gamma^{-1})$.*

(Note: It is easily checked from the definition that the ideal A_v does not depend on the choice of the Frobenius σ_v. Again, A_v is also the Fitting ideal of H_v.)

Let us now write $G = \Delta \times G_\ell$ with G_ℓ the maximal ℓ-subgroup of G. Let ε denote the idempotent $|\Delta|^{-1} s(\Delta)$, and $\varepsilon' = 1 - \varepsilon$. The functor "$M \mapsto \varepsilon' M$" is exact on the category of $\mathbb{Z}_\ell[G]$-modules, and will often be simply written by a prime: $\varepsilon' M = M'$. It has the effect of eliminating the component belonging to the trivial character of Δ.

Let us assume that the Galois group G_0 belonging to the core curve X_0 contains Δ. We label this assumption as (C1); it is much weaker than demanding that $\bar{X}$ be connected. This simplifies things for us, since we then have $T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))' = \bigoplus_{v \in \Sigma/G} (\tilde{H}_v)'$. (Without the prime, the left hand side would arise from the right hand side by factoring out a product of copies of $\mathbb{Z}_\ell(1)$, on which G_0 acts trivially.)

Let $R = \mathbb{Z}_\ell[G][[\Gamma]]$. The preceding remarks show that $T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))'$ is of projective dimension 1 (or zero) over R' and

$$\mathrm{Fitt}_{R'}(T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))') = \Big(\prod_{v \in \Sigma/G} \tilde{e}_v(\gamma^{-1}) \Big). \tag{1}$$

We repeat, for emphasis, that this product is a unit in the case $\ell = p$, corresponding to the fact that $T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))' = 0$.

For the next result we are going to use that Fitting ideals are multiplicative in short exact sequences of R-modules of projective dimension at most one ([CG]). It should also be pointed out that the factors $\tilde{e}_v(\gamma^{-1})$ are nonzerodivisors in R, so one may cancel them.

The module $T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))$ injects into $T_\ell(M_{S,\Sigma})$, so we have a short exact sequence

$$0 \to T_\ell(\tau_\Sigma(\bar{\mathbb{F}})) \to T_\ell(M_{S,\Sigma}) \to T_\ell(M_S) \to 0, \tag{2}$$

where $T_\ell(M_S)$ denotes $T_\ell(\mathcal{M})/T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))$ (the Tate module of the whole motive modulo the bottom piece of the filtration). We write Θ_S for $\Theta_S^\emptyset$ (recall that Θ_S^Σ is short for $\Theta_S^\Sigma(\gamma^{-1})$ in all of this section). As pointed out prior to Lemma 2.1, we have $\Theta_S^\Sigma = \prod_{v\in\Sigma/G} \tilde{e}_v(\gamma^{-1})\cdot\Theta_S$. The element Θ_S might not be in R (only in the full ring of quotients), but the following lemma shows as a byproduct that $\varepsilon'\Theta_S$ is in R'. From formula (1) and from Thm. 1.1 (b) we obtain the following result (using multiplicativity of Fitting ideals on sequence (2)):

Lemma 2.3. *Under assumption (C1), we have*

$$\mathrm{Fitt}_{R'}(T_\ell(M_S)') = \Big(\varepsilon'\Theta_S\Big).$$

The next step is more difficult: we now want to get rid of S as well. We begin by recalling that there is a short exact sequence

$$0 \to T(X) = T_\ell(J_X(\bar{\mathbb{F}})) \to T_\ell(M_S) \to L \to 0. \tag{3}$$

The lattice L is connected to the modules H_v as follows: L is the kernel of $\mathbb{Z}_\ell\bar{S} \to \mathbb{Z}_\ell$, and $\mathbb{Z}_\ell\bar{S} = \bigoplus_{v\in S/G} H_v$. To proceed, we need a somewhat more general result on Fitting ideals. We restate a result (Proposition 1) from [Gr]. One has to use here that for Iwasawa modules without $\mathbb{Z}$-torsion, the Iwasawa adjoint is canonically isomorphic to the $\mathbb{Z}_\ell$-dual (easy proof using the definitions). We note right away that the lemma holds for R in the place of R' as well, and could also be stated for all rings of type $\mathbb{Z}_\ell(\chi)[G_\ell][[\Gamma]]$, with χ an arbitrary character of Δ. (We also mention that loc.cit. uses $\mathbb{Z}_\ell(\chi)$-duals instead of $\mathbb{Z}_\ell$-duals, but these two functors are naturally isomorphic by a trace argument.)

Lemma 2.4. *Let $0 \to X_1 \to X_2 \to X_3 \to X_4 \to 0$ be an exact sequence of R'-modules, which are finitely generated free over $\mathbb{Z}_\ell$. Assume further that X_2 and X_3 have projective dimension 0 or 1 over R'. (The value 0 can occur only if the module is zero.) Then we have:*

$$\mathrm{Fitt}_{R'}(X_1^*)\,\mathrm{Fitt}_{R'}(X_3) = \mathrm{Fitt}_{R'}(X_2)\,\mathrm{Fitt}_{R'}(X_4),$$

with * standing for $\mathbb{Z}_l$-dual.

Neither this nor the result of [CG] mentioned just above can be applied directly in our situation, since in the primed sequence (3) there is no reason for either of the outer terms $T(X)'$ and L' to be of projective dimension one. The thing to do here is to resolve L', thereby producing a four-term sequence with the required properties. Again it helps that we apply $()'$, since after applying this, L and $\mathbb{Z}_\ell\bar{S}$ become equal. On the other hand $\mathbb{Z}_\ell\bar{S}$ is just the direct sum of the H_v with $v\in S$. We find a resolution of H_v as follows. First we identify H_v with the quotient $R/(e(v), \tau-1 : \tau\in I_v)$ where $e(v)$ is short for $e_v(\gamma^{-1}) = 1-\sigma_v^{-1}\gamma^{-d}$. Second, we map H_v into $R/e(v)R$ by sending 1 to $s(I_v)$. This map is well defined since $(\tau-1)s(I_v)$ is zero for all $\tau\in I_v$. We claim that the resulting sequence

$$0 \to H_v \to R/e(v)R \to R/(e(v), s(I_v)) \to 0$$

is exact. Everything is clear except the injectivity of $H_v \to R/e(v)R$. We show this as follows. Since $e(v)$ is a degree d polynomial in γ^{-1} with leading term a unit in $\mathbb{Z}_\ell[G]$, the module $R/e(v)R$ is $\mathbb{Z}_\ell$-free of rank $d|G|$. Similarly, H_v is $\mathbb{Z}_\ell$-free of rank

$d \cdot |G/I_v|$. One can rewrite the right hand module as $(\mathbb{Z}_\ell[G]/(s(I_v))[[\Gamma]])/(e(v))$. Now $\mathbb{Z}_\ell[G]/(s(I_v))$ is free of rank $|G| - |G/I_v|$ over $\mathbb{Z}_\ell$, and hence $R/(e(v), s(I_v))$ is free of rank $d(|G| - |G/I_v|)$, by the same reasoning as a few lines ago. Hence all terms in the sequence are $\mathbb{Z}_\ell$-free, and the rank of H_v is the difference of the rank of the module in the middle and of the module at the right. This forces the map starting from H_v to be injective.

We now put these resolutions together (recall $\mathbb{Z}_\ell \bar{S} = \bigoplus_{v \in S/G} H_v$):

$$0 \to \mathbb{Z}_\ell \bar{S} \to X_3 := \bigoplus_{v \in S/G} R/e(v)R \to X_4 := \bigoplus_{v \in S/G} R/(e(v), s(I_v)) \to 0.$$

We resume sequence (3), giving new names to two of its terms:

$$0 \to X_1 := T(X) \to X_2 := T_\ell(M_S) \to L \to 0,$$

and here L' is canonically isomorphic to $(\mathbb{Z}_\ell \bar{S})'$ (we repeat: this uses assumption C1), so that we can string the primed versions of the two sequences together:

$$0 \to X_1' \to X_2' \to X_3' \to X_4' \to 0$$

is exact. So we can apply Lemma 2.4 to find the Fitting ideal of the $\mathbb{Z}_\ell$-dual of X_1'. To this end we need the Fitting ideals of the other three modules. For X_2', this is given by Lemma 2.3, and for X_3' and X_4' they are easily determined. To wit, $\mathrm{Fitt}_{R'}(X_3')$ is generated by the product $\prod_{v \in S} e(v)$, a nonzerodivisor, and $\mathrm{Fitt}_{R'}(X_4')$ is the product of the ideals $(e(v), s(I_v))$ over $v \in S$. When applying 2.4, we can hence formally divide by $\prod_{v \in S} e(v)$, and we obtain the following result.

Theorem 2.5. *If assumption (C1) holds, then*

$$\mathrm{Fitt}_{R'}(T(X)'^*) = \Theta_S \cdot \prod_{v \in S/G} (1, \frac{s(I_v)}{e(v)}).$$

(The term $(1, \frac{s(I_v)}{e(v)})$ means the fractional R'-ideal generated by the two quantities 1 *and $\frac{s(I_v)}{e(v)}$.)*

The preceding result (which is, by the way, formally similar to a result in [Gr]) systematically ignores the trivial character of the non-ℓ-part Δ of our Galois group G. In particular, it is void for ℓ-groups G! This issue will be addressed in detail in §4. For now, we show that in case $p = \ell$ we also have a result over the trivial character, that is, concerning $\varepsilon T(X)^*$. It is equivalent (and it saves notation) to formulate and prove such a result just for ℓ-groups G, since $\varepsilon T(X)^*$ is canonically isomorphic to $T(X^\Delta)^*$, and $X^\Delta \to X$ is a covering with group G_ℓ.

The result in this case runs as follows (it certainly could be generalized at extra technical expense):

Theorem 2.6. *Suppose $p = \ell$ and G is an ℓ-group. If X is connected and there is a point $v_1 \in S/G$ which is $\mathbb{F}_q$-rational and totally ramified in X/Y, then*

$$\mathrm{Fitt}_{R'}(T(X)'^*) = \Theta_S \cdot \prod_{v \in S/G, v \neq v_1} (1, \frac{s(I_v)}{e(v)}).$$

Proof: As already pointed out, the lowest piece of the three-step filtration is now zero, so (looking at sequence (2) above) we see that the statement of Lemma 2.3 is now true, without the primes:

$$\mathrm{Fitt}_R(T_\ell(M_S)) = (\Theta_S).$$

(In particular Θ_S has to lie in R now.) Since H_{v_1} is just $\mathbb{Z}_\ell$ with trivial G- and Γ-action, we get a simple description of the lattice $L = \ker(\mathbb{Z}_\ell \bar{S} \to \mathbb{Z}_\ell)$ in our situation:

$$L \cong \bigoplus_{v \in S/G, v \neq v_1} H_v.$$

We find a resolution of L quite analogously as we did for $\mathbb{Z}_\ell \bar{S}$: we get $0 \to L \to X_3 \to X_4 \to 0$, with X_3 and X_4 defined just as before, only with the summand for $v = v_1$ missing. We now can string together the previous sequence and the sequence (3) directly, no priming is needed. The rest of the proof (application of 2.4) is the same as for Theorem 2.5. QED

As a corollary (to what we said about the validity of Lemma 2.3) we record:

Corollary 2.7. *If $p = \ell$, then Θ_S is in $R = \mathbb{Z}_\ell[G][[\Gamma]]$, furthermore it annihilates $T_\ell(M_S)$ and hence $T(X)$.*

§3 *Fitting ideals of class groups, I*

All notations and assumptions of the last section will be kept. Descent from $\bar{\mathbb{F}}$ to $\mathbb{F}_q$ is easy, in sharp contrast with the general number field case. However, working out the resulting ideals at finite level is a bit more cumbersome since the expression obtained in the right hand side of Theorem 2.5 (and 2.6) contains denominators which may well go to zero upon descent. But, again in contrast to the number field case (trivial zero problem), this issue can be overcome just by appropriately rewriting terms.

The $\mathbb{Z}_\ell$-dual occurring on the left hand side of these theorems will have the effect that our final result is in terms of the Pontryagin dual of $cl^0(X)$ (while the undualized object is the primary object of interest). We recall our shorthand $T(X)$ for $T_\ell(J_X(\bar{\mathbb{F}}))$. There is a functorial isomorphism for all $\mathbb{Z}_\ell$-modules N of finite corank:

$$T_\ell(N)^* \cong \mathrm{Hom}(N, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Setting $N = J_X(\bar{\mathbb{F}})\{\ell\}$ and noting that $N^\Gamma = J_X(\mathbb{F}_q) = cl^0(X)$ we get (using that Pontryagin dualization is an antiequivalence of categories)

$$\begin{aligned}
\big(T(X)^*\big)_\Gamma &= \mathrm{Hom}(J_X(\bar{\mathbb{F}}), \mathbb{Q}_\ell/\mathbb{Z}_\ell)_\Gamma \\
&= \mathrm{Hom}(J_X(\bar{\mathbb{F}})^\Gamma, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\
&= \mathrm{Hom}(J_X(\mathbb{F}_q), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\
&= \big(cl^0(X)\{\ell\}\big)^\vee,
\end{aligned}$$

the Pontryagin dual of the ℓ-part of the degree zero class group of the curve X. An analogous formula holds in the χ-parts for any character χ of the non-ℓ-part of G, and of course also after applying a prime (see previous section) to every module.

Let θ'_S denote the image of Θ_S in $A' := \varepsilon' \mathbb{Z}_\ell[G]$. (This makes sense, since $\varepsilon' \Theta_S$ is in R'.) Since Fitting ideals behave well under base change, the Fitting ideal of $\big(cl^0(X)\{\ell\}'\big)^\vee$ is simply given by the image of the right hand side of the equality of Theorem 2.5 in the ring A'. The product of the fractional ideals can of course be expanded as an ideal generated by $2^{|S|}$ terms, containing various subproducts of $e(v)$'s in the denominator. Any element $e(v)$ might map map to zero in A'. It will, for instance, if v is totally ramified in X and $\mathbb{F}_q$-rational.

We need some more notation in this situation. Recall that K and k are the function fields of X and Y respectively. For any $W \subset S/G$ let K_W be the fixed field of the compositum I_W of all inertia groups I_v with $v \in W$; in other words, K_W is maximal in K with the property of being unramified at all places in W. To K_W/k one attaches an element $\Theta(K_I)_{S\setminus W}$ at infinite level exactly the same way as Θ_S is attached to K/k. It lives in the quotient ring of $\mathbb{Z}_\ell[G_W][[\Gamma]]$, where $G_W = \mathrm{Gal}(K_W/k) = G/I_W$, and again $\varepsilon'\Theta(K_I)_{S\setminus W}$ is in $\varepsilon'\mathbb{Z}_\ell[G_W][[\Gamma]]$. (Recall again that we just write Θ_S to mean $\Theta_S(\gamma^{-1})$.) Finally let $g_W = \prod_{v\in W}|I_v|/|\prod_{v\in W} I_v|$. In other words, g_W is the integer satisfying $g_W s(I_W) = \prod_{v\in W} s(I_v)$. The usual Euler relations give us

$$\Big(\prod_{v\in W} s(I_v)\Big)\Theta_S = g_W \cdot \prod_{v\in W}(1-F_v^{-1})\cdot \mathrm{cor}^G_{G/I_W}\Theta(K_W)_{S\setminus W}.$$

Here cor denotes corestriction, $S\setminus W$ is abuse of notation for S with the places above W removed, and F_v denotes the Frobenius of v in $G_W\times\Gamma$ (one could say, the total Frobenius). We make the obvious but very important remark that $F_v = \sigma_v\gamma^{d_v}$, so that the Euler factor $1-F_v^{-1}$ is nothing but $e_v(\gamma^{-1})$. That is, we are able to multiply Θ_S^Σ with $\prod_{v\in W} s(I_v)/e_v(\gamma^{-1})$ so that no denominators are left, and this makes it possible to project the resulting term into $\mathbb{Z}_\ell[G]$. (The priming is not needed here but only for the validity of 2.3 and 2.5.) We are thus able to express the image of the right hand side in Theorem 2.5 in $A' = \mathbb{Z}_\ell[G]'$ without denominators. For this, let $\theta(K_W)'_{S\setminus W}$ denote the image of $\Theta(K_W)_{S\setminus W}$ in $(\mathbb{Z}_\ell[G_W])'$. Assembling all this we obtain the following result:

Theorem 3.1. *Assume that condition (C1) holds (i.e. the core group G_0 contains the non-ℓ-part of G). Then we have:*

$$\mathrm{Fitt}_{\mathbb{Z}_\ell[G]'}(cl^0(X)\{\ell\}'^\vee) = \Big\langle g_W cor^G_{G/I_W}\theta(K_W)'_{S\setminus W}\ \Big|\ W\subset S/G\Big\rangle.$$

Remark: (1) The term with $W=\emptyset$ is of course special: $g_\emptyset = 1$, the corestriction is identity, and we just get θ_S. This is called "the element at top level".

(2) The Pontryagin dual of the G-module on the left is taken in the covariant sense (note G is abelian; this comes from Proposition 1 in [Gr]). If the ℓ-part of G is cyclic, Prop. 1 in the appendix of [MW] gives that we can omit the dualization (since it does not change the Fitting ideal).

As said before, Theorem 3.1 unfortunately says nothing if G is an ℓ-group, since we had to take primed modules. It is however possible to treat groups G whose ℓ-part is cyclic. This will be done in the next section, under some technical restrictions. Before we come to that, we turn to the ε-part for $\ell = p$, just as in Theorem 2.6. Exactly the same proof as for Theorem 3.1 yields (using 2.6 instead of 2.5):

Theorem 3.2. *Assume that $\ell = p$, that G is an ℓ-group and X is connected, and that the point $v_1\in S/G$ is $\mathbb{F}_q$-rational and totally ramified. Then:*

$$\mathrm{Fitt}_{\mathbb{Z}_\ell[G]}(cl^0(X)\{\ell\}^\vee) = \Big\langle g_W cor^G_{G/I_W}\theta(K_I)_{S\setminus W}\ \Big|\ W\subset (S/G)\setminus\{v_1\}\Big\rangle.$$

Remark: (1) The second part of the remark following Theorem 3.1 applies here just as well.

(2) Some readers will notice that in 3.2 and 2.6 the condition that G is an ℓ-group is superfluous. We left it in for the following reason: The bigger G becomes, the more restrictive the hypothesis concerning a totally ramified place will be (if the non-ℓ-part of G is not cyclic, there is no chance!). Hence it is better, in the case of mixed groups G, to work with both 3.1 and 3.2; the former to cover the ε'-part, and the latter to cover the ε-part. It thus suffices to have a point whose inertia is

the complete ℓ-part of G. Since we are looking at $\ell = p$ here, this does allow a lot of noncyclic groups G_ℓ.

(3) The final result in the next section will be of a similar form, but we need more restrictive hypotheses, and instead of one, there will be two places v_1, v_2 that get excluded.

§4 *Fitting ideals of Tate modules and class groups, II*

In this section our goal is to eliminate the priming operation that occurs in Theorem 2.5 and 3.1 and weakens the statements. The focus will be on $\ell \neq p$ since for $\ell = p$ we already proved something (see Theorem 2.6 and 3.2). This will be a technical argument, and to keep notation and formalism within bounds, we will impose considerable restrictions. The final outcome is given in Thm. 4.8 and Thm. 4.9; right after 4.8 we discuss a setting where the formula of 4.8 simplifies a lot.

First and foremost we assume that G (still abelian) is a cyclic ℓ-group. (We have no idea how to eliminate cyclicity; on the other hand the assumption that we have an ℓ-group is quite harmless as we shall see.) We assume that $\mathbb{F}_q$ is the field of constants of k as well as of K; we now insist that X be connected (that is, K is a field), and that Σ is the set of places above v_0, where v_0 is a point of degree one on Y, which is totally split in X. (It would be o.K. to admit some more points of this type into Σ.) We finally assume that every point in S is of degree one. (Some but not all of the conditions we just imposed can be forced by moving to a larger base field. We will not discuss this further.)

We repeat that we now assume $\ell \neq p$. We remind the reader that S is assumed nonempty all the time. It easily follows from class field theory that if there is (tame!) ramification at all, the maximal ramification degree occurring in X/Y occurs for at least two points in S/G. As a last thing, we need to *assume* $|S/G| \geq 2$ if X/Y happens to be unramified everywhere.

We recall from earlier sections that the ℓ-adic Tate module $T_\ell(\mathcal{M}) = T_\ell(M^X_{\bar{S},\bar{\Sigma}})$ comes with a filtration

$$0 \subset T'(\mathcal{M}) \subset T''(\mathcal{M}) \subset T_\ell(\mathcal{M}),$$

where $T'(\mathcal{M}) = T_\ell(\tau_\Sigma(\bar{\mathbb{F}}))$ (the Tate module of a torus), and $T''(\mathcal{M}) = T_\ell(J_{X,\Sigma}(\bar{\mathbb{F}}))$ (the Tate module of a generalized Jacobian). The middle quotient of this filtration is $T(X) = T_\ell(J_S(\bar{\mathbb{F}}))$, and the rightmost quotient is $L = \ker(\mathbb{Z}_\ell\bar{S} \to \mathbb{Z}_\ell)$. But under our simplifying assumptions, $\mathbb{Z}_\ell\bar{S}$ is the same as $\mathbb{Z}_\ell S$, and this is isomorphic to $\bigoplus_{v\in S/G}\mathbb{Z}_\ell[G/G_v]$, with trivial Γ-action. (We have $I_v = G_v$ for all $v \in S$.) Similarly, $T'(\mathcal{M})$ is isomorphic to $(\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell[G])(1)$, where the twist indicates that the Frobenius $\gamma \in \Gamma$ acts as multiplication by q.

From our cyclicity assumptions we infer that $\mathbb{Z}[G]/\mathbb{Z}s(G) \cong \mathbb{Z}[G]^+$ (the augmentation ideal in $\mathbb{Z}[G]$), and hence an isomorphism $T'(\mathcal{M}) \cong (\mathbb{Z}_\ell[G])^+(1)$. This results in an imbedding $T'(\mathcal{M}) \to \mathbb{Z}_\ell[G](1)$ which we will treat as an inclusion.

Proposition 4.1. *(a) There exists an extension module $\tilde{Z} \supset T_\ell(\mathcal{M})$ such that $\tilde{Z}/T_\ell(\mathcal{M}) \cong \mathbb{Z}_\ell[G](1)^n$ over $\Lambda = \mathbb{Z}_\ell[G][[\Gamma]]$ for some $n \in \mathbb{N}$, with the property that the inclusion map $T'(\mathcal{M}) \to \tilde{Z}$ can be extended to an injective map $\alpha : \mathbb{Z}_\ell[G](1) \to \tilde{Z}$.*

(b) These data can be chosen such that $n = 1$ and the map j which makes

$$\begin{tikzcd} \mathbb{Z}_\ell[G](1) \arrow[r, "\alpha"] \arrow[d] & \tilde{Z} \arrow[d] \\ \mathbb{Z}_\ell(1) \arrow[r, "j"] & \mathbb{Z}_\ell[G](1) \end{tikzcd}$$

commute sends 1 to $s(G)$.

PROOF: (a) We write $T_\ell(\mathcal{M}) = P/U$ with P and U Λ-free of the same rank n. Define $\tilde{T} \in \Lambda[G]$ to be $\gamma - q$; then $\Lambda[G]/\tilde{T}\Lambda[G] = \mathbb{Z}_\ell[G](1)$. We put $\tilde{Z} = P/\tilde{T}U$; the injection $Z \to \tilde{Z}$ is given as multiplication by $\tilde{T}$. Then $\tilde{Z}/Z \cong P/\tilde{T}P \cong \mathbb{Z}_\ell[G](1)^n$, and it remains to show that α exists.

The obstruction to extendability of the map $T'(\mathcal{M}) = \mathbb{Z}_\ell[G]^+(1) \to T_\ell(\mathcal{M})$ to a map $\mathbb{Z}_\ell[G](1) \to T_\ell(\mathcal{M})$ is an element w of $\mathrm{Ext}^1_\Lambda(\mathbb{Z}_\ell(1), T_\ell(\mathcal{M}))$. Likewise the obstruction to extendability of the resulting inclusion map $T'(\mathcal{M}) \to \tilde{Z}$ to a map $\mathbb{Z}_\ell[G](1) \to \tilde{Z}$ is an element $\tilde{w} \in \mathrm{Ext}^1_\Lambda(\mathbb{Z}_\ell(1), \tilde{Z})$, and $\tilde{w}$ is the image of w under the natural map $\eta : \mathrm{Ext}^1_\Lambda(\mathbb{Z}_\ell(1), T_\ell(\mathcal{M})) \to \mathrm{Ext}^1_\Lambda(\mathbb{Z}_\ell(1), \tilde{Z})$. Hence it will certainly suffice to show that η is null. A quick calculation shows that $\mathrm{Ext}^1_\Lambda(\mathbb{Z}_\ell(1), N) \cong N^G/\tilde{T}N^G$ functorially in N. For $N = T_\ell(\mathcal{M})$ we also have $N^G = s(G)N$ since $T_\ell(\mathcal{M})$ is c.t. by §2. From this and the fact that the image of $T_\ell(\mathcal{M})$ in $\tilde{Z}$ is divisible by $\tilde{T}$, the nullity of η follows.

If α is injective, we are done. If not, then since α restricted to $T'(\mathcal{M})$ is injective, we must have $\alpha(T'(\mathcal{M})) = Im(\alpha)$, that is, α has already values in $T_\ell(\mathcal{M})$. It is then easy, replacing $\tilde{Z}$ by a split extension $T_\ell(\mathcal{M}) \oplus \mathbb{Z}_\ell[G](1)$, to modify α so as to make it injective.

(b) Let us first explain why j exists. Consider the map α' arising from α by following up with $\tilde{Z} \to \mathbb{Z}_\ell[G](1)$ (call that map π). Then $\mathbb{Z}_\ell[G]^+$ is in the kernel of α' since $\alpha(\mathbb{Z}_\ell[G]^+) \subset T_\ell(\mathcal{M}) = \ker(\pi)$, so j exists. Let $a = j(1) \in \mathbb{Z}_\ell[G](1)^n$. Then $a = rb$ with $r \in \mathbb{Z}_\ell$ nonzero and b an element of a $\mathbb{Z}_\ell$-basis of $(\mathbb{Z}_\ell[G](1)^n)^G = s(G)\mathbb{Z}_\ell[G](1)^n$. By a base change we can achieve that this basis is the standard basis $s(G)e_1, \ldots, s(G)e_n$ and b is the first element in it. We look at the copy U of $\mathbb{Z}_\ell[G](1)$ inside $\mathbb{Z}_\ell[G](1)^n$ generated by re_1. We replace $\tilde{Z}$ by the full preimage of U under $\tilde{Z} \to \mathbb{Z}_\ell[G](1)^n$. This reduces n to the value 1 and has the effect that now j is onto the fixed elements of $U \cong \mathbb{Z}_\ell[G](1)$; we even may suppose that $j(1) = s(G)$. QED

We remark that the module $\tilde{Z}$ just constructed is c.t. over G, because $T_\ell(\mathcal{M})$ and $\mathbb{Z}_\ell[G](1)$ both are c.t. over G. From both parts of Proposition 4.1 we now obtain an important diagram with short exact rows and columns (the twelve zeros at the four edges are omitted):

$$\begin{tikzcd} T'(\mathcal{M}) \arrow[r, "i"] \arrow[d] & T_\ell(\mathcal{M}) \arrow[r, "\pi'"] \arrow[d] & T_\ell(\mathcal{M})/T'(\mathcal{M}) \arrow[d] \\ \mathbb{Z}_\ell[G](1) \arrow[r, "\alpha"] \arrow[d] & \tilde{Z} \arrow[r, "\pi"] \arrow[d, "\gamma"] & \tilde{Z}/\alpha(\mathbb{Z}_\ell[G](1)) \arrow[d, "\gamma'"] \\ \mathbb{Z}_\ell(1) \arrow[r, "j"] & \mathbb{Z}_\ell[G](1) \arrow[r] & \mathbb{Z}_\ell[G](1)/s(G)\mathbb{Z}_\ell(1). \end{tikzcd}$$

We now retain the right hand column of this 3×3-diagram, and we use it to construct a new diagram. Recall $T''(\mathcal{M})/T'(\mathcal{M}) = T(\mathcal{M})$ and $T_\ell(\mathcal{M})/T''(\mathcal{M}) = L$.

The quantity $\tilde{L}$ is defined by the diagram; again the first and second row and the rightmost column represent short exact sequences (bordering zeros omitted).

$$\begin{array}{ccccc}
T(X) & \longrightarrow & T_\ell(\mathcal{M})/T'(\mathcal{M}) & \longrightarrow & L \\
\downarrow{=} & & \downarrow & & \downarrow \\
T(X) & \longrightarrow & \tilde{Z}/\alpha(\mathbb{Z}_\ell[G](1)) & \longrightarrow & \tilde{L} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell)(1) & \xrightarrow{=} & (\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell)(1).
\end{array}$$

The program is now to calculate $\tilde{L}$ with the help of information coming from cohomology, and then calculate $\mathrm{Fitt}_\Lambda(T(X))$ by the method used in the proof of Theorem 2.5, applied to the middle row. Note that the central module $\tilde{Z}/\alpha(\mathbb{Z}_\ell[G](1))$ in the above diagram is again c.t. (since α is injective), making the method applicable; we cannot use the top row since $T_\ell(\mathcal{M})/T'(\mathcal{M})$ need not be cohomologically trivial. The main problem is that we do not know a priori the extension class given by the right hand column of the diagram. Some of the ideas in the following proof are inspired by recent work of Mejía-Huguet and Rzedowski-Calderón [MR]. We begin with a simple lemma.

Lemma 4.2. *Let M be a $\mathbb{Z}[G]$-module without $\mathbb{Z}$-torsion. Then $\mathrm{H}^{-1}(G, M)$ is canonically isomorphic to the torsion submodule of M_G (the coinvariant module).*

Proof: By definition, $\mathrm{H}^{-1}(G, M)$ is the kernel of the map $\mathrm{N}_G : M_G \to M^G$ coming from multiplication by $s(G)$. Since this maps becomes an isomorphism after tensoring with $\mathbb{Q}$ and since its target is torsion-free, its kernel must be exactly the torsion submodule of M_G. QED

By definition, $T(X) = T_\ell(J_X(\bar{\mathbb{F}}))$. From this it follows that

$$\hat{\mathrm{H}}^0(G, T(X)) \cong \mathrm{H}^{-1}(G, J_X(\bar{\mathbb{F}}))$$

(use the standard sequence $0 \to T(X) \to \mathbb{Q}_\ell T(X) \to J_X(\bar{\mathbb{F}}) \to 0$). From a result of Rzedowski-Calderón, Villa-Salvador and Madan [RVM] (we are using $\ell \neq p$ in quoting this) and the cyclicity of G we know that $\mathrm{H}^{-1}(J_X(\bar{\mathbb{F}}))$ has order $\ell^{2(n-n_1)}$ where $|G| = \ell^n$ and the maximal ramification degree occurring in K/k is ℓ^{n_1}. (Recall that this maximal degree is attained at least for two places in the set $S/G \subset Y$.) Hence $\hat{\mathrm{H}}^0(G, T(X))$ has the same order.

Now we have the short exact sequence

$$0 \to T(X) \to \tilde{Z}/\alpha(\mathbb{Z}_\ell[G](1)) \to \tilde{L} \to 0$$

from above. As its middle term is c.t. over G, this implies: $\mathrm{H}^{-1}(G, \tilde{L})$ has order $\ell^{2(n-n_1)}$. By Lemma 4.2, this cohomology group identifies with the torsion subgroup of $\tilde{L}_G$. It turns out that this numerical information is enough to determine the G-structure of $\tilde{L}$, and this in turn will suffice to calculate the Fitting ideal of $T(X)$ over Λ.

First we list the set $S/G \subset Y$ in the form $S/G = \{v_1, \ldots, v_r\}$ (no repeated elements), such that for the the ramification indices ℓ^{n_i} of the v_i we have $n_1 \geq n_2 \geq \ldots \geq n_r$. (Obviously the unramified places come last.) We noted above that $n_1 = n_2$. (In the unramified case both are zero.) Then L is the kernel of the augmentation map $L_0 \to \mathbb{Z}_\ell$, with $L_0 = \bigoplus_{i=1}^r \mathbb{Z}_\ell[G/G_i]$, $G_i = G_{v_i}$. It is not hard to show that the natural exact sequence

$$0 \to \mathbb{Z}_\ell[G/G_1]^+ \to L \to \bigoplus_{i=2}^{r} \mathbb{Z}_\ell[G/G_i] \to 0$$

is split. Moreover there are no nontrivial extensions of $\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell I_{G/G_1}$ (this module has no nonzero G-fixed elements), and therefore $\tilde{L}$ can also be split as the direct sum of $\mathbb{Z}_\ell[G/G_1]^+$ and L', which is an extension of $\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell$ by $A := \bigoplus_{i=2}^r \mathbb{Z}_\ell[G/G_i]$. (Note that we are neglecting the Tate twist; at this stage we are only interested in the G-structure.) Moreover a quick calculation shows: the torsion subgroup of $(\mathbb{Z}_\ell[G/G_1]^+)_G$ is of order ℓ^{n-n_1}. Since the torsion of $\tilde{L}_G$ is of order $\ell^{2(n-n_1)}$, we get that the torsion subgroup of L'_G is again of order ℓ^{n-n_1}.

We will show: This information implies that L' is a direct sum $L'' \oplus \bigoplus_{i=3}^r \mathbb{Z}_\ell[G/G_i]$, where L'' is an explicitly described extension:

$$0 \to \mathbb{Z}_\ell[G/G_2] \to L'' \to \mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell \to 0,$$

and as a G-module, $L'' \cong \mathbb{Z}_\ell[G/G_2]^+ \oplus \mathbb{Z}_\ell[G]$.

For any G-module A, every extension of $\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell$ by A is given in the form $E_x := (M \oplus \mathbb{Z}_\ell[G])/\mathbb{Z}_\ell \cdot (x, s(G))$ for some $x \in A^G$, and the class of the extension only depends on x modulo $s(G)A$. In our case, let $(u_2, \dots, u_r) = x$ be an element of $A = \bigoplus_{i=2}^r \mathbb{Z}_\ell[G/G_i]$ defining the extension L'. We can write $u_i = z_i s(G/G_i)$ with $z_i \in \mathbb{Z}_\ell$. The coinvariant module of the extension E_x is then

$$\frac{\mathbb{Z}_\ell^{r-1} \oplus \mathbb{Z}_\ell}{\mathbb{Z}_\ell([G:G_2]z_2, \dots, [G:G_r]z_r, [G:1])}.$$

The torsion part of this has at least $\gcd([G:G_2]z'_2, \dots, [G:G_r]z'_r)$ elements, where z'_i is the largest ℓ-power dividing z_i. On the other hand we know from above that the order of the torsion part is exactly $[G:G_2]$. Moreover $[G:G_2]$ is minimal among the $[G:G_i]$. This means that we can reorder the places such that $[G:G_2]$ is still minimal, and $z'_2 = 1$, that is, z_2 is a unit.

We claim that there is an automorphism j of A such that

$$x = j(s(G/G_2), 0, \dots, 0).$$

Indeed, let j be the identity on the summands of A with indices 3 to r, and let $j(1, 0, \dots, 0) = (z_i s(G_2/G_i))_{2 \le i \le r}$. (Note that indeed $G_i \subset G_2$!) Then if ρ is any lift of $s(G/G_2)$ to $\mathbb{Z}_\ell[G]$, we get $j(s(G/G_2), 0, \dots, 0) = \rho j(1, 0, \dots, 0) = (z_i s(G/G_i))_{2 \le i \le r} = x$.

Thus we may assume that x is actually equal to $(s(G/G_2), 0, \dots, 0)$. This shows that L' can be written as $L'' \oplus \bigoplus_{i=3}^r \mathbb{Z}_\ell[G/G_i]$, and L'' is the extension of $\mathbb{Z}_\ell[G]/s(G)\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell[G/G_2]$ given by:

$$L'' = \frac{\mathbb{Z}_\ell[G/G_2] \oplus \mathbb{Z}_\ell[G]}{\mathbb{Z}_\ell(s(G/G_2), s(G))}.$$

As a G-module, this is the direct sum of the free module generated by (the image of) $(0,1)$, and a copy of $\mathbb{Z}_\ell[G/G_2]^+$, generated by (the image of) $(1, s(G_2))$. Let us put this on record, recalling that $\tilde{L} = \mathbb{Z}_\ell[G/G_1]^+ \oplus L'$ and that the middle module in the short exact sequence leading from $T(X)$ to $\tilde{L}$ is G-c.t. and torsion-free, hence projective:

Proposition 4.3. *There is a projective $\mathbb{Z}[G]$-module Q and a short exact sequence*

$$0 \to T(X) \to Q \to \tilde{L} \to 0,$$

where $\tilde{L} := \mathbb{Z}_\ell[G/G_1]^+ \oplus \mathbb{Z}_\ell[G/G_2]^+ \oplus \mathbb{Z}_\ell[G] \oplus \bigoplus_{i=3}^r \mathbb{Z}_\ell[G/G_i]$. (In fact, G_1 and G_2 coincide; we just keep them apart for clarity.)

This does not yet take the Γ-action into account. For this we have to back up a little. The automorphism of A constructed above is also Γ-linear, since the whole of A is a trivial Γ-module. However, the module $B := (\mathbb{Z}_\ell[G]/\mathbb{Z}_\ell s(G))(1)$ is not a trivial Γ-module; the generator γ of Γ acts as multiplication by q. We set $A_1 = \mathbb{Z}_\ell[G/G_1]^+$ and $A_i = \mathbb{Z}_\ell[G/G_i]$ for $i = 2, \ldots, r$ and we recall $\tilde{L} = A_1 \oplus L'' \oplus \bigoplus_{i=3}^r A_i$, where L'' sits in a sequence $0 \to A_2 \to L'' \to B \to 0$, and the submodule $L = \oplus_{i=1}^r A_i$ in $\tilde{L}$ has trivial Γ-action. Let e_1 resp. e_2 denote the images of $(1,0)$ resp. $(0,1)$ in L'' (see construction of L'' above). Then e_1 is fixed by Γ again (since it comes from L). Let us set $T = \gamma - 1$ and recall $\tilde{T} = \gamma - q.$, noting that $\tilde{T} = \gamma \cdot \tilde{e}_{v_0}(\gamma^{-1})$. Then B is annihilated by $\tilde{T}$. Therefore we have:

$$\gamma e_2 = (a_1, ae_1 + ze_2, a_3, \ldots, a_r)$$

with $z \in \mathbb{Z}_\ell[G]$ congruent to q mod $s(G)$, $a \in \mathbb{Z}_\ell[G/G_2]$, and $a_i \in A_i$ $(i \neq 2)$. Since we are free to change z modulo $s(G)$ (using the relation $s(G/G_2)e_1 + s(G)e_2 = 0$; note that a changes as well) we may assume that $z = q$. Let ε denote the augmentation map.

Lemma 4.4. *We have $1 + \varepsilon(a)|G_2| = q$, and the a_i are annihilated by $s(G)$.*

Proof: This follows from the welldefinedness of γ on L'': we have $0 = F(s(G/G_2)e_1 + s(G)e_2)$. This is equal to $(s(G/G_2) + s(G)a)e_1 + s(G)qe_2 = s(G/G_2)[1 + |G_2|\varepsilon(a)] + s(G/G_2)qe_2$; if this is to be zero in L'', we must have $1 + |G_2|\varepsilon(a) = q$. To see the second statement, note that $s(G)e_2$ comes from L, which has trivial γ-action. QED

Lemma 4.5. *One can replace L'' inside $\tilde{L}$ by an isomorphic copy such that the decomposition of $\tilde{L}$ in a direct sum and the short exact sequence $0 \to A_2 \to L'' \to B \to 0$ remain valid, and such that the a_i (with $i = 1$, and $i = 3, \ldots, r$) are replaced by zero. As a consequence, we have that*

$$\tilde{L} = A_1 \oplus L'' \oplus \bigoplus_{i=3}^r A_i$$

as $\mathbb{Z}_\ell[G][[\Gamma]]$-modules.

Proof: Let L_1 be the $\mathbb{Z}_\ell[G]$-submodule of L'' (written as a direct sum as above) generated by $e_1' := (0, e_1, 0, \ldots, 0)$ and $e_2' := (-a_1, e_2, 0, \ldots, 0)$. The difference $e_2' - e_2$ is annihilated by $s(G)$ according to Lemma 4.4. Therefore the relation $s(G/G_2)e_1' + s(G)e_2'$ is again valid, and we have a natural sequence $0 \to A_2 \to L_1 \to B \to 0$. It is easy to see that $\gamma e_2' = (0, \gamma e_2, 0, \ldots, 0)$. Moreover it is clear that $\tilde{L}$ is again the direct sum of L_1 and the A_i (with $i = 1, 3, 4, \ldots, r$). We rename L_1 into L'' again. QED

In the next step, we embed $\tilde{L}$ into a $\mathbb{Z}_\ell[G][[\Gamma]]$-module P which is free over $\mathbb{Z}_\ell[G]$. Thanks to the last lemma, it suffices to embed A_i into P_i and L'' into P'' and then let P be the direct sum of the P_i and P''. The monomorphisms $A_i \to P_i$ will be denoted j_i; the monomorphism $L'' \to P''$ will be written j''; and the total monomorphism $\tilde{L} \to P$ will be written j.

We begin with the easy parts. For $i \geq 3$, let $P_i = \mathbb{Z}_\ell[G]$ with trivial Γ-action and map A_i to P_i via $j_i(1) = s(G_i)$. For $i = 1$, take $P_1 = \mathbb{Z}_\ell[G]$ with trivial Γ-action and map A_1 to P_1 by $j(\overline{\sigma - 1}) = (\sigma - 1)s(G_1)$. Here and in the sequel, σ is a chosen generator of G.

We now take $P'' = \mathbb{Z}_\ell[G] \oplus \mathbb{Z}_\ell[G]$ and define $j'' : L'' \to P''$ by $j''(e_1) = (s(G_2), 0)$ and $j''(e_2) = (-1, \sigma - 1)$. It is immediate that the relation $s(G/G_2)e_1 + s(G)e_2$ maps to zero, so j'' is well-defined. As said above, $P = P_1 \oplus P'' \oplus P_3 \oplus \ldots \oplus P_r$, and the embedding $L \to P$ is j'' on P'' and j_i on P_i.

We let Γ act on P'' by $\gamma(1,0) = ((1,0)$ and

$$\gamma((0,1) = (\tilde{a}, q)$$

with $\tilde{a}$ yet to be determined.

Lemma 4.6. *(a) The equation $(\sigma - 1)\tilde{a} = s(G_2)a - q + 1$ is solvable in $\mathbb{Z}_\ell[G]$. (Note that the term $s(G_2)a$ makes sense in $\mathbb{Z}_\ell[G]$.)*

(b) If $\tilde{a}$ is a solution of the equation in (a), then $j'' : L'' \to P''$ is Γ-equivariant.

PROOF: (a) It suffices to show that the augmentation map applied to the right hand side produces zero. Applying augmentation gives $|G_2|\varepsilon(a) - q + 1$, and this is indeed zero by the last lemma.

(b) The formula $j''(\gamma e_1) = \gamma j''(e_1)$ follows from the definitions and has nothing to do with $\tilde{a}$. We calculate:

$$\begin{aligned} j''(\gamma e_2) &= j''(ae_1 + qe_2) \\ &= \big(s(G_2)a - q, q(\sigma - 1)\big) \end{aligned}$$

and

$$\begin{aligned} \gamma j''(e_2) &= F(-1, \sigma - 1) \\ &= (-1, 0) + F(0, \sigma - 1) \\ &= \big((\sigma - 1)\tilde{a} - 1, q(\sigma - 1)\big). \end{aligned}$$

Therefore the γ-linearity of j'' is actually equivalent to equation (a). QED

Now we calculate $\mathrm{Fitt}(P)$. (Unadorned Fitt always means the Fitting ideal over $\Lambda = \mathbb{Z}_\ell[G][[\Gamma]]$.) The Fitting ideal of P_i is simply (T) for $i = 1$ and $i = 3, \ldots, r$. Moreover $\mathrm{Fitt}(P'')$ is fairly easy to determine: P'' is free of rank 2 over $\mathbb{Z}_\ell[G]$, and γ acts via some triangular matrix with diagonal entries 1 and q. This implies $\mathrm{Fitt}(P'') = (T\tilde{T})$. Finally, the Fitting ideal of P is the product of the Fitting ideals of the P_i and of P''. (It is well-known and easy that the Fitting ideal of a finite direct sum is the product of the Fitting ideals of the summands.) So we get

$$\mathrm{Fitt}(P) = (T^r\tilde{T}).$$

The last step in this calculation is to determine the Fitting ideal of the cokernel $P/j(\tilde{L})$. The main point is the calculation of the Fitting ideal of $P''/j(L'')$.

For any subgroup H of G, let $e(H)$ denote the idempotent $s(H)/|H|$.

Lemma 4.7. $\mathrm{Fitt}(P''/j''(L''))$ *is generated by $T - (q-1)e(G_2)$ and $(\sigma - 1)s(G_2)$. (It is part of the statement that $(q-1)e(G_2)$ has integral coefficients.)*

PROOF: Let $e = e(G_2)$ and $e' = 1 - e$. The relations defining $P''/j''(L'')$ from the two generators $(1,0)$ and $(0,1)$ are given by the following matrix, the first two rows of which are already relations of P'', the other two arise by factoring out $j''(L'')$:

$$\begin{pmatrix} T & 0 \\ -\tilde{a} & \tilde{T} \\ s(G_2) & 0 \\ -1 & \sigma - 1 \end{pmatrix}.$$

Let δ be the determinant of the square submatrix formed by the second and fourth row, so $\delta = \tilde{T} - \tilde{a}(\sigma - 1) = \tilde{T} - s(G_2)a + q - 1 = T - s(G_2)a$. Since the Fitting ideal also contains $(\sigma - 1)s(G_2)$ (take third and fourth row), we can replace a by $\varepsilon(a)$ and still get an element $\delta' = T - s(G_2)\varepsilon(a)$ of the Fitting ideal. But by Lemma 5.4, $\varepsilon(a) = (q-1)/|G_2|$, so $\delta' = T - (q-1)e(G_2)$.

It suffices now to show that the Fitting ideal is generated by the two elements given. It is generated by δ' and the product $(T, s(G_2))(\tilde{T}, \sigma - 1)$, as can be seen by considering all 2×2-minors. But $\mathbb{Z}_\ell[G][[\Gamma]]/(\delta')$ is isomorphic to $\mathbb{Z}_\ell[G]$, via the map η sending T to $(q-1)e(G_2)$. Applying η to the above product of ideals gives

$$((q-1)e, s(G_2))\cdot((q-1)e', \sigma-1).$$

The left hand ideal is simply generated by $s(G_2)$ (since $(q-1)/|G_2|$ is an integer), so the product has the value $(\sigma-1)s(G_2)$, since $s(G_2)e'=0$. This shows that the Fitting ideal taken modulo δ' can be generated by $(\sigma-1)s(G_2)$ alone, and we are done. QED

The Fitting ideal of $P_i/j_i(A_i)$ is easily computed: we get $(T, s(G_i))$ for $i\geq 3$ and $(T,(\sigma-1)s(G_1))$ for $s=1$. Since $P/j(\tilde{L})$ is given as a direct sum, we know that the Fitting ideal of $P/j(\tilde{L})$ is the product of the Fitting ideals of $P''/j(L'')$ and of the $P_i/j_i(A_i)$ for $i=1,3,4,\ldots,r$, that is:

$$\mathrm{Fitt}(P/j(\tilde{L})) = \big(T-(q-1)e(G_2), (\sigma-1)s(G_2)\big)\cdot\big(T,(\sigma-1)s(G_1)\big)\cdot\prod_{i=3}^{r}\big(T,s(G_i)\big).$$

We now put everything together, using Lemma 2.4 on the four-term sequence

$$0\to T(X)\to \tilde{Z}/\alpha(\mathbb{Z}_\ell[G](1))\to P\to P/j(\tilde{L})\to 0$$

which comes from Proposition 4.3 and the resolution

$$0\to\tilde{L}\to P\to P/j(\tilde{L})\to 0.$$

The Fitting ideal of the second term is the same as the Fitting ideal of $T_\ell(\mathcal{M})$ (determined at the end of §3), since $\tilde{Z}/T_\ell(\mathcal{M})\cong\mathbb{Z}_\ell[G](1)$, α is injective, and as said in §4, Fitt is multiplicative on short exact sequences of modules having projective dimension 1. This gives:

Theorem 4.8. *Under all assumptions stated at the beginning of the section (in particular G is a cyclic ℓ-group),* $\mathrm{Fitt}_{\mathbb{Z}_\ell[G][[\Gamma]]}(T(X)^*)$ *is given by*

$$\Theta_S^{\Sigma}(\gamma^{-1})\frac{1}{T\tilde{T}}\big(1,\frac{(\sigma-1)s(G_1)}{T}\big)\big(T-(q-1)e(G_2),(\sigma-1)s(G_2)\big)\cdot\prod_{i=3}^{r}\big(1,\frac{s(G_i)}{T}\big).$$

This formula looks clumsy. One simplification is possible: since $\Theta_S^{\Sigma}(\gamma^{-1}) = \tilde{e}_{v_0}(\gamma^{-1})\Theta_S(\gamma^{-1}) = \gamma\tilde{T}\Theta_S(\gamma^{-1})$, we may simultaneously omit $\tilde{T}$ in the denominator and the superscript Σ in the above statement. Caution is due since we may not assume that $\Theta_S(\gamma^{-1})$ is in R (i.e. free of denominator; in fact it is not).

We discuss the formula in the case where $G_1=G_2=G$. In other words, we are imposing the condition that at least two places are totally ramified. We repeat that the existence of one such place forces the existence of a second one, since G_1 and G_2 are equal. Then the second generator in both fractional ideals $\big(1,\frac{(\sigma-1)s(G_1)}{T}\big)$ and $\big(T-(q-1)e(G_2),(\sigma-1)s(G_2)\big)$ disappears. Let $e=e(G)$ and $e'=1-e$. Then $T-(q-1)e(G_2)=e\tilde{T}+e'T$. Moreover we can use what was said in the preceding paragraph and rewrite the expression $\frac{1}{T}(e\tilde{T}+e'T)$ to $\eta := e\tilde{T}T^{-1}+e'$. Then the whole expression for the Fitting ideal becomes

$$\eta\Theta_S(\gamma^{-1})\cdot\prod_{i=3}^{r}\big(1,\frac{s(G_i)}{T}\big),$$

which looks a lot simpler. Warning: Neither $\Theta_S(\gamma^{-1})$ nor η are in R, both have a denominator. But since the product of fractional ideals that constitutes the third factor contains 1, it is clear that the product $\tilde{\Theta}_S := \eta\Theta_S(\gamma^{-1})$ must be in R. In this situation it is possible to write down the result at finite level; the reasoning is very similar to the arguments used for Theorem 3.1 and need not be given in detail again. However there is one notable difference: Since G is assumed cyclic, Fitting ideals over the group ring do not change when the module is dualized (Prop. 1 in the appendix to [MW]), so we may omit the Pontryagin dual ! Just a little more notation: Let $\tilde\theta(K_W)_{S\setminus W}$ denote the image of $\eta_W\Theta(K_W)_{S\setminus W}(\gamma^{-1}) \in \mathbb{Z}_\ell[G_W][[\Gamma]]$ in $\mathbb{Z}_\ell[G_W]$. (Notation: see §3.) Here the element $\eta_W \in \mathbb{Q}_\ell[G_W]$ is defined in analogy with η, and $\eta \mapsto \eta_W$. The result at finite level is:

Theorem 4.9. *Under the assumptions of Thm. 4.8 and the extra assumptions that $G_1 = G_2 = G$ (recall $S = \{v_1, \dots, v_r\}$ and $G_i = G_{v_i}$), we have*

$$\mathrm{Fitt}_{\mathbb{Z}_\ell[G]}(cl^0(X)\{\ell\}) = \left\langle g_W\, cor^G_{G_W}\tilde\theta(K_W)_{S\setminus W} \;\middle|\; W \subset (S/G)\setminus\{v_1, v_2\}\right\rangle.$$

(Remark: Again, the generator corresponding to empty W is very special: it simply equals $\tilde\theta(K)_S$, and we will just write $\tilde\theta_S$ for this top level generator.)

Now we illustrate this by a few examples.

(1) If $r = 2$, then the desired Fitting ideal needs only one generator, obtained from $\tilde\Theta_S(\gamma^{-1})$ by putting $\gamma = 1$. In particular, $cl^0(X)\{\ell\}$ is then c.t. by a result in [CG]. An example may be constructed as follows: assume $\ell = 3$, $p \equiv 1$ modulo 3, let $Y/\mathbb{F}_q$ be an elliptic curve, and suppose $P \in Y(\mathbb{F}_q)$ has order 2. Then $2(P)-2(\infty)$ is principal, equal to $\mathrm{Div}(f)$, $f \in \mathbb{F}_q(y)$, say. Let X be the curve defined by $x^3 - f(y) = 0$. Then the degree 3 covering X/Y ramifies in exactly two points, and X is of genus 3. So $cl^0(X)\{\ell\}$ is G-cohomologically trivial (and not zero if $\mathbb{F}_q$ is large enough). At the level of Tate modules we even find that $T(X)$ is $\mathbb{Z}_\ell[G]$-free of rank 2.

(2) Now let us assume $r \geq 3$ and $G_1 = \ldots = G_r (= G)$. One can show that $e\Theta(K_W)_{S\setminus W} \in e\,Quot(R) \cong \mathbb{Q}_\ell(T)$ is divisible by T exactly $r - 1 - |W|$ times, and hence $e\,cor^G_{G/G_W}\tilde\Theta(K_W)_{S\setminus W}$ evaluates to 0 for $W \neq (S/G)\setminus\{v_1, v_2\}$ and to something different from 0 for W being the maximal choice $(S/G)\setminus\{v_1, v_2\}$. Since G_W is trivial for all nonempty W, the fields K_W are all just k for $W \neq \emptyset$, and the corestrictions $cor^G_{G/G_W}\tilde\theta(K_W)_{S\setminus W}$ are all equal, and annihilated by e'. We thus obtain just two generators, one in $e\mathbb{Z}_\ell[G]$ and the other in $e'\mathbb{Z}_\ell[G]$, to wit: $|G|^{r-2}\cdot cor^G_1\tilde T T^{-1}\Theta(k)_{\{v_1,v_2\}}|_{T=0})$ (coming from W as large as possible) and $e'\Theta_S|_{T=0}$ (coming from $W = \emptyset$). This ideal cannot be principal (since $\mathbb{Z}_\ell[G]$ is local, and the ideal is a nontrivial direct sum of its e- and e'-components), and therefore the class group is not c.t. in this case. An easy instance of this is found by taking Y to be the projective line and X a degree 2 covering with four $\mathbb{F}_q$-rational ramified points, so $\ell = 2$ and X is an elliptic curve. Then the norm element $s(G)$ of the group G (which has order 2) annihilates $cl^0(X)\{2\}$ completely; on the other hand this class group is at least of order 4, and hence visibly not c.t. over G.

Concluding remark: If G is any abelian group with cyclic ℓ-part, then the Fitting ideal of $cl^0(X)\{\ell\}$ can be computed by character decomposition along the characters of the non-ℓ-part of G. For the trivial character one uses the results established in this section, and for the others one uses what was shown in §3. It does not seem necessary to write down the exact technical hypotheses and the outcome in detail.

§5 *Fitting ideals with the set* Σ *retained*

In this section we explain what happens to the Fitting ideals dealt with in previous sections if we do not eliminate the auxiliary set Σ. (Such results are useful in treating the Brumer-Stark conjecture.) From a logical viewpoint, these results might have preceded those in which Σ is eliminated. However, the results with Σ in place are parallel to, and much simpler than those with Σ eliminated, so we can deal with them quickly now, referring back where appropriate.

We let $T(X)_\Sigma = T_\ell(J_{X,\Sigma}(\bar{\mathbb{F}}))$, and we have a short exact sequence analogous to (3) in §2:

$$0 \to T(X)_\Sigma \to T_\ell(M_{S,\Sigma}) \to L \to 0.$$

Now the middle term has principal Fitting ideal (given by Theorem 1.1 (b)), and we already discussed the module L in §2. We proceed exactly as there: resolve L, obtain a 4-term sequence with middle terms of projective dimension at most one, and appeal to Lemma 2.4. The only difference (actually, a simplification) is that we do not need to apply the priming operation. The outcome is, in complete analogy with Theorem 2.5 (recall $R = \mathbb{Z}_\ell[[\Gamma]][G]$):

Theorem 5.1. *If assumption (C1) holds, then*

$$\mathrm{Fitt}_R(T(X)^*_\Sigma) = \Theta_S^\Sigma \cdot \prod_{v \in S/G} (1, \frac{s(I_v)}{e(v)}).$$

The descent to finite level also works as before. Denote by $cl^0_\Sigma(X)$ the degree zero ray class group of X with respect to Σ; this is also the group of $\mathbb{F}_q$-rational points on the Σ-Jacobian of X. Thus we have, just as at the beginning of §3:

$$(T(X)^*_\Sigma)_\Gamma \cong (cl^0_\Sigma(X)\{\ell\})^\vee.$$

By exactly the same arguments as in the sequel of §3 (again, minus the complications that arose from the priming operations) we are led to the following analog of Theorem 3.1:

Theorem 5.2. *Assume that condition (C1) holds (i.e. the core group G_0 contains the non-ℓ-part of G). Then we have:*

$$\mathrm{Fitt}_{\mathbb{Z}_\ell[G]}(cl^0(X)\{\ell\}^\vee) = \left\langle g_W cor^G_{G/I_W} \theta(K_W)^\Sigma_{S\setminus W} \;\middle|\; W \subset S/G \right\rangle.$$

(The quantities $\theta(K_W)^\Sigma_{S\setminus W}$ used in this theorem are obtained from the analogous Θ quantities by projecting from $\mathbb{Z}_\ell[G/I_W][[\Gamma]]$ to $\mathbb{Z}_\ell[G/I_W]$.)

§6 *Brumer's conjecture, and more examples*

Let us make the simplifying assumption that $\mathbb{F}_q$ is the constant field of X. Brumer's conjecture states (with notations as previously) that $A\theta_S$ annihilates the class group $cl(X)$ (not only the degree 0 part), where A is the annihilator ideal of $\mathbb{F}_q^\times$ (the "roots of unity") in $\mathbb{Z}[G]$. This is known to hold, since the Brumer-Stark conjecture was completely proved in this setting by Deligne (see [Ta]), and the Brumer conjecture can be deduced from this (see [Ro] chapter 15). Here we show that we can also deduce it from the results in the preceding section (under our assumptions), and we sometimes get a little more.

We assume to begin with that G and X/Y satisfy all assumptions of the last section (in particular G is cyclic of ℓ-power order), and that $\ell \neq p$.

We claim that $A\theta_S$ is contained in the $\mathbb{Z}_\ell[G]$-span of $\tilde{\theta}_S$ (the generator in the last theorem corresponding to empty W). To show this, we recall that σ is a generator of G. Then $(\sigma - 1)\eta = (\sigma - 1)e' = \sigma - 1$. On the other hand, using that $|G|$ divides $q-1$, one easily checks that $\beta := eT + e'\tilde{T}$ is in $\mathbb{Z}_\ell[G]$, and clearly we have $\beta\eta = \tilde{T}$. Therefore $R\tilde{\Theta}_S$ contains $(\sigma-1)\Theta_S$ and $\tilde{T}\Theta_S$. We now go from R to $\mathbb{Z}_\ell[G]$ (setting $T = 0$) and obtain that $\mathbb{Z}_\ell[G]\tilde{\theta}_S$ contains $(\sigma - 1)\theta_S$ and $(q-1)\theta_S$. Since the $\mathbb{Z}_\ell[G]$-annihilator of the roots of unity in K is generated by $\sigma - 1$ and $q - 1$ under our assumptions on K/k, we have established our claim.

Thus $A\theta_S$ is a subset of the Fitting ideal of $cl^0(K)\{\ell\}$, and hence also a subset of the $\mathbb{Z}_\ell[G]$-annihilator of this module. So we have proved that $A\theta_S$ annihilates $cl^0(K)\{\ell\}$, for G and X/Y as above and $\ell \neq p$. The latter restriction can be eliminated immediately, with the help of Corollary 2.7. (This result involves a dual, but of course the annihilator of a module and its dual agree.)

Now we replace the condition on G and X/Y by the following: G has cyclic ℓ-part, and the subcovering X^Δ/Y satisfies all conditions in 4.9 (recall Δ is the non-ℓ-part of G). The fact that $A\theta_S$ annihilates $\varepsilon' cl^0(X)\{\ell\}$ is an easy consequence of Theorem 3.1. Thus $A\theta_S$ annihilates all of $cl^0(X)$.

Mimicking the argument on p. 125 in [Ta], we can show that the last statement, together with the fact that θ_S annihilates the module generated by degree zero divisors supported on S, actually implies: $A\theta_S$ annihilates $cl(X)$. (Note that $|S/G| \geq 2$.) So we have reproved Brumer's conjecture, under some technical conditions.

This is at least useful as a checkup on our results. However, a little more can be said. Let χ_0 denote the trivial character of G. Extend the construction of $\tilde{\theta}_S$ to the case where G is not necessarily an ℓ-group, just by letting $\tilde{\theta}$ and θ agree in the ε'-part. So the only character value which changes when putting a tilde on θ_S is the value at χ_0. A similar statement applies to Θ_S and $\tilde{\Theta}_S$. So if Θ_S has a "double zero at χ_0", i.e. $\chi_0(\Theta_S)$ is divisible by T^2, we find that θ_S and $\tilde{\theta}_S$ are equal (both evaluate to zero under χ_0). On the other hand we know by 4.9 and §3 that $\tilde{\theta}_S$ annihilates the degree zero class group. Since $\tilde{\Theta}_S$ still has a zero at χ_0, it annihilates the whole of $\mathbb{Z}\tilde{S}$ (not only the augmentation kernel). So a slight variant of the argument on p. 125 in [Ta] again allows to extend the annihilation statement to the whole class group. What have we gained? Brumer's conjecture predicts annihilation by $A\theta_S$, which equals $A\tilde{\theta}_S$ under the double zero assumption; we have proved annihilation by the strictly larger principal ideal $(\theta_S) = (\tilde{\theta}_S)$. (Here we focus on the top level annihilator; as said before, on the right hand side of the equation in Theorem 4.9 many other annihilators appear.) The double zero hypothesis is fulfilled as soon as S/G has at least three points, that is: very frequently.

Let us discuss a few more examples.

Take $k = \mathbb{F}_q(x)$, and K to be the field obtained by adjoining a $(q-1)$-th root of $-x$ to k. In fancier language, K is the "cyclotomic" extension of k of conductor $m = (x)$, see chapter 15 in [Ro]. We put $S = \{m, \infty\}$; this is just the set of ramified places. The Galois group G is identified with $(\mathbb{F}_q[x]/(x))^\times = \mathbb{F}_q^\times$; the automorphism corresponding to $a \in \mathbb{F}_q^\times$ is written σ_a. The partial zeta function $\zeta_S(u, \sigma_a)$ (we put $u = q^{-s}$, Rosen has w instead of s) is given by: $1 + u/(1-qu)$ if a is the constant 1, and just $u/(1-qu)$ if a is any other nonzero constant. Hence $\Theta_S(u) = 1 + N_G \cdot u/(1-qu)$, and $\theta_S = 1 - N_G/(q-1)$ on setting $u = 1$. We already see that $\chi_0(\theta_S) = 0$. Moreover $cl^0(K) = 0$ since K is again rational, so the optimal annihilator for $cl^0(K)$ would be 1.

Let us determine $\tilde{\theta}$. For all nontrivial characters χ we have $\chi(\tilde{\theta}_S) = \chi(\theta_S)$ by construction, and $\chi(\theta_S) = 1$ by inspection. Applying χ_0 to Θ_S gives the rational function $1 - (q-1)u/(qu-1)$. This equals the fraction $(u-1)/(qu-1)$, and this gives exactly to $T/\tilde{T}$ on replacing u by γ^{-1}. Since at χ_0, $\tilde{\Theta}_S$ is gotten from Θ_S by multiplying with $\tilde{T}/T$, we get that $\chi_0(\tilde{\Theta}_S)$ is again 1, so $\tilde{\Theta}_S = 1$, the best possible result in this special case. (Note that we cannot conclude here that $1 = \tilde{\Theta}_S$ annihilates the whole of $cl(K)$ (it obviously doesn't), and this is because $\tilde{\theta}_S$ does not vanish at χ_0.) By example (1) at the end of §4, the Fitting ideal of $cl^0(K)$ is generated by the top level element $\tilde{\theta}_S$ in this case; but this is clear anyway since $cl^0(K) = 0$.

One can do a similar calculation when K/k is the "cyclotomic" extension of conductor $m = (x^2)$. Here it turns out that $\tilde{\theta}_S$ and the element η ([Ro] p.272) are associated in $\mathbb{Z}_\ell[G]$ for all $\ell \neq p$. (In the previous example, $\eta = \tilde{\theta}_S = 1$.) It is interesting to note that Gross gave a direct proof that η annihilates $cl^0(K)$, but his argument makes heavy use of the fact that $cl^0(k) = 0$.

Finally we look at the "cyclotomic" extension $\tilde{K}/k$ of conductor $m = (x(x-1))$ with Galois group $\tilde{G}$. Unfortunately this is not cyclic, so we look at a "diagonal" subfield K. More precisely: we identify $\tilde{G} = \big(k[x]/(x(x-1))\big)^\times$ with $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ via $\bar{f} \mapsto (f(0), f(1))$, we let Δ be the kernel of the obvious multiplication map $\mathbb{F}_q^\times \times \mathbb{F}_q^\times \to \mathbb{F}_q^\times$, and we let K be the fixed field of Δ. Then K/k is cyclic with Galois group $G \cong \mathbb{F}_q^\times$, (x) and $(x-1)$ are totally ramified, and the point at infinity has splitting and ramification degree both equal to 2. The genus of K is 1.

To be as concrete as possible, we set $q = 5$, and we obtain (details omitted) that

$$\Theta(u) = 1 + u + 2u\sigma_2^{-1} + 4u^2 \cdot s(G)/(1-5u).$$

Applying the trivial character of $G = \{1, \sigma_2, \sigma_3, \sigma_4\} \cong (\mathbb{Z}/5)^\times$ to this, we get

$$1 + 3u + 16u^2/(1-5u),$$

which equals $(1-u)^2/(1-5u)$. So we have a double zero as predicted by the three ramified points, and the elements θ_S and $\tilde{\theta}_S$ agree, both having the value

$$2 + 2\sigma_2^{-1} - s(G).$$

On the other hand, the element η in this context is $2 + 2\sigma_2^{-1}$. Our results say that $\tilde{\theta}_S$ is an annihilator of $cl^0(K)$; since $s(G)$ is certainly an annihilator (k has genus 0), we see that η also has to annihilate $cl^0(K)$. Actually one can make our result 4.9 totally explicit, say for $\ell = 2$: The only eligible W's are empty or the singleton comprising just the infinite place of k. For the latter choice K_W/k is quadratic and of genus 0, so $\tilde{\theta}(K_W)_{S\setminus W}$ (which generates the Fitting ideal of $cl^0(K_W)$ since only two places ramify now) must be a unit. This produces

$$\mathrm{Fitt}_{\mathbb{Z}_2[G]}(cl^0(K)\{2\}) = (2 + 2\sigma_2^{-1} - s(G), 1 + \sigma_4).$$

The element $1 + \sigma_4 = s(\mathrm{Gal}(K/K_W))$ arises from corestriction. Since $s(G)$ is a multiple of it, we can simplify the expression to $(2 + 2\sigma_2^{-1}, 1 + \sigma_4)$.

Finally it is clear that σ_4 acts as inversion on $cl^0(K)$ (since K_W has genus 0). Thus we may think of the class group as a module over $\mathbb{Z}_2[G]/(1 + \sigma_4)$, which is isomorphic to $\mathbb{Z}_2[i]$. The Fitting ideal of $cl^0(K)\{2\}$ over that ring then is just the principal ideal $(2(1+i))$. From this one can deduce that $cl^0(K)\{2\}$ is cyclic over $\mathbb{Z}_2[i]$ and isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/2$ as an abelian group. This can be double-checked: $K = k\big(\sqrt[4]{x(x-1)}\big)$, so one just has to count points on the elliptic curve $w^4 = x(x-1)$ over $\mathbb{F}_5$; a little care is needed at infinity, but we do get exactly 8 points.

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