On the Coates-Sinnott Conjecture

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In [5], Coates and Sinnott formulated a far reaching conjecture linking the values $\Theta_{F/k,S}(1-n)$ for even integers $n \geq 2$ of an S–imprimitive, Galois-equivariant L–function $\Theta_{F/k,S}$ associated to an abelian extension F/k of totally real number fields to the annihilators over the group ring $\mathbb{Z}[G(F/k)]$ of the even Quillen K–groups $K_{2n-2}(O_F)$ associated to the ring of integers O_F of the top field F. In the same paper, Coates and Sinnott essentially prove the ℓ-adic étale cohomological version of their conjecture, in which $K_{2n-2}(O_F)$ is replaced by $H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$, for all primes $\ell > 2$, under the hypothesis that $k = \mathbb{Q}$. Refinements of this result for $k = \mathbb{Q}$, involving Fitting ideals rather than annihilators of $H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$, were obtained in particular cases by Cornacchia-Østvaer [7] and in general by Kurihara [14]. More recently, Burns and Greither [3] proved the same type of refinements (involving Fitting ideals of étale cohomology groups) for arbitrary totally real base fields k, but working under the very strong hypothesis that the Iwasawa μ–invariants $\mu_{F,\ell}$ vanish for all odd primes ℓ. In this paper, we study a class of abelian extensions of an arbitrary totally real base field k including, for example, subextensions of real cyclotomic extensions of type $k(\zeta_{p^m})^+/k$, where p is an odd prime. For this class of extensions, we prove similar refinements of the étale cohomological version of the Coates-Sinnott conjecture, under no vanishing hypotheses for the Iwasawa μ-invariants in question. Our methods of proof are different from the ones employed in [3], [14] and [7]. We build upon ideas developed by Greither in [10] and Wiles in [23] and [22], in the context of Brumer's Conjecture. If the Quillen-Lichtenbaum Conjecture is proved (and a proof seems to be within reach), then we have canonical $\mathbb{Z}_\ell[G(F/k)]$–module isomorphisms $K_{2n-i}(O_F) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^i_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$, for all $n \geq 2$, all $i = 1, 2$, and all primes $\ell > 2$, and all these results will yield proofs of the original K-theoretic version of the Coates-Sinnott Conjecture, in the cases and under the various hypotheses mentioned above.

1 Introduction and Notation

Let F/k be an abelian extension of number fields of Galois group $G := \mathrm{Gal}(F/k)$. The group of complex valued characters of G will be denoted by $\widehat{G}$. For every $\chi \in \widehat{G}$, we denote by $e_\chi := |G|^{-1} \cdot \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}$ the corresponding idempotent element in the group ring $\mathbb{C}[G]$. Let S be a finite set of primes (places) in k, containing the set S_∞ of all the infinite primes and the set $S_{\text{ram}} = S_{\text{ram}}(F/k)$ of those primes which ramify in F/k. To the set of data $(F/k, S)$, one associates the S-imprimitive G-equivariant L-function

$$\Theta_{F/k,S} : \mathbb{C} \longrightarrow \mathbb{C}[G]\,, \qquad \Theta_{F/k,S}(s) := \sum_{\chi \in \widehat{G}} L_S(\chi, s) \cdot e_\chi^{-1}\,,$$

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where $L_S(\chi, s)$ is the Artin L–function associated to χ with Euler factors at primes in S removed. The G–equivariant L–function above is holomorphic outside $s = 1$ and has a simple pole at $s = 1$ (imposed by the simple pole of the S–imprimitive Dedekind zeta function $\zeta_{k,S}(s) = L_S(\mathbf{1}_G, s)$, where $\mathbf{1}_G$ denotes the trivial character of G.) In particular, the values $\Theta_{F/k,S}(1-n)$ make sense in the group ring $\mathbb{C}[G]$, for all $n \in \mathbb{Z}_{\geq 1}$. A classical result of Klingen and Siegel [19] shows that

$$\Theta_{F/k,S}(1-n) \in \mathbb{Q}[G]\,, \text{ for all } n \in \mathbb{Z}_{\geq 1}\,.$$

Additional, more refined information on the special values $\Theta_{F/k,S}(1-n)$, for $n \in \mathbb{Z}_{\geq 1}$, is provided by a deep theorem proved independently and with different methods by Deligne-Ribet [8], Pi. Cassou-Noguès [4] and Barsky [2]. Next, we will restate this theorem in terms of certain étale cohomology groups associated to the ring of integers O_F in the top field F. Later, we will remind the reader of the links between these étale cohomology groups and the Quillen K–theory groups associated to O_F, established by Soulé in [20]. For the basic definitions and properties of the étale cohomology groups and Quillen K–theory groups associated to a ring of algebraic integers, the reader is strongly advised to read Kolster's excellent survey article [13].

Let ℓ be a prime number and $n \in \mathbb{Z}_{\geq 1}$. Then $\mathbb{Z}_\ell(n)$ denotes the usual (Tate twisted) ℓ–adic sheaf on the étale site associated to the scheme $\mathrm{Spec}(O_F[1/\ell])$, obtained by inverting all the primes above ℓ in O_F. As usual, $H^i_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$ denotes the i-th étale cohomology group associated to this sheaf, for all $i \geq 0$. Under the current hypotheses, these cohomology groups come endowed with canonical $\mathbb{Z}_\ell[G]$–module structures. Results of Quillen and Soulé [20] imply that these cohomology groups are all finitely generated $\mathbb{Z}_\ell$–modules. In particular, their torsion subgroups are all finite. If $i \geq 3$, $H^i_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$ vanish if $\ell > 2$ and are finite, 2-primary groups, if $\ell = 2$ (see [13], §2.) Also, work of Quillen, Borel and Soulé [20] implies that $H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$ is finite, for all $n \geq 2$.

Theorem 1.1 (Deligne-Ribet) *For all F/k and S as above, all $n \in \mathbb{Z}_{\geq 1}$ and all primes ℓ, we have*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{F/k,S}(1-n) \subseteq \mathbb{Z}_\ell[G]\,.$$

In order to help the reader get a feel for the deep theorem stated above and gain some motivation for the conjectures which will ensue, we will have a brief closer look at the case $n = 1$. In that case, we have a canonical $\mathbb{Z}_\ell[G]$–module isomorphism (see [13], §2)

$$H^1_{et}(O_F[1/\ell], \mathbb{Z}_\ell(1))_{\mathrm{tors}} \xrightarrow{\sim} \mu(F) \otimes \mathbb{Z}_\ell\,,$$

where $\mu(F)$ is the group of roots of unity in F. Consequently, it is well known that the annihilator showing up in the theorem above is the ideal of $\mathbb{Z}_\ell[G]$ generated by the elements $(1 - \sigma_v^{-1} \cdot \mathbf{N}v)$, where v runs through all the primes of k which are not in S, $\sigma_v \in G$ is the Frobenius morphism associated to v and $\mathbf{N}v$ is the cardinality of the residue field of v. Therefore, if $k = \mathbb{Q}$ and $F = \mathbb{Q}(\zeta_m)$, where $\zeta_m = e^{2\pi i/m}$, for some $m > 1$, the left hand side of the inclusion in the theorem above is the (ℓ–primary part of the) classical Stickelberger ideal. This is why we set

$$\mathrm{Stick}_{F/k,S,\ell}(1-n) := \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{F/k,S}(1-n)$$

and call these the higher (ℓ-adic) Stickelberger ideals associated to the data $(F/k, S, n, \ell)$. The analogy with the classical theorem of Stickelberger can be pushed further by observing that there is a canonical $\mathbb{Z}_\ell[G]$–module isomorphism (see [13], §2)

$$H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(1))_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{Cl}(O_F[1/\ell]) \otimes \mathbb{Z}_\ell\,,$$

where $\mathrm{Cl}(O_F[1/\ell])$ is the ideal–class group of the Dedekind domain $O_F[1/\ell]$. Consequently, (a slightly weaker version of) the classical conjecture of Brumer (the natural generalization of Stickelberger's Theorem, see [23]) can be restated in terms of ℓ-adic étale cohomology groups as follows.

Conjecture 1.2 (Brumer) *For all F/k, S and ℓ as above, we have*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{et}(O_F[1/\ell], \mathbb{Z}_\ell(1))_{\mathrm{tors}}) \cdot \Theta_{F/k,S}(0) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(1))_{\mathrm{tors}})\,.$$

This cohomological restatement of Brumer's Conjecture (and Stickelberger's Theorem, in particular) led Coates and Sinnott in [5] to formulate (and essentially prove, if $k = \mathbb{Q}$) the following natural generalization.

Conjecture 1.3 (Coates–Sinnott, cohomological version) *For all F/k, S and ℓ as above, and all $n \in \mathbb{Z}_{\geq 2}$, we have*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{F/k,S}(1-n) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n)))\,.$$

At this point, the strongest piece of evidence in support of this conjecture is the following (see [3].)

Theorem 1.4 (Burns–Greither) *The conjecture above is true, assuming that k is totally real, F is either totally real or CM, $\ell > 2$, S contains all the primes above ℓ in F and (most importantly) the Iwasawa μ–invariant $\mu_{F,\ell}$ vanishes.*

In fact, in [3], the authors compute the actual Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}_\ell[G]}$ of a certain direct $\mathbb{Z}_\ell[G]$–summand of $H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n))$ in terms of special values of G–equivariant L–functions. This Fitting ideal is included in and is a finer invariant than the annihilator $\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{et}(O_F[1/\ell], \mathbb{Z}_\ell(n)))$ in the statement of the conjecture. Also, it is worth noting that since the μ–invariants $\mu_{F,\ell}$ are known to vanish in the case where $F/\mathbb{Q}$ is abelian (a well-known theorem of Ferrero-Washington), the results in [3] capture as particular cases the previous theorems of Cornacchia-Østvaer [7] on extensions of type $\mathbb{Q}(\zeta_{p^m})^+/\mathbb{Q}$, with p prime, and Kurihara [14] on arbitrary abelian extensions of $\mathbb{Q}$. The methods used in [3] are based on a systematic use of the equivariant Iwasawa theory of complexes, initiated by Kato [12] and extended by Nekovàř [16].

The main goal of this paper is to eliminate the extra hypotheses in the theorem above (most importantly, eliminate the very strong hypothesis $\mu_{F,\ell} = 0$, for all primes $\ell > 2$), for a special class of extensions of arbitrary totally real number fields. One of our main results is the following (see §5, Proposition 5.1, Theorem 5.2 and Theorem 5.4 for more complete versions.)

Theorem 1.5 *Let F/k be an abelian extension of totally real number fields of Galois group G. Let $p > 2$ be a prime number and assume that $p \nmid [F : F \cap k_\infty]$, where k_∞ is the cyclotomic $\mathbb{Z}_p$–extension of k. Then, for any finite set S of primes in k containing all the infinite primes and those which ramify in F/k and for all even $n \in \mathbb{Z}_{\geq 2}$, we have the following.*

$$\text{i.}\qquad \begin{aligned} \imath(\mathrm{Stick}_{F/k,S,p}(1-n)) &= \mathrm{Fitt}_{\mathbb{Z}_p[G]}(H^2_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n))) \\ &\subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]}(H^2_{et}(O_F[1/p], \mathbb{Z}_p(n)))\,, \end{aligned}$$

where $\imath$ is the involution of the group ring $\mathbb{Z}_p[G]$, given by $\imath(\sigma) = \sigma^{-1}$, for all $\sigma \in G$.

$$\text{ii.}\qquad \begin{aligned} \mathrm{Stick}_{F/k,S,p}(1-n) &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(H^2_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n))) \\ &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(H^2_{et}(O_F[1/p], \mathbb{Z}_p(n)))\,. \end{aligned}$$

The methods we use for the proof of the theorem above are equivariant Iwasawa theoretic and build upon ideas developed in [23] and [10] in the context of Brumer's Conjecture. They are quite different from the methods of [3] or [14]. In particular, we take full advantage of Wiles's proof of the Main Conjecture (away from the prime 2) in Iwasawa Theory [22], including his proof of the equality between the analytic and algebraic μ–invariants associated to the relevant p–adic L–functions and Iwasawa modules, respectively.

Remark 1.6 Let k be a totally real number field. Let χ be any continuous (1-dimensional) character of its absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ and let $k_\chi := \bar{k}^{\ker\chi}$ (the maximal subfield of the algebraic closure $\bar{k}$ of k fixed under the action of $\ker\chi$.) For obvious reasons, we will call χ even if k_χ is totally real and odd if k_χ is a CM field. Then, as an immediate consequence of the functional equation satisfied by the primitive Artin L–function $L(\chi, s)$ associated to χ, one can prove that, for all $n \in \mathbb{Z}_{\geq 2}$ and all finite sets S of primes in k containing $S_\infty \cup S_{\text{ram}}(k_\chi/k)$, we have $L_S(\chi, 1-n) \neq 0$ if and only if χ and n have the same parity. This shows that for fields F as in the theorem above (i.e. totally real), $\Theta_{F/k,S}(1-n) = 0$, for all odd $n \in \mathbb{Z}_{\geq 2}$. Therefore, the above theorem with the equality in (i) replaced by inclusion is trivially true for n odd. This explains why the theorem above settles a refined version of the p–primary part of the Coates–Sinnott conjecture (cohomological version) for the class of extensions described in its hypotheses.

We conclude the introductory section with a brief discussion of the K–theoretic version of the Coates–Sinnott conjecture and its relation to the cohomological version. For a number field F, $K_n(O_F)$ will denote the n–th Quillen K–group associated to the ring of integers O_F of F, for all $n \in \mathbb{Z}_{\geq 0}$. By definition, $K_0(O_F)$ is the usual Grothendieck group classifying isomorphism classes of finitely generated projective O_F–modules and it is canonically isomorphic to $\mathrm{Cl}(O_F) \oplus \mathbb{Z}$. For $n \geq 1$, $K_n(O_F)$ is the n–th homotopy group $\pi_n(BGL(O_F)^+)$ of Quillen's "enlarged classifying space" $BGL(O_F)^+$ associated to $GL(O_F) = \cup_m GL_m(O_F)$. It is well known that for $n = 1, 2$, Quillen's $K_n(O_F)$ are canonically isomorphic to Bass's $K_1(O_F)$ and Milnor's $K_2(O_F)$, respectively. In particular, $K_1(O_F)$ is canonically isomorphic to the group of global units $O_F^\times$ of O_F and $K_2(O_F)$ is the kernel of the so–called tame symbol on Milnor's $K_2(F)$. A deep theorem of Quillen shows that $K_n(O_F)$ is a finitely generated (abelian) group, for all $n \geq 0$. An equally deep theorem of Borel shows that

$$K_{2n-2}(O_F) \text{ is finite, for all } n \in \mathbb{Z}_{\geq 2},$$

$$\mathrm{rk}_{\mathbb{Z}} K_{2n-1}(O_F) = d_n := \mathrm{ord}_{s=(1-n)}\zeta_F(s) = \begin{cases} r_1 + r_2 - 1, & n = 1; \\ r_1 + r_2, & n > 1 \text{ odd}; \\ r_2, & n \geq 2 \text{ even}, \end{cases}$$

where r_1 and r_2 denote the number of real and half the number of complex embedding of F, respectively. Of course, the right-most equality above is a direct consequence of the functional equation satisfied by $\zeta_F(s)$. Borel goes further and defines $\mathbb{R}$–linear regulator maps

$$\rho_n : K_{2n-1}(O_F) \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{d_n},$$

for all $n \geq 1$, and defines regulators $R_n^B := \mathrm{Vol}(\mathbb{R}^{d_n}/\rho_n(K_{2n-1}(O_F)))$, generalizing Dirichlet's regulator (which can be recovered by setting $n = 1$.) Moreover, Borel goes a long way towards generalizing Dirichlet's classical class number formula, by showing that, for all $n \in \mathbb{Z}_{\geq 1}$, we have

$$\zeta_F^*(1-n) = q_n \cdot R_n^B, \text{ for some } q_n \in \mathbb{Q}^\times ,$$

where $\zeta_F^*(1-n)$ denotes the leading term in the Taylor expansion at $s = (1-n)$ of the Dedekind zeta function $\zeta_F(s)$. The conjectural generalization of the classical class–number formula was later formulated by Lichtenbaum as follows.

Conjecture 1.7 (Lichtenbaum) *For all F as above and all $n \in \mathbb{Z}_{\geq 2}$, we have*

$$\zeta_F^*(1-n) \sim \pm \frac{|K_{2n-2}(O_F)|}{|K_{2n-1}(O_F)_{\mathrm{tors}}|} \cdot R_n^B \,,$$

where $\sim$ means "equal up to an integral power of 2" and the sign $\pm$ is easily determined from the functional equation of $\zeta_F(s)$.

The link between Quillen's K–groups and the ℓ–adic étale cohomology groups of O_F introduced earlier was made by Soulé in [20]. Building upon earlier work of Grothendieck on Chern classes, Soulé constructed canonical ℓ–adic étale Chern character morphisms

$$\mathrm{ch}_{n,i}^{(\ell)} : K_{2n-i}(O_F) \otimes \mathbb{Z}_\ell \longrightarrow H_{et}^i(O_F[1/\ell], \mathbb{Z}_\ell(n)) \,, \text{ for all } \ell > 2 \text{ prime, } i = 1, 2 \text{ and } n \in \mathbb{Z}_{\geq 2} \,,$$

and, most importantly, showed that they are surjective for all i and n as above and all primes $\ell > 2$. Later, Dwyer and Friedlander constructed similar morphisms at the prime $\ell = 2$, which are now known to be neither surjective nor injective, in general. If F/k is a Galois extension of number fields, of Galois group G, then the étale Chern maps $\mathrm{ch}_{n,i}^{(\ell)}$ are $\mathbb{Z}_\ell[G]$–linear (functoriality !) One of the major conjectures in contemporary mathematics is the following.

Conjecture 1.8 (Quillen–Lichtenbaum) *The ℓ–adic étale Chern maps $\mathrm{ch}_{n,i}^{(\ell)}$ are isomorphisms for all i, n and $\ell > 2$ as above.*

The first significant piece of evidence in favor of this conjecture was provided by Tate [21], who established the required isomorphisms for $2n - i = 2$. It is known, due to work of Suslin, that this conjecture is a consequence of a conjecture of Bloch-Kato, linking the Milnor K–theory of the field F to étale cohomology. Recent work of Voievodsky, Rost, Weibel and others seems to have led to a proof of the Bloch-Kato Conjecture. This result would settle the Quillen-Lichtenbaum Conjecture. If the Quillen–Lichtenbaum Conjecture is indeed proved, then the following K–theoretic version of the Coates-Sinnott Conjecture, formulated in [5], is equivalent to the previously discussed cohomological version.

Conjecture 1.9 (Coates–Sinnott, K-theoretic version) *For all F/k, S and $\ell > 2$ as above, and all $n \in \mathbb{Z}_{\geq 2}$, we have*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(K_{2n-1}(O_F)_{\mathrm{tors}} \otimes \mathbb{Z}_\ell) \cdot \Theta_{F/k,S}(1-n) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(K_{2n-2}(O_F) \otimes \mathbb{Z}_\ell) \,.$$

Obviously, we can play the same "K–theory versus étale cohomology" game with Lichtenbaum's Conjecture stated above. In particular, note that Borel's results show that if F is a totally real number field and $n \in \mathbb{Z}_{\geq 2}$ is even, then Borel's regulator R_n^B is trivial (equal to 1) and the groups $K_{2n-i}(O_F)$ are finite, for all $i = 1, 2$. Also, $\zeta_F^*(1-n) = \zeta_F(1-n)$ in this case. This observation, combined with Soulé's result on the surjectivity of the étale Chern maps shows that, under these hypotheses, the groups $H_{et}^i(O_F[1/\ell], \mathbb{Z}_\ell(n))$ are finite, for all $i = 1, 2$ and primes $\ell > 2$. Therefore, in this case, the cohomological version of Lichtenbaum's Conjecture can be stated as follows.

Conjecture 1.10 (Lichtenbaum, cohomological version, particular case) *For F totally real, $n \in \mathbb{Z}_{\geq 2}$ even, and $\ell > 2$ prime, we have*

$$\zeta_F(1-n) \sim_\ell \pm \frac{|H_{et}^2(O_F[1/\ell], \mathbb{Z}_\ell(n))|}{|H_{et}^1(O_F[1/\ell], \mathbb{Z}_\ell(n))|} \,,$$

where $\sim_\ell$ means "equal up to an ℓ–adic unit" and the sign $\pm$ is as before.

As a consequence of his proof of the Main Conjecture (away from 2) in Iwasawa Theory over arbitrary totally real number fields [22], Wiles settled this last conjecture (see Theorem 1.6 in [22].)

Theorem 1.11 (Wiles) *Conjecture 1.10 holds true.*

Finally, we conclude by noting that, in the absence of a published proof for the Quillen-Lichtenbaum Conjecture, the only evidence in favor of the K–theoretic version of the Coates–Sinnott Conjecture for $2n-2>2$ and F/k non-trivial is due to Banaszak (see [1].) In [1], the author proves the desired annihilation for the subgroup of $K_{2n-2}(O_F)$ consisting of its so-called divisible elements, for all $n\in\mathbb{Z}_{\geq 2}$ even, all primes $\ell>2$, and for $F/\mathbb{Q}$ abelian, under some mild extra-hypotheses.

2 Group rings, projective dimension and Fitting ideals

In what follows, we fix an odd prime number p. If O is a finite unramified extension of $\mathbb{Z}_p$ of maximal ideal m_O, then $\Lambda_O := O[[T]]$ denotes the usual one variable (local) Iwasawa O–algebra, whose maximal ideal m_{Λ_O} is generated by m_O and T. For simplicity, we let $\Lambda := \Lambda_{\mathbb{Z}_p}$. In this section, we discuss several properties (mostly of homological nature) of finitely generated modules over rings of type $O[G]$ and $\Lambda_O[G]$, where G is a fixed, finite, abelian group. Some of these results have appeared in print before (although perhaps not in the precise form in which they are presented here), see [9], [10], [3], and some are improvements upon already existing results.

If M is a finitely generated module over a Noetherian ring R, then $\mathrm{Fitt}_R(M)$ denotes the first Fitting ideal of M over R. For the definition and properties of Fitting ideals the reader can consult the Appendix of [15]. As usual, $\mathrm{pd}_R(M)$ and $\mathrm{cod}_R(M)$ denote the projective dimension and codimension (i.e. the length of the maximal M–regular sequence in R) of M, respectively (see [17].)

If G is a p–group, then the rings $O[G]$ and $\Lambda_O[G]$ are local, Noetherian, reduced rings, of maximal ideals generated by m_O and I_G and m_{Λ_O} and I_G, respectively, where I_G is the augmentation ideal associated to G (generated by $\{g-1\mid g\in G\}$.) If G is not a p–group, then these group rings are finite direct sums of local, Noetherian, reduced rings, and these direct sum decompositions go as follows. We let $G=P\times\Delta$, where P is the p–Sylow subgroup of G. If $Q(O)$ denotes the ring of fractions of O ($Q(O)$ is a finite, unramified extension of $\mathbb{Q}_p$.), then $\widehat{\Delta}(Q(O))$ denotes the set of irreducible $Q(O)$–valued characters of Δ, which is in obvious one-to-one correspondence with the orbits of $\widehat{\Delta}(\overline{\mathbb{Q}}_p)$ under the usual $\mathrm{Gal}(\overline{\mathbb{Q}}_p/Q(O))$-action. Throughout, by $\psi\in\widehat{\Delta}(Q(O))$ we also mean that ψ is a chosen representative in the corresponding orbit of $\widehat{\Delta}(\overline{\mathbb{Q}}_p)$. All characters ψ as above induce obvious surjective morphisms of $O[P]$-algebras and $\Lambda_O[P]$–algebras $\psi: O[G]\twoheadrightarrow O(\psi)[P]$ and $\psi:\Lambda_O[G]\twoheadrightarrow\Lambda_{O(\psi)}[P]$, respectively. These lead to the usual isomorphisms of semi-local rings

$$O[G]\xrightarrow{\sim}\bigoplus_{\psi\in\widehat{\Delta}(Q(O))}O(\psi)[P]\,,\quad \Lambda_O[G]\xrightarrow{\sim}\bigoplus_{\psi\in\widehat{\Delta}(Q(O))}\Lambda_{O(\psi)}[P]\,,$$

where $O(\psi)$ is the O-algebra generated by the values of ψ (an unramified finite extension of $\mathbb{Z}_p$ in its own right.) As a consequence, every module M over $O[G]$ or $\Lambda_O[G]$ splits canonically into a direct sum $M=\oplus_\psi M^\psi$, for all ψ as above, where $M^\psi := M\otimes_{O[G]}O(\psi)[P]$ has an obvious structure of module over $O(\psi)[P]$ or $\Lambda_{O(\psi)}[P]$, respectively.

Lemma 2.1 *For O and G as above and a finite $O[G]$–module M with $\mathrm{pd}_{O[G]}(M)\leq 1$, we have:*

i. *The ideal $\mathrm{Fitt}_{O[G]}(M)$ is principal, generated by a non-zero divisor in $O[G]$.*

ii. $|M| = [O[G] : \mathrm{Fitt}_{O[G]}(M)] = |O/pO|^{\mathrm{ord}_p \prod_\chi \chi(f)}$, *where f is any generator of* $\mathrm{Fitt}_{O[G]}(M)$ *and the product is taken with respect to all* $\chi \in \widehat{G}(\overline{\mathbb{Q}}_p)$.

P r o o f. First, let us remark that due to the (character component) direct sum decompositions discussed above, it suffices to prove (i) and (ii) above under the hypothesis that G is a p–group. If that is the case, then the ring $O[G]$ is local. Therefore, any finitely generated projective $O[G]$–module is free of finite rank. This fact, combined with the finiteness of M (implying $M \otimes_O Q(O) = 0$), shows that, since $\mathrm{pd}_{O[G]}(M) \leq 1$, we have an exact sequence of $O[G]$–modules

$$0 \longrightarrow O[G]^m \xrightarrow{F} O[G]^m \longrightarrow M \longrightarrow 0\,.$$

By definition, $\mathrm{Fitt}_{O[G]}(M)$ is principal, generated by $f_{O[G]} := \det_{O[G]}(F)$. This is a non-zero divisor, because $F \otimes_O \mathbf{1}_{Q(O)}$ is an isomorphism of free $Q(O)[G]$–modules of rank m and therefore its determinant (which is equal to $f_{O[G]}$) is a non-zero divisor in $Q(O)[G]$. This settles (i) above.

In order to prove (ii), one makes the elementary observation that $|M| = [O : (\det_O(F))]$, where $\det_O(F)$ is the determinant of F viewed as a morphism of free O–modules of rank $m \cdot |G|$. Now, the canonical decomposition $\overline{\mathbb{Q}}_p[G] \xrightarrow{\sim} \oplus_\chi \overline{\mathbb{Q}}_p$, with $\chi \in \widehat{G}(\overline{\mathbb{Q}}_p)$, implies right away that

$$\det_O(F) = \det_{\overline{\mathbb{Q}}_p}(F \otimes_O \mathbf{1}_{\overline{\mathbb{Q}}_p}) = \prod_\chi \chi(\det_{\overline{\mathbb{Q}}_p[G]}(F \otimes_O \mathbf{1}_{\overline{\mathbb{Q}}_p})) = \prod_\chi \chi(f_{O[G]})\,.$$

Finally, one makes the equally elementary observation that for all non-zero divisors $f \in O[G]$, if we let $F_f : O[G] \longrightarrow O[G]$ denote the multiplication by f map, we have $\det_{O[G]}(F_f) = f$ and therefore

$$[O[G] : (f)] = [O : (\det_O(F_f))] = [O : (\prod_\chi \chi(f))]\,,$$

with χ running through $\widehat{G}(\overline{\mathbb{Q}}_p)$, as above. This settles (ii). □

Recall that a classical theorem of Nakayama states that a finitely generated $O[G]$–module M satisfies the condition $\mathrm{pd}_{O[G]}(M) \leq 1$ (respectively if $\mathrm{pd}_{O[G]}(M) = 0$) if and only if M is G–cohomologically trivial, i.e. the Tate cohomology groups $\widehat{H}^i(H, M)$ are trivial, for all $i \in \mathbb{Z}$ and all subgroups H of G (respectively if M is G-coh. trivial and O-free.) (See [18], Chapter IX, where this theorem is stated and proved for $\mathbb{Z}[G]$–modules. Obviously, the same proof goes through for $O[G]$–modules, with O as above.) In what follows, we state and prove a similar criterion for $\Lambda_O[G]$–modules M with $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$, extending a result of Greither (see [10] and [9].)

Proposition 2.2 *A finitely generated $\Lambda_O[G]$–module M satisfies* $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$ *if and only if the following conditions are simultaneously met.*

i. $\mathrm{pd}_{\Lambda_O}(M) \leq 1$.

ii. *M is G–cohomologically trivial.*

P r o o f. As in the proof of the previous Lemma, it suffices to prove the Proposition under the assumption that G is a p–group. (Note that if $G = P \times \Delta$, where P is the p–Sylow subgroup of G, and M is a $\mathbb{Z}_p$–module, then M is automatically Δ–coh. trivial. Therefore, by a standard inflation-restriction argument, M is G–coh. trivial if and only if it is P–coh. trivial.) So, throughout the proof we assume that G is a p–group. Consequently, $\Lambda_O[G]$ and $O[G]$ are local rings.

First, assume that $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$. Therefore, we have an exact sequence of $\Lambda_O[G]$–modules

$$0 \longrightarrow \Lambda_O[G]^m \longrightarrow \Lambda_O[G]^n \longrightarrow M \longrightarrow 0\,.$$

Since $\Lambda_O[G]$ is a free Λ_O–module, this implies that $\mathrm{pd}_{\Lambda_O}(M) \leq 1$. Also, since we have $\Lambda_O[G] \xrightarrow{\sim} \Lambda_O \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, the $\mathbb{Z}[G]$–module $\Lambda_O[G]$ is induced, therefore G–coh. trivial (see [18].) Consequently, the long exact sequence of H–cohomology associated to the short exact sequence above shows that the H–coh. groups of M vanish, for all subgroups H of G. Therefore, M is G–coh. trivial.

Second, let us assume that (i) and (ii) hold. We will prove that $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$. For the arguments which follow, compare with [10]. Conditions (i) and (ii) imply that M has a presentation

$$0 \longrightarrow A \longrightarrow \Lambda_O[G]^n \longrightarrow M \longrightarrow 0\,,$$

for some $\Lambda_O[G]$–module A which is G–coh. trivial (because M and $\Lambda_O[G]^n$ are G-coh. trivial) and Λ_O–projective (therefore Λ_O–free). We will prove that A is $\Lambda_O[G]$-projective. For that, it suffices to consider a presentation

$$0 \longrightarrow B \longrightarrow \Lambda_O[G]^n \longrightarrow A \longrightarrow 0\,, \tag{2.1}$$

and show that the exact sequence above splits in the category of $\Lambda_O[G]$–modules. Obviously, B and A are Λ_O-free, therefore the sequence above is Λ_O-split. Consequently, it induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_O}(A, B) \longrightarrow \mathrm{Hom}_{\Lambda_O}(A, \Lambda_O[G]^n) \longrightarrow \mathrm{Hom}_{\Lambda_O}(A, A) \longrightarrow 0\,.$$

Consequently, since $\mathrm{Hom}_{\Lambda_O[G]}(X, Y) = \mathrm{Hom}_{\Lambda_O}(X, Y)^G$, for all $\Lambda_O[G]$–modules X and Y, the sequence (2.1) is $\Lambda_O[G]$-split if $H^1(G, \mathrm{Hom}_{\Lambda_O}(A, B)) = 0$. However, since A and B are Λ_O-free, we have two obvious exact sequences of $\Lambda_O[G]$–modules

$$0 \longrightarrow B \xrightarrow{T} B \longrightarrow B/TB \longrightarrow 0\,,$$

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_O}(A, B) \xrightarrow{T} \mathrm{Hom}_{\Lambda_O}(A, B) \longrightarrow \mathrm{Hom}_{\Lambda_O}(A, B/TB) \longrightarrow 0\,.$$

Now, A/TA is G–coh. trivial (because A is) and O–free (because A is Λ_O–free). Consequently, A/TA is $O[G]$–free (Nakayama's criterion). Therefore, $\mathrm{Hom}_{\Lambda_O}(A, B/TB) = \mathrm{Hom}_O(A/TA, B/TB)$ is an induced, therefore coh. trivial G–module (see [18], Chpt. IX, §3). Consequently, the last exact sequence implies that multiplication by T induces automorphisms on the finitely generated Λ_O–modules $\widehat{H}^i(G, \mathrm{Hom}_{\Lambda_O}(A, B))$, for $i \in \mathbb{Z}$. Since $T \in m_{\Lambda_O}$, the classical Nakayama Lemma implies that $\widehat{H}^i(G, \mathrm{Hom}_{\Lambda_O}(A, B)) = 0$, for all i. In particular, $H^1(G, \mathrm{Hom}_{\Lambda_O}(A, B)) = 0$. □

The reader will note right away that the criterion above is in perfect analogy with Nakayma's. The difference is that the first condition in the proposition above is automatically met if one replaces Λ_O by O (or any PID for that matter.) So, the natural question which arises is when the condition $\mathrm{pd}_{\Lambda_O}(M) \leq 1$ is met by a finitely generated Λ_O–module M. The answer is as follows.

Lemma 2.3 *Let M be a finitely generated Λ_O–module. Then* $\mathrm{pd}_{\Lambda_O}(M) \leq 1$ *if and only if M does not contain non–trivial finite Λ_O–submodules.*

Proof. Since Λ_O is a regular local ring of dimension 2, the Auslander-Buchsbaum Theorem (see [17], Chapter 9) implies that $\mathrm{pd}_{\Lambda_O}(M) + \mathrm{cod}_{\Lambda_O}(M) = \dim(\Lambda_O) = 2$.

First, assume that $\mathrm{pd}_{\Lambda_O}(M) \leq 1$. Consequently, $\mathrm{cod}_{\Lambda_O}(M) \geq 1$. This means that there exist $\lambda \in m_{\Lambda_O}$, such that the multiplication by λ map $M \xrightarrow{\lambda} M$ is injective. On the other hand, if M has a non-trivial finite submodule N, assumed without loss of generality to be cyclic, say isomorphic to Λ_O/I, for some non-trivial ideal I in Λ_O, then the multiplication by λ map $\Lambda_O/I \xrightarrow{\lambda} \Lambda_O/I$ is also injective. Since N is finite, this map is bijective. Therefore, there exists $\lambda' \in \Lambda_O$, such that $(\lambda\lambda' - 1) \in I \subseteq m_{\Lambda_O}$. Therefore $1 \in m_{\Lambda_O}$, which is false.

Now, assume that M has no non-trivial finite Λ_O–submodules. Then, the well known structure theorem for finitely generated Λ_O–modules implies that there is an injective Λ_O–module morphism $M \hookrightarrow \mathcal{S}$ with finite cokernel, where $\mathcal{S}$ is a so-called standard Iwasawa module of the type

$$\mathcal{S} = \Lambda_O^r \oplus (\oplus_i \Lambda_O/f_i)\,,$$

for a unique $r \in \mathbb{Z}_{\geq 0}$ and unique (finitely many) $f_i \in \Lambda_O$, which are either powers of (monic) irreducible Weierstrass polynomials in $O[T]$ or powers of a fixed uniformizer π_O of O. Now, recall that Λ_O is a UFD (any regular local ring is), whose irreducible elements are (up to units) the irreducible Weierstrass polynomials in $O[T]$ and the uniformizer π_O. This implies that we can always choose $\lambda \in m_{\Lambda_O}$, coprime to the Iwasawa characteristic polynomial $\mathrm{char}_{\Lambda_O}(M) := \prod_i f_i$ and that for any such λ the multiplication by λ map $\mathcal{S} \xrightarrow{\lambda} \mathcal{S}$ is injective. This means that multiplication by λ restricted to M is injective. This shows that $\mathrm{cod}_{\Lambda_O}(M) \geq 1$. The Auslander-Buchsbaum theorem comes to the rescue again to imply that $\mathrm{pd}_{\Lambda_O}(M) \leq 1$. □

Lemma 2.4 *Assume that M is a finitely generated, torsion Λ_O–module, such that* $\mathrm{pd}_{\Lambda_O}(M) \leq 1$. *Then the ideal* $\mathrm{Fitt}_{\Lambda_O}(M)$ *is principal generated by the characteristic polynomial* $\mathrm{char}_{\Lambda_O}(M) \in O[T]$.

Proof. Since Λ_O is a local ring, the argument at the beginning of the proof of Lemma 2.1 shows that $\mathrm{Fitt}_{\Lambda_O}(M)$ is principal, generated by a non-zero divisor, say f, in Λ_O. As in the proof of Lemma 2.3, we embed M in a (unique up to an isomorphism) torsion standard Iwasawa module $\mathcal{S}$,

$$0 \longrightarrow M \longrightarrow \mathcal{S} \longrightarrow F \longrightarrow 0\,.$$

where F is a finite Λ_O–module. From the definition of Fitting ideals, $\mathrm{Fitt}_{\Lambda_O}(\mathcal{S}) = (\mathrm{char}_{\Lambda_O}(M))$. We remind the reader that in the regular, local ring Λ_O of Krull dimension 2, the maximal ideal $m_{\Lambda_O} = (\pi_O, T)$ is the only prime of height 2, whereas the primes of height one are all principal, generated by the irreducible elements of Λ_O (i.e. the irreducible Weierstrass polynomials in $O[T]$ and π_O, up to units.) Also, if F is a finite Λ_O–module, then the localizations $F_{\mathfrak{p}}$ vanish, for all primes $\mathfrak{p} \subset \Lambda_O$ of height 1. This means that the exact sequence above gives rise to $\Lambda_{O,\mathfrak{p}}$–module isomorphisms $M_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{S}_{\mathfrak{p}}$ by localization, for all primes $\mathfrak{p}$ of height 1. Since Fitting ideals commute with localizations, this implies that $(f)_{\mathfrak{p}} = (\mathrm{char}_{\Lambda_O}(M))_{\mathfrak{p}}$, for all primes $\mathfrak{p}$ of height 1. This implies that the factorizations of f and $\mathrm{char}_{\Lambda_O}(M)$ into products of irreducibles in the UFD Λ_O are (up to units and permutations of factors) the same. Consequently, $\mathrm{Fitt}_{\Lambda_O}(M) = (f) = (\mathrm{char}_{\Lambda_O}(M))$. □

Lemma 2.5 *Let M be a finitely generated $\Lambda_O[G]$–module, which is a torsion Λ_O–module. Assume that* $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$ *and M/fM is finite, for some $f \in \Lambda_O$. Then* $\mathrm{pd}_{\Lambda_O[G]/f}(M/fM) \leq 1$.

P r o o f. As before, it obviously suffices to prove the statement in the lemma under the assumption that G is a p–group. Then, $\Lambda_O[G]$ is a local, Noetherian ring. Since $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$ and M is Λ_O–torsion, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda_O[G]^m \arrow[r] \arrow[d, "f"] & \Lambda_O[G]^m \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Lambda_O[G]^m \arrow[r] & \Lambda_O[G]^m \arrow[r] & M \arrow[r] & 0
\end{tikzcd} ,$$

where $m \in \mathbb{Z}_{>0}$ and the vertical maps are multiplication by f. It is well known that for a finitely generated torsion Λ_O–module M and $f \in \Lambda_O$, M/fM is finite if and only if $\ker(M \xrightarrow{f} M)$ is finite (and this is equivalent to f being coprime to $\mathrm{char}_{\Lambda_O}(M)$.) Since M has no finite Λ_O–submodules (see Lemma 2.3 and Proposition 2.2), this kernel is trivial in our case. Consequently, the snake lemma applied to the commutative diagram above produces a short exact sequence of $\Lambda_O[G]/f$–modules at the level of the cokernels of the multiplication by f maps

$$0 \longrightarrow (\Lambda_O[G]/f)^m \longrightarrow (\Lambda_O[G]/f)^m \longrightarrow M/fM \longrightarrow 0 .$$

This shows that, indeed $\mathrm{pd}_{\Lambda_O[G]/f}(M/fM) \leq 1$.

An alternative proof in the case $f = T$ goes as follows. Let H be a subgroup of G. Since $\mathrm{pd}_{\Lambda_O[G]}(M) \leq 1$, we have $\widehat{H}^i(H, M) = 0$, for all $i \in \mathbb{Z}$ (Proposition 2.2.) The long exact sequence of Tate H–cohomology groups associated to the exact sequence of $O[G] = \Lambda_O[G]/T$–modules

$$0 \longrightarrow M \xrightarrow{T} M \longrightarrow M/TM \longrightarrow 0$$

(see argument above for exactness) produces $\widehat{H}^i(H, M/TM) = 0$, for all $i \in \mathbb{Z}$. Via Nakayama's criterion, this has the desired consequence. □

3 Tate twists, Iwasawa modules and p–adic étale cohomology

Let k be a number field and let $p > 2$ be a fixed prime number. We denote by $G_k := \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k, by μ_{p^m} the roots of unity of order dividing p^m in $\overline{k}$ and we let $\mu_{p^\infty} := \cup_{m \geq 0} \mu_{p^m}$. We have a canonical group isomorphism $\mathrm{Aut}(\mu_{p^\infty}) \xrightarrow{\sim} \mathbb{Z}_p^\times$. If composed with the usual G_k–action on μ_{p^∞}, this isomorphism leads to the so–called p–cyclotomic character of G_k

$$c_p : G_k \longrightarrow \mathrm{Aut}(\mu_{p^\infty}) \xrightarrow{\sim} \mathbb{Z}_p^\times ,$$

which is continuous (Krull topology on G_k and p–adic topology on $\mathbb{Z}_p^\times$) and canonically defined. Of course, c_p factors through $G(\mathcal{K}/k)$, for any Galois extension $\mathcal{K}/k$, such that $k(\mu_{p^\infty}) \subseteq \mathcal{K} \subseteq \overline{k}$. In particular, c_p factors through the continuous morphism $c_p : G(k(\mu_{p^\infty})/k) \longrightarrow \mathbb{Z}_p^\times$ (note and excuse the abusive notation.) As usual, the maximal subfield $k_\infty := k(\mu_{p^\infty})^{c_p^{-1}((\mathbb{Z}_p^\times)_{\mathrm{tors}})}$ of $k(\mu_{p^\infty})$ fixed under the action of $c_p^{-1}((\mathbb{Z}_p^\times)_{\mathrm{tors}})$ is called the cyclotomic $\mathbb{Z}_p$–extension of k. Obviously, we have a group isomorphism $G(k(\mu_{p^\infty})/k) \xrightarrow{\sim} G(k(\mu_p)/k) \times G(k_\infty/k)$, given by Galois restriction. By restriction, the cyclotomic character c_p induces (topological) isomorphisms κ_p between $\Gamma_k := G(k_\infty/k)$ and an open subgroup of finite index of $(1 + p\mathbb{Z}_p)$ and ω_p between $\Delta_k := G(k(\mu_p)/k)$ and a subgroup of $(\mathbb{Z}_p^\times)_{\mathrm{tors}} = \mu_{p-1}$. If composed with the appropriate scalar multiple of the p–adic logarithm map, κ_p induces a topological isomorphism between Γ_k and $\mathbb{Z}_p$ (additive structure.) The character ω_p is called the Teichmuller character associated to k and p.

In what follows, if $\mathcal{G}$ is a profinite group (usually an infinite Galois group), and O is a finite, unramified extension of $\mathbb{Z}_p$, we denote by $O[[\mathcal{G}]]$ the (compact) profinite group algebra with coefficients in O, associated to $\mathcal{G}$. In particular, after choosing a (topological) generator γ_k of Γ_k, we have the well-known continuous O–algebra isomorphism $O[[\Gamma_k]] \xrightarrow{\sim} O[[T]]$, sending γ_k to $(T+1)$. Also, when we say that M is an $O[[\mathcal{G}]]$–module, we mean this in the topological sense, and M will be endowed with either a discrete or compact topology. Let $\mathcal{G}$ be any closed subgroup of a (topological) quotient of G_k through which c_p factors (e.g. $\mathcal{G} = G(k(\mu_{p^\infty})/k)$, $\mathcal{G} = \Gamma_k$, $\mathcal{G} = \Delta_k$ etc.) Then, for all $n \in \mathbb{Z}$, we denote by $\mathbb{Z}_p(n)$ the (compact) $\mathbb{Z}_p[[\mathcal{G}]]$–module, which is free of rank 1 over $\mathbb{Z}_p$ (with the usual $\mathbb{Z}_p$–topology) and endowed with the $\mathcal{G}$–action given by c_p^n, i.e. $\sigma * x = c_p^n(\sigma) \cdot x$, for all $x \in \mathbb{Z}_p(n)$ and all $\sigma \in \mathcal{G}$. If M is an arbitrary (compact or discrete) $O[[\mathcal{G}]]$–module then, for all $n \in \mathbb{Z}$, we let $M(n) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$ denote the $O[[\mathcal{G}]]$–module endowed with the diagonal $\mathcal{G}$–action and the canonical O–module structure. Under the obvious O–module isomorphism $M(n) \xrightarrow{\sim} M$, the new action of $\mathcal{G}$ on M is obtained from the old one by twisting with the n–the power of c_p, i.e. $\sigma * x = c_p^n(\sigma)\sigma \cdot x$, for all $x \in M$ and $\sigma \in \mathcal{G}$. The module $M(n)$ is called the n-th Tate twist of M. Note the obvious $\mathbb{Z}_p[[\mathcal{G}]]$–module isomorphisms

$$\varprojlim_m \mu_{p^m} \xrightarrow{\sim} \mathbb{Z}_p(1)\,, \qquad \varinjlim_m \mu_{p^m} \xrightarrow{\sim} (\mathbb{Q}_p/\mathbb{Z}_p)(1)\,,$$

where the projective limit is taken with respect to the p–power maps and $\mathbb{Q}_p/\mathbb{Z}_p$ is viewed as a discrete $\mathbb{Z}_p[[\mathcal{G}]]$–module with the trivial $\mathcal{G}$–action.

Next, we explain a similar twisting procedure at the level of profinite group rings. Let O be as above and let $\mathcal{G}$ be any abelian closed subgroup of a (topological) quotient of G_k through which c_p factors. Then, we define the (continuous) O–algebra isomorphisms

$$t_n : O[[\mathcal{G}]] \longrightarrow O[[\mathcal{G}]]\,, \quad \imath : O[[\mathcal{G}]] \longrightarrow O[[\mathcal{G}]]$$

by letting $t_n(\sigma) = c_p(\sigma)^n \cdot \sigma$ and $\imath(\sigma) = \sigma^{-1}$, for all $\sigma \in \mathcal{G}$ and $n \in \mathbb{Z}$. At times, we will extend the above maps to (isomorphisms of) the total rings of fractions $Q(O[[\mathcal{G}]])$ of $O[[\mathcal{G}]]$. Obviously, $\imath$ makes sense even if c_p does not factor through $\mathcal{G}$. It is useful to note that $t_n \circ \imath = \imath \circ t_{-n}$, for all $n \in \mathbb{Z}$. The following lemma is very useful in computations with Tate twists of modules.

Lemma 3.1 *If O and $\mathcal{G}$ be as above, and M is a finitely presented $O[[\mathcal{G}]]$–module, then*

i. $\mathrm{Fitt}_{O[[\mathcal{G}]]}(M(-n)) = t_n(\mathrm{Fitt}_{O[[\mathcal{G}]]}(M))$ *and* $\mathrm{Ann}_{O[[\mathcal{G}]]}(M(-n)) = t_n(\mathrm{Ann}_{O[[\mathcal{G}]]}(M))$*, for all $n \in \mathbb{Z}$.*

ii. *If M is finite and $\widehat{M} := Hom(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is its Pontrjagin dual (endowed with the canonical $O[[\mathcal{G}]]$–action), then* $\mathrm{Ann}_{O[[\mathcal{G}]]}(\widehat{M}) = \imath(\mathrm{Ann}_{O[[\mathcal{G}]]}(M))$.

Proof. We prove the first equality in (i), leaving the rest to the reader. The proof is an immediate consequence of the observation that there is an $O[[\mathcal{G}]]$–module isomorphism $\phi_{-n} : O[[\mathcal{G}]] \longrightarrow O[[\mathcal{G}]](-n)$, given by $\phi_{-n}(a) = t_{-n}(a) \otimes 1$, for all $a \in O[[\mathcal{G}]]$. Consequently, any $O[[\mathcal{G}]]$–module presentation

$$O[[\mathcal{G}]]^r \xrightarrow{f} O[[\mathcal{G}]]^s \xrightarrow{\pi} M \longrightarrow 0$$

gives rise to an $O[[\mathcal{G}]]$–module presentation

$$O[[\mathcal{G}]]^r \xrightarrow{t_n(f)} O[[\mathcal{G}]]^s \xrightarrow{\pi'} M(-n) \longrightarrow 0\,,$$

where $\pi' := (\pi \otimes \mathbf{1}_{\mathbb{Z}_p(-n)}) \circ \phi^s_{-n}$ and the matrix of $t_n(f)$ is obtained by applying t_n to the entries of the matrix associated to f. Now, one applies the definition of the Fitting ideal and the fact that t_n is a ring morphism. $\square$

The modules M whose Tate twists will be considered in this paper arise from specific Iwasawa theoretic considerations in the following manner. Assume that F is the top field in an abelian extension F/k of number fields of Galois group G. Fix a prime $p > 2$ as above, and let $\widetilde{F} := F(\mu_p)$, $\widetilde{G} := G(\widetilde{F}/k)$, $\mathcal{G} := G(F_\infty/k)$, $\widetilde{\mathcal{G}} := G(\widetilde{F}_\infty/k)$. We denote by S_p the set of primes in k sitting above p. As usual, for a finite set S of primes in k containing S_∞, we let $\mathcal{X}_S := \mathrm{Gal}(M_\infty(S)/F_\infty)$, where $M_\infty(S)$ is the maximal abelian pro-p extension of F_∞ which is unramified away from primes above $S \cup S_p$. For simplicity, if $S = S_\infty$ we let $M_\infty(S) = M_\infty$ and $\mathcal{X}_S = \mathcal{X}$. Then, classical results of Iwasawa show that $\mathcal{X}_S$ is endowed with the standard $\mathbb{Z}_p[[\mathcal{G}]]$–module structures, is finitely generated over this group ring (equivalently, finitely generated over $\mathbb{Z}_p[[\Gamma_F]]$) and consequently compact, and contains no finite non-trivial $\mathbb{Z}_p[[\Gamma_F]]$–submodules (see Theorem 18 in [11].) It is also known that $\mathcal{X}_S$ is a torsion $\mathbb{Z}_p[[\Gamma_F]]$–module if and only if F is a totally real field.

Working assumption. From now on, we work under the following assumption: $p > 2$ is a fixed prime number, F is a totally real number field, such that $[F(\mu_p) : F] = 2$.

Remark 3.2 Note that the assumption above is equivalent to $\mathbb{Q}(\mu_p)^+ \subseteq F$. Also, note that, under this assumption, $\Delta_F := \mathrm{Gal}(\widetilde{F}/F)$ (which can and will be identified, via Galois restriction, to $\mathrm{Gal}(\widetilde{F}_\infty/F_\infty)$) is of order 2. Therefore, Δ_F acts trivially on any Tate twist $\mathbb{Z}_p(n)$, with $n \in \mathbb{Z}$ even. This means that these "even" Tate twists have natural $\mathcal{G}$–module structures. Consequently, for any $\mathbb{Z}_p[[\mathcal{G}]]$–module M, the Γ_F–coinvariants and invariants $M(n)_{\Gamma_F}$ and $M(n)^{\Gamma_F}$ inherit natural $\mathbb{Z}_p[G]$–module structures, for all $n \in \mathbb{Z}$ even.

The following theorem makes the link between the p–adic étale cohomology groups introduced in §1 and the Iwasawa modules defined above (see [13], §3 for the proof.)

Theorem 3.3 *Let F/k and $p > 2$ be as above. Let S be a finite set of primes in k containing $S_{\mathrm{ram}}(F/k) \cup S_\infty$ and let $n \in \mathbb{Z}_{\geq 2}$ be an even integer. Under our working assumption, we have canonical $\mathbb{Z}_p[G]$–module isomorphisms*

i. $H^2_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n)) \cong Hom(\mathcal{X}_S(-n), \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_F} = Hom(\mathcal{X}_S(-n)_{\Gamma_F}, \mathbb{Q}_p/\mathbb{Z}_p)\,;$

ii. $H^1_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n)) \cong Hom(\mathbb{Z}_p(-n), \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_F} = Hom(\mathbb{Z}_p(-n)_{\Gamma_F}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Q}_p/\mathbb{Z}_p(n)^{\Gamma_F}\,,$

where $O_{F,S}$ denotes the ring of S–integers in F and the equalities in (i) and (ii) are obvious from the definitions of twists and (co)invariants.

Remark 3.4 Note that, under our assumptions, the étale cohomology groups in the theorem above are both finite. Indeed, we have already seen this for $S = S_\infty$ (see our comments preceding Conjecture 1.10). To see this for a general set S, one uses the localization sequence in p–adic étale cohomology due to Soulé [20], which leads to the following isomorphism and short exact sequence.

$$H^1_{et}(O_F[1/p], \mathbb{Z}_p(n)) \xrightarrow{\sim} H^1_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n))$$

$$0 \longrightarrow H^2_{et}(O_F[1/p], \mathbb{Z}_p(n)) \longrightarrow H^2_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n)) \longrightarrow \oplus_w H^1_{et}(F(w), \mathbb{Z}_p(n-1)) \longrightarrow 0\,.$$

The direct sum is taken with respect to all primes $w \in S(F) \setminus (S_\infty(F) \cup S_p(F))$, and $F(w)$ is the residue field of w. Now, if $\mathbb{F}_q$ is a finite field with q elements, of characteristic different from p, and $m \in \mathbb{Z}$, then there is a canonical isomorphism $H^1_{et}(\mathbb{F}_q, \mathbb{Z}_p(m)) \cong (\mathbb{Q}_p/\mathbb{Z}_p(m))^{G_{\mathbb{F}_q}}$, where the Tate twist is taken with respect to the p–cyclotomic character of $G_{\mathbb{F}_q}$. For $m \geq 1$, this group is easily seen to be isomorphic to the group of p–power roots of unity in the degree m extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$, and therefore it is finite. Now, the finiteness statement we made earlier follows immediately.

As a consequence, the $\mathbb{Z}_p[G]$–modules $\mathcal{X}_S(-n)_{\Gamma_F}$ and $\mathbb{Z}_p(-n)_{\Gamma_F}$ are finite. This implies that the corresponding Γ_F–invariant submodules are finite. However, since $\mathcal{X}_S(-n)$ and $\mathbb{Z}_p(-n)$ have no finite $\mathbb{Z}_p[[\Gamma_F]]$–submodules (deep theorem of Iwasawa for the first, obvious for the second), this shows that $\mathcal{X}_S(-n)^{\Gamma_F} = 0$ and $\mathbb{Z}_p(-n)^{\Gamma_F} = 0$.

Remark 3.5 With some care, part (ii) of the theorem above can be extended to arbitrary values of $n \in \mathbb{Z}_{\geq 1}$. Namely, for all $n \in \mathbb{Z}_{\geq 1}$, one has canonical isomorphisms of $\mathbb{Z}_p[G]$–modules

$$H^1_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n))_{\text{tors}} \cong Hom(\mathbb{Z}_p(-n), \mathbb{Q}_p/\mathbb{Z}_p)^{G_F} = Hom(\mathbb{Z}_p(-n)_{G_F}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Q}_p/\mathbb{Z}_p(n)^{G_F}\,,$$

where $G_F = \mathrm{Gal}(\overline{F}/F)$. The main difference is that, for odd values n, the étale cohomology groups $H^1_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n))$ are not finite, but their torsion subgroups are.

Lemma 3.6 *We have the following.*

i. *For all $n \in \mathbb{Z}$, $\mathrm{Ann}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p(-n)) = \mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p(-n)) = t_n(I_{\mathcal{G}})$, where $I_{\mathcal{G}}$ is the augmentation ideal of $\mathbb{Z}_p[[\mathcal{G}]]$, generated by $\{(\sigma - 1) \mid \sigma \in \mathcal{G}\}$.*

ii. *For all $n \in \mathbb{Z}_{\geq 2}$ even, $\mathrm{Ann}_{\mathbb{Z}_p[G]}(H^1_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n))) = \mathrm{Fitt}_{\mathbb{Z}_p[G]}(H^1_{et}(O_{F,S}[1/p], \mathbb{Z}_p(n)))$ and they are equal to the ideal of $\mathbb{Z}_p[G]$ generated by $\{(1 - \sigma_v \cdot Nv^{-n}) \mid v \notin S \cup S_p\}$, where $\sigma_v \in G$ and Nv denote the Frobenius morphism and the cardinality of the residue field associated to the prime v in k, respectively.*

P r o o f. First, the equalities between annihilators and Fitting ideals are consequences of the fact that all modules involved are cyclic over the corresponding group rings. (The cyclicity is a direct consequence of Theorem 3.3 (ii) and of the fact that $\mathbb{Z}_p(-n)$ itself is a cyclic $\mathbb{Z}_p[[\mathcal{G}]]$–module, for all integers n.) Now, (i) above is a direct consequence of Lemma 3.1 (i) and of the obvious equality $\mathrm{Ann}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathbb{Z}_p) = I_{\mathcal{G}}$. (Here, $\mathbb{Z}_p$ is viewed as $\mathbb{Z}_p(0)$.) A direct consequence of (i) is the equality $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathbb{Z}_p(-n)_{\Gamma_F}) = \pi_G(t_n(I_{\mathcal{G}}))$, where $\pi_G : \mathbb{Z}_p[[\mathcal{G}]] \twoheadrightarrow \mathbb{Z}_p[G]$ is the usual projection morphism induced by Galois restriction. Tchebotarev's density theorem applied to the (abelian) Galois extension F/k gives an equality between $\pi_G(t_n(I_{\mathcal{G}}))$ and the ideal of $\mathbb{Z}_p[G]$ generated by $\{(1 - \sigma_v \cdot Nv^n) \mid v \notin S \cup S_p\}$. (Note that one could replace $S \cup S_p$ by any finite set of primes in k containing $S \cup S_p$.) Now, a direct application of Lemma 3.1 (ii) and Theorem 3.3 (ii) leads to the proof of (ii) above. $\square$

4 Equivariant Iwasawa power series and the main conjecture

We fix a totally real number field k and a prime number $p > 2$. We keep the notations of the beginning of the last section. Since k and p are fixed, we let $c := c_p$, $\omega := \omega_p$, $\kappa := \kappa_p$, $\Gamma := \Gamma_k$. As usual, we fix a topological generator $\gamma := \gamma_k$, which leads to a canonical topological an O–algebra isomorphism $O[[\Gamma]] \xrightarrow{\sim} O[[T]]$, sending γ to $(1+T)$, for all finite extensions O of $\mathbb{Z}_p$. We set $u := \kappa(\gamma)$. Note that $u \in (1 + p\mathbb{Z}_p) \subseteq \mathbb{Z}_p^\times$. We fix an embedding $\mathbb{C} \hookrightarrow \mathbb{C}_p$ and think of all the complex valued (one dimensional, finite order) complex valued characters ψ of all the intervening groups (e.g. $\psi \in \widehat{G}_k(\mathbb{C})$) as $\mathbb{C}_p$–valued characters (e.g. $\psi \in \widehat{G}_k(\mathbb{C}_p)$). For any such character, we let $k_\psi := \overline{k}^{\ker \psi}$ and call ψ even if k_ψ is totally real. As in [22], we call ψ of type $\mathcal{S}$ if $k_\psi \cap k_\infty = k$ and of type $\mathcal{W}$ if $k_\psi \subseteq k_\infty$. Let $\psi \in \widehat{G}_k(\mathbb{C})$ be an even character and S be a set of primes in k containing S_∞ and the primes which ramify in k_ψ/k. As Wiles shows in [22], work of Deligne–Ribet [8] produces power series $G_{\psi,S}(T), H_{\psi,S}(T) \in \mathbb{Z}_p(\psi)[[T]]$, for all even characters $\psi \in \widehat{G}_k(\mathbb{C})$, uniquely determined by the following properties.

i. $H_{\psi,S}(T) = (\psi(\gamma)(1+T)-1)$, if ψ is of type $\mathcal{W}$ and $H_{\psi,S}(T) = 1$ otherwise;

ii. $\frac{G_{\psi,S}(u^n-1)}{H_{\psi,S}(u^n-1)} = L_{S\cup S_p}(\psi\omega^{-n}, 1-n)$ for all $n \in \mathbb{Z}_{\geq 1}$;

iii. $G_{\psi\rho,S}(T) = G_{\psi,S}(\rho(\gamma)(1+T)-1)$ and $H_{\psi\rho,S}(T) = H_{\psi,S}(\rho(\gamma)(1+T)-1)$, for all $\rho \in \widehat{G}_k(\mathbb{C})$ of type $\mathcal{W}$.

Here, $L_{S\cup S_p}(\psi\omega^{-n}, s)$ is the usual global, $(S \cup S_p)$–imprimitive L–function associated to $\psi\omega^{-n} \in \widehat{G}_k(\mathbb{C})$. If we let $g_{\psi,S}(T) := \frac{G_{\psi,S}(T)}{H_{\psi,S}(T)}$ and view it as an element in the total ring of fractions $Q(\mathbb{Z}_p(\psi)[[T]])$ of $\mathbb{Z}_p(\psi)[[T]]$, then the S–imprimitive p–adic L–function associated to ψ is uniquely determined by

$$L_{p,S}(\psi, 1-s) := g_{\psi,S}(1-s)\,,$$

for all $s \in \mathbb{Z}_p$.

Now, let F/k be a totally real, finite abelian extension of k of Galois group G, such that $F \cap k_\infty = k$. Let S be a fixed finite set of primes in k, containing S_∞ and $S_{\mathrm{ram}}(F/k)$. Let $\widetilde{F} := F(\mu_p)$ and $\widetilde{G} := G(\widetilde{F}/k)$. Obviously, $\widetilde{F} \cap k_\infty = k$ and therefore all the characters ψ of $\widehat{\widetilde{G}}$ except for the trivial one are of type $\mathcal{S}$. For simplicity, let us assume that $[\widetilde{F} : F] = 2$ (see our working assumption in the previous section), meaning that the even characters of $\widetilde{G}$ are precisely those which factor through G. We let $\mathcal{G} = \mathrm{Gal}(F_\infty/k)$ and $\widetilde{\mathcal{G}} = \mathrm{Gal}(\widetilde{F}_\infty/k)$. Since $\mathcal{G} \cong G \times \Gamma$, for all finite extensions O of $\mathbb{Z}_p$, we have obvious (continuous) O–algebra isomorphisms $O[[\mathcal{G}]] \cong O[G][[T]]$ (and similarly for $O[[\widetilde{\mathcal{G}}]] \cong O[\widetilde{G}][[T]]$.) If O contains the values of all the characters $\chi \in \widehat{\widetilde{G}}(\mathbb{C})$, then we have an obvious character decomposition $Q(O[[\widetilde{\mathcal{G}}]]) \cong \oplus_\chi Q(O[[T]])e_\chi$. Under the automorphisms t_n and $\imath$ $Q(O[[\widetilde{\mathcal{G}}]])$ defined in the previous section, note that $t_n(T+1) = u^n(T+1)$, $t_n(e_\chi) = e_{\chi\omega^{-n}}$, for all $n \in \mathbb{Z}$, $\imath(T+1) = (T+1)^{-1}$ and $\imath(e_\chi) = e_{\chi^{-1}}$, for all $\chi \in \widetilde{G}$. We fix an extension O which is large enough in this sense and define the following ($\widetilde{G}$–equivariant) power series.

$$\begin{aligned} g_S(T) &:= \sum_{\chi \text{ odd}} g_{\chi^{-1}\omega,S}(u(1+T)^{-1}-1)\cdot e_\chi \in Q(O[[\widetilde{\mathcal{G}}]]) \\ G_S(T) &:= \sum_{\chi \text{ odd}} G_{\chi^{-1}\omega,S}(u(1+T)^{-1}-1)\cdot e_\chi \in Q(O)[[\widetilde{\mathcal{G}}]] \\ H_S(T) &:= \sum_{\chi \text{ odd}} H_{\chi^{-1}\omega,S}(u(1+T)^{-1}-1)\cdot e_\chi \in Q(O)[[\widetilde{\mathcal{G}}]] \end{aligned}$$

Obviously, we have $g_S(T) = G_S(T)/H_S(T)$ and since the trivial character is the only character of $\widetilde{G}$ of type $\mathcal{W}$, we also have the equality

$$H(T) = (u(1+T)^{-1} - 1) \cdot e_\omega + 1 \cdot (1 - e_\omega) \quad \in \frac{1}{|\widetilde{G}|}\, \mathbb{Z}_p[\widetilde{G}][[T]]\,. \tag{4.1}$$

The link between the equivariant power series $g_S(T)$ and the equivariant global L–functions $\Theta_{F/k,S}$ defined in §1 is as follows. For every $m \in \mathbb{Z}_{>0}$, we let k_m be the unique intermediate extension of k_∞/k satisfying $[k_m : k] = p^m$. We consider the field composita $F_m := k_m \cdot F$ and $\widetilde{F}_m := \widetilde{F} \cdot k_m$. We let $G_m := G(F_m/k) \cong G \times \Gamma/\Gamma^{p^m}$ and $\widetilde{G}_m := G(\widetilde{F}_m/k) \cong \widetilde{G} \times \Gamma/\Gamma^{p^m}$. Also, we let $\Theta_m(1-n) := \Theta_{\widetilde{F}_m/k, S\cup S_p}(1-n) \in Q(\mathbb{Z}_p[\widetilde{G}_m])$, for all $n \in \mathbb{Z}_{\geq 1}$ (see Theorem 1.1.) Then, the inflation property of Artin L–functions implies that the elements $\{\Theta_m(1-n)\}_m$ are coherent with respect to the usual projection maps $Q(\mathbb{Z}_p[\widetilde{G}_{m+1}]) \twoheadrightarrow Q(\mathbb{Z}_p[\widetilde{G}_m])$. This means that we can define $\Theta_\infty(1-n) := \varprojlim_m \Theta_m(1-n) \in Q(\mathbb{Z}_p[[\widetilde{\mathcal{G}}]])$, for all $n \in \mathbb{Z}_{\geq 1}$.

Proposition 4.1 *With notations as above, we have the following.*

i. $\Theta_\infty(0) = g_S(T)$. *In particular,* $g_S(T), G_S(T), H_S(T) \in Q(\mathbb{Z}_p[[\widetilde{\mathcal{G}}]])$.

ii. $\Theta_\infty(1-n) = t_{1-n}(g_S(T))$, *for all* $n \in \mathbb{Z}_{\geq 1}$.

Proof. First, we have to show that if $\pi_m : Q(O[[\widetilde{\mathcal{G}}]]) \twoheadrightarrow Q(O[[\widetilde{G}_m]])$ denotes the usual projection at the level of group rings, then $\pi_m(g_S(T)) = \Theta_m(0)$, for all m. Indeed,

$$\begin{aligned} \pi_m(g_S(T)) &= \sum_{\chi,\rho} g_{\chi^{-1}\omega}(u\rho(\gamma^{-1}) - 1)e_{\chi\rho} \\ &= \sum_{\chi,\rho} g_{\chi^{-1}\rho^{-1}\omega}(u-1)e_{\chi\rho} \\ &= \sum_{\chi,\rho} L_{S\cup S_p}(\chi^{-1}\rho^{-1}, 0)e_{\chi\rho} = \Theta_m(0)\,, \end{aligned}$$

where the sums are taken with respect to the characters $\chi \in \widehat{\widetilde{G}}$ and $\rho \in \widehat{\Gamma/\Gamma^{p^m}}$. The second and third equalities above are direct consequences of properties (iii) and (ii) above, respectively. This proves the first statement in (i) of the proposition. The second statement is a direct consequence of the first combined with the fact that $H_S(T) \in Q(\mathbb{Z}_p[[\widetilde{\mathcal{G}}]])$ (see (4.1) above.) The proof of (ii) in the proposition involves a calculation similar to the one above. We leave it to the reader. □

Proposition 4.2 *Let us assume that p does not divide $[F : k]$ and let $n \in \mathbb{Z}_{\geq 1}$. Then:*

i. $H_S(T) \in \mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$ *and the ideal generated by* $t_{1-n}(H_S(T))$ *in* $\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$ *equals* $\mathrm{Ann}_{\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]}(\mathbb{Z}_p(n))$.

ii. $G_S(T) \in \mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$ *and the ideal generated by* $t_{1-n}(G_S(T))$ *in* $\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$ *equals*

$$\mathrm{Stick}_\infty(1-n) := \varprojlim_m \mathrm{Stick}_{\widetilde{F}_m/k, S\cup S_p, p}(1-n)\,.$$

P r o o f. Since $p \nmid |\widetilde{G}|$, equlaity (4.1) implies that $H_S(T) \in \mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$. Now, in view of Lemma 3.1, it suffices to show the equality of $\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$-ideals $(t_1(H_S(T))) = \mathrm{Ann}_{\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]}(\mathbb{Z}_p(0))$. However, $\mathrm{Ann}_{\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]}(\mathbb{Z}_p(0)) = I_{\widetilde{\mathcal{G}}}$, where $I_{\widetilde{\mathcal{G}}}$ is the augmentation ideal of $\mathbb{Z}_p[[\widetilde{\mathcal{G}}]]$. Now, (4.1) above implies that $t_1(H_S(T)) = ((T+1)^{-1} - 1)e_{\mathbf{1}_{\widetilde{G}}} + (1 - e_{\mathbf{1}_{\widetilde{G}}}) = (\gamma^{-1} - 1)e_{\mathbf{1}_{\widetilde{G}}} + (1 - e_{\mathbf{1}_{\widetilde{G}}})$, which is clearly a generator of $I_{\widetilde{\mathcal{G}}}$. In order to prove (ii), we start with the obvious exact sequences

$$0 \longrightarrow \mathbb{Z}_p(n) \xrightarrow{1-\gamma^{p^m}} \mathbb{Z}_p(n) \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma^{p^m}} \longrightarrow 0\,.$$

Combined with (i), this shows that $\pi_m(t_{1-n}(H_S(T)))$ generates the ideal $\mathrm{Ann}_{\mathbb{Z}_p[\widetilde{G}_m]}((\mathbb{Q}_p/\mathbb{Z}_p(n))^{\Gamma^{p^m}})$ in $\mathbb{Z}_p[\widetilde{G}_m]$, for all m. Now, we apply Theorem 1.1, Theorem 3.3 and Remark 3.5, as well as Proposition 4.1 to conclude that $\pi_m(t_{1-n}(G_S(T))) = \pi_m(t_{1-n}(H_S(T))) \cdot \Theta_m(1-n) \in \mathbb{Z}_p[\widetilde{G}_m]$ and also that this is a generator of $\mathrm{Stick}_{\widetilde{F}_m/k, S\cup S_p, p}(1-n)$. The proof of (ii) concludes by taking the projective limit of the $\mathbb{Z}_p[\widetilde{G}_m]$–elements $\pi_m(t_{1-n}(H_S(T)))\cdot\Theta_m(1-n) = \pi_m(t_{1-n}(H_S(T)\cdot g_S(T))) = \pi_m(t_{1-n}(G_S(T)))$ and the ideal generated by these inside $\mathbb{Z}_p[[\widetilde{\mathcal{G}}]] = \varprojlim_m \mathbb{Z}_p[\widetilde{G}_m]$. $\square$

Now, we turn again to the Iwasawa modules $\mathcal{X}_S$ associated to the field F and the set S, as in the previous section. In [22], Wiles proves the Main Conjecture in Iwasawa Theory over arbitrary totally real number fields k. We will only need a weaker version of Wiles's result, which we will state next. For this, assume that F/k and $p > 2$ are as above. Additionally, assume that p does not divide $[F : k]$. Let O be an unramified finite extension of $\mathbb{Z}_p$ containing the values of all the characters of G. Then, as in §2, we have canonical splitting $O[[\mathcal{G}]] = \oplus_\chi O[[T]]e_\chi = \oplus_\chi \Lambda_O \cdot e_\chi$ and $\mathcal{X}_S \otimes O = \oplus_\chi (\mathcal{X}_S \otimes O)^\chi$, where χ runs through all the characters of G and $(\mathcal{X}_S \otimes O)^\chi := e_\chi(\mathcal{X}_S \otimes O)$.

Theorem 4.3 (Wiles) *Assume that F/k, $p > 2$ and O are as above. Then, we have an equality of $\Lambda_O = O[[T]]$–ideals* $(\mathrm{char}_{\Lambda_O}(\mathcal{X}_S \otimes O)^\chi) = (G_{\chi,S}(T))$, *for all characters χ of G.*

P r o o f. See Theorems 1.2, 1.3, 1.4 in [22] and keep in mind that there is an equality of μ–invariants $\mu((\mathcal{X}_S \otimes O)^\chi) = \mu((\mathcal{X} \otimes O)^\chi) = \mu((X \otimes O)^{\chi^{-1}\omega})$, where X is as in [22] and χ is as above. $\square$

Corollary 4.4 *Assume that F/k and $p > 2$ are as above. Then the $\mathbb{Z}_p[[\mathcal{G}]]$–module $\mathcal{X}_S$ has projective dimension ≤ 1 and its Fitting ideal is principal generated by $\imath(t_1(G_S(T)))$*

P r o o f. Since $\mathcal{X}_S$ has no finite nontrivial Λ–submodules (Iwasawa's Theorem) and it is G–cohom. trivial (because $p \nmid G$), Proposition 2.2 and Lemma 2.3 imply that we have $\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]}\mathcal{X}_S \leq 1$. Let O be as above. Since $\mathcal{X}_S \otimes O$ contains no finite Λ_O–submodules, we have $\mathrm{pd}_{\Lambda_O}(\mathcal{X}_S \otimes O)^\chi \leq 1$, for all χ (see Lemma 2.3). Consequently, Lemma 2.4 combined with Theorem 4.3 shows that

$$\mathrm{Fitt}_{\Lambda_O}((\mathcal{X}_S \otimes O)^\chi) = (\mathrm{char}_{\Lambda_O}(\mathcal{X}_S \otimes O)^\chi) = (G_{\chi,S}(T))\,,$$

for all χ. Now, a simple calculation combined with Proposition 4.2 shows that

$$\imath(t_1(G_S(T))) = \sum_{\chi\in\widehat{G}} G_{\chi,S}(T)e_\chi \in \mathbb{Z}_p[[\mathcal{G}]]$$

(Note how $\imath \circ t_1$ turns the idempotents e_χ associated to odd characters χ of $\widetilde{G}$ in the definition of $G_S(T)$ into idempotents $e_{\chi^{-1}\omega}$ of the even characters $\chi^{-1}\omega$ of $\widetilde{G}$, i.e. characters of G.) Consequently, we have

$$\mathrm{Fitt}_{O[[\mathcal{G}]]}(\mathcal{X}_S \otimes O) = (\imath(t_1(G_S(T)))\,.$$

Finally, since $\imath(t_1(G_S(T))) \in \mathbb{Z}_p[[G]]$, O is a faithfully flat $\mathbb{Z}_p$-algebra and Fitting ideals commute with extensions of the rings of coefficients, we have $\mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathcal{X}_S) = (\imath(t_1(G_S(T))))$. □

5 The étale cohomological version of the Coates-Sinnott conjecture

We are ready to formulate and prove our results on the étale cohomological version of the Coates-Sinnott Conjecture (see Conjecture 1.3.) First, we assume that F/k is an abelian extension of totally real number fields of Galois group G, and p is an odd prime such that $[\widetilde{F} : F] = 2$ and $p \nmid |G|$ (which automatically implies that $F \cap k_\infty = k$.) The notations are the same as in the previous section. Eventually (see Theorem 5.4 below), we will drop some of the above assumptions on the set of data $(F/k\,,\,p)$.

Proposition 5.1 *Assume that F/k and $p > 2$ are as above, $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 2}$ is even. Then*

i. $\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathcal{X}_S(-n)) \leq 1$ *and* $\mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathcal{X}_S(-n)) = \imath(\mathrm{Stick}_\infty(1-n))$.

ii. $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(\mathcal{X}_S(-n)_{\Gamma^{p^m}}) \leq 1$ *and, if S contains all the primes which ramify in F_m/k, we have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\mathcal{X}_S(-n)_{\Gamma^{p^m}}) = \imath(\mathrm{Stick}_{F_m/k,S,p}(1-n))\,.$$

P r o o f. The fact that $\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathcal{X}_S(-n)) \leq 1$ is a direct consequence of Proposition 2.2 and Lemma 2.3 ($\mathcal{X}_S$ has no nontrivial finite Λ–submodules, therefore the same is valid for $\mathcal{X}_S(-n)$, for all n. Also, $\mathcal{X}_S(-n)$ is G–cohom. trivial because $p \nmid |G|$.) Now, a direct application of Lemma 2.5, with $O = \mathbb{Z}_p$ and $f = 1 - \gamma^{p^m}$ implies that $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(\mathcal{X}_S(-n)_{\Gamma^{p^m}}) \leq 1$. The successive application of the above Corollary, Lemma 3.1, equality $t_n \circ \imath = \imath \circ t_{-n}$, and Proposition 4.2 leads to the following

$$\begin{aligned}
\mathrm{Fitt}_{\mathbb{Z}_p[[\mathcal{G}]]}(\mathcal{X}_S(-n)) &= (t_n \circ \imath \circ t_1(G_S(T))) \\
&= \imath(t_{1-n}(G_S(T))) \\
&= \imath(\mathrm{Stick}_\infty(1-n))\,,
\end{aligned}$$

which conclude the proof of the second statement in (i). In particular, this implies, by Γ^{p^m} descent and the definition of $\mathrm{Stick}_\infty(1-n)$, that $\mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\mathcal{X}_S(-n)_{\Gamma^{p^m}}) = \imath(\mathrm{Stick}_{F_m/k,S\cup S_p,p}(1-n))$. This is almost the second statement in (ii) above, except for the fact that we want the S–imprimitive rather than the $S \cup S_p$–imprimitive higher Stickelberger ideals. However, from the definitions, the difference between the two is given by a product of Euler factors as follows

$$\mathrm{Stick}_{F_m/k,S\cup S_p,p}(1-n) = \mathrm{Stick}_{F_m/k,S,p}(1-n) \cdot (\prod_{v \in S_p \setminus S} (1 - \sigma_v^{-1} \cdot Nv^{n-1}))\,.$$

Since $n \neq 1$ and Nv is a p–power, for all $v \in S_p$, each of these Euler factors $(1 - \sigma_v^{-1} \cdot Nv^{n-1})$ is a divisor of $(1 - p^a)$, for some $a \in \mathbb{Z}_{>0}$, and it is therefore invertible in $\mathbb{Z}_p[G_m]$. Consequently, the two Stickelberger ideals are equal and this concludes the proof. □

Theorem 5.2 *Assume that F/k, p, m and n are as in the proposition above. Also, assume that the set S contains all the primes which ramify in F_m/k. Then, we have*

i. $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n))^{\frown}) \leq 1$, $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(H^2_{et}(O_{F_m}[1/p],\mathbb{Z}_p(n))^{\frown}) \leq 1$ *and*

$$\begin{aligned} \imath(\mathrm{Stick}_{F_m/k,S,p}(1-n)) &= \mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\,H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n))^{\frown}) \\ &= \mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\,H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n))) \\ &\subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\,H^2_{et}(O_{F_m}[1/p],\mathbb{Z}_p(n))) \\ &= \mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\,H^2_{et}(O_{F_m}[1/p],\mathbb{Z}_p(n))^{\frown})\,; \end{aligned}$$

where $\frown$ *stands for Pontrjagin dual.*

ii. $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n))) \leq 1$, $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(H^2_{et}(O_{F_m}[1/p],\mathbb{Z}_p(n))) \leq 1$ *and*

$$\begin{aligned} \mathrm{Stick}_{F_m/k,S,p}(1-n) &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G_m]}(\,H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n))\,) \\ &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G_m]}(\,H^2_{et}(O_{F_m}[1/p],\mathbb{Z}_p(n))\,. \end{aligned}$$

In particular, the p–primary part of the Coates–Sinnott conjecture holds for F_m/k, for all $m \in \mathbb{Z}_{\geq 0}$.

P r o o f. First, we prove the statements on projective dimension in both (i) and (ii) above. Recall that we have isomorphisms of $\mathbb{Z}_p[G_m]$–modules

$$H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n))^{\frown} \cong \mathcal{X}_S(-n)_{\Gamma^{p^m}}\,, \qquad H^2_{et}(O_{F_m}[1/p],\mathbb{Z}_p(n))^{\frown} \cong \mathcal{X}(-n)_{\Gamma^{p^m}}$$

(see Theorem 3.3.) Both $\mathcal{X}_S(-n)$ and $\mathcal{X}(-n)$ satisfy $\mathrm{pd}_{\mathbb{Z}_p[[\mathcal{G}]]} \leq 1$ (a direct application of Proposition 2.2.) Now, the projective dimension statement in (i) follows by an application of Lemma 2.5 to both $\mathcal{X}_S(-n)$ and $\mathcal{X}(-n)$ and $f = 1-\gamma^{p^m}$. The projective dimension statement in (ii) follows from above, based on the following observation. If M is a finite $\mathbb{Z}_p[H]$ module, for some finite group H and $\mathrm{pd}_{\mathbb{Z}_p[H]}(M) \leq 1$, then if $\widehat{M} := Hom(M,\mathbb{Q}_p/\mathbb{Z}_p)$, we also have $\mathrm{pd}_{\mathbb{Z}_p[H]}(\widehat{M}) \leq 1$. Indeed, this follows directly from Theorem 9, Chpt. IX, §5 in [18], combined with the fact that $\mathrm{Ext}^1_{\mathbb{Z}_p}(M,\mathbb{Q}_p/\mathbb{Z}_p) = 0$, as $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mathbb{Z}_p$–injective. Now, apply this observation to the group $H := G_m$ and the modules $M := \mathcal{X}_S(-n)_{\Gamma^{p^m}}$ and $M := \mathcal{X}(-n)_{\Gamma^{p^m}}$, respectively.

The first equality in (i) is a direct consequence of Proposition 5.1 (ii) and the $\mathbb{Z}_p[G_m]$–module isomorphism $H^2_{et}(O_{F_m,S}[1/p],\mathbb{Z}_p(n)) \cong \mathcal{X}_S\widehat{(-n)_{\Gamma^{p^m}}}$ (see Theorem 3.3.) For completing the proof of (i), we are going to use the following.

Lemma 5.3 *Let H be a finite abelian group, such that its p–Sylow subgroup P is cyclic. Let O be a finite extension of $\mathbb{Z}_p$. Then*

i. *For any finite $O[H]$–module M, we have* $\mathrm{Fitt}_{O[H]}(M) = \mathrm{Fitt}_{O[H]}(\widehat{M})$.

ii. *If* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is an exact sequence of finite $O[G]$–modules, such that either* $\mathrm{pd}_{O[H]}A \leq 1$ *or* $\mathrm{pd}_{O[H]}C \leq 1$, *we have*

$$\mathrm{Fitt}_{O[H]}(B) = \mathrm{Fitt}_{O[H]}(A)\cdot\mathrm{Fitt}_{O[H]}(C)\,.$$

P r o o f. (sketch) If $H = P \times H_0$, then one can split the $O[H]$–modules M, A, B, C into character components with respect to the characters $\chi \in \widehat{H_0}(Q(O))$, as in §2. These character components will be modules over the rings $O(\chi)[P]$, satisfying the same properties as the original modules M, A, B, C. Also, the exact sequence above splits into corresponding exact sequences for the character components, for all characters χ as above. Therefore, we can assume from the very beginning that H is a finite, cyclic p–group. Now, (i) above is a direct consequence of Proposition 1 in the Appendix of [15] (see also [6], equality (1), p. 464, but note that the assumption P cyclic has to be made in loc. cit.). Part (ii) of our Lemma is a direct consequence of (i), Lemma 5 in [6] and the observation that if $\mathrm{pd}_{O[H]}(C) \leq 1$, then $\mathrm{pd}_{O[H]}(\widehat{C}) \leq 1$ (see above.) $\square$

We return to the proof of (i) in the Theorem. Let us observe that the p–Sylow subgroup of G_m is $\Gamma/\Gamma^{p^m} \cong \mathbb{Z}/p^m\mathbb{Z}$, therefore it is cyclic. Consequently, the second and third equalities in (i) are direct applications of Lemma 5.3 (i) for $H = G_m$, $M = H^2_{et}(O_{F_m,S}[1/p], \mathbb{Z}_p(n))$ and $M = H^2_{et}(O_{F_m}[1/p], \mathbb{Z}_p(n))$, respectively. The inclusion in (i) is obtained by applying Lemma 5.3 (ii) to the (second) localization sequence in Remark 3.4.

Now, we prove (ii) in the theorem. A successive application of the first equality in (i), Proposition 5.1 (ii), Theorem 3.3, Lemma 3.1 and the localization sequence in Remark 3.4 leads to the following equalities and inclusions of ideals.

$$\begin{aligned}
\mathrm{Stick}_{F_m/k,S,p}(1-n) &= \imath(\mathrm{Fitt}_{\mathbb{Z}_p[G_m]}(\mathcal{X}_S(-n)_{\Gamma^{p^m}})) \\
&\subseteq \imath(\mathrm{Ann}_{\mathbb{Z}_p[G_m]}(\mathcal{X}_S(-n)_{\Gamma^{p^m}})) \\
&= \mathrm{Ann}_{\mathbb{Z}_p[G_m]}(H^2_{et}(O_{F_m,S}[1/p], \mathbb{Z}_p(n))) \\
&\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G_m]}(H^2_{et}(O_{F_m}[1/p], \mathbb{Z}_p(n)))\,.
\end{aligned}$$

This concludes the proof of (ii) as well. The p–primary part of the Coates-Sinnott Conjecture for F_m/k is the last inclusion in (ii). $\square$

Theorem 5.4 *Let F/k be an abelian extension of totally real number fields of Galois group G. Let $p > 2$ be a prime number and assume the $p \nmid [F : F \cap k_\infty]$, where k_∞ is the cyclotomic $\mathbb{Z}_p$–extension of k. Then the conclusions of Theorem 5.2 hold true for F/k, p, all $m \in \mathbb{Z}_{\geq 0}$ and all even $n \in \mathbb{Z}_{\geq 2}$. In particular, the p–primary part of the Coates-Sinnott Conjecture holds true for F_m/k, for all $m \in \mathbb{Z}_{\geq 0}$.*

P r o o f. We will derive this statement from Proposition 5.1 and Theorem 5.2 in two steps.

In **step 1**, we assume that $[\tilde{F} : F] = 2$ and show that the conclusions of both Theorem 5.2 and Proposition 5.1 hold for the set of data $(F/k,\, p)$. As above, we let $G = \mathrm{Gal}(F/k)$, $\mathcal{G} = \mathrm{Gal}(F_\infty/k)$ and $\Gamma := \Gamma_k = \mathrm{Gal}(k_\infty/k)$. Also, let $G' := \mathrm{Gal}(F_\infty/k_\infty)$. Then, the Galois restriction map $\pi := \mathrm{res}_{F_\infty/k_\infty}$ leads to an exact sequence of topological groups

$$0 \longrightarrow G' \longrightarrow \mathcal{G} \xrightarrow{\pi} \Gamma \longrightarrow 0\,.$$

Since Γ is free of rank one in the category of abelian topological groups (of free generator $\gamma := \gamma_k$), the exact sequence above splits. Let $s : \Gamma \longrightarrow \mathcal{G}$ be a splitting map, let $\Gamma' := s(\Gamma)$ and $F' := (F_\infty)^{\Gamma'}$. Then, Galois restriction induces an isomorphisms $G' \cong \mathrm{Gal}(F/F \cap k_\infty) \cong \mathrm{Gal}(F'/k)$ and the hypothesis $p \nmid [F : F \cap k_\infty]$ implies right away that $p \nmid |G'|$. Therefore the conclusions of Proposition 5.1 and Theorem 5.2 hold true for F'_m/k, p, and all m and as above. However, it is easy to see that, under our hypotheses, there exists an $m_0 \in \mathbb{Z}_{\geq 0}$, such that $F = F'_{m_0}$. Therefore, $F_m = F'_{m \cdot m_0}$, for all $m \in \mathbb{Z}_{\geq 0}$. Now, our assertion follows immediately by observing that $F_\infty = F'_\infty$ and that $\mathrm{Stick}_\infty(1-n)$ and the modules $\mathcal{X}(-n)$

and $\mathcal{X}_S(-n)$ are canonically associated to p, n, S and $F_\infty = F'_\infty$ and do not depend of the finite level subfields of $F_\infty = F'_\infty$. This completes step 1.

Step 2. Now, we drop the assumption $[\widetilde{F} : F] = 2$. However, if we let $F' := F(\mu_p)^+$ be the maximal totally real subfield of $F(\mu_p)$, we obviously have $\widetilde{F'} = \widetilde{F}$, $[\widetilde{F'} : F'] = 2$, and $p \nmid [F' : F' \cap k_\infty]$. According to Step 1 above, the conclusions of Theorem 5.2 holds for the set of data $(F'/k,\, p)$. We let $\Delta := \mathrm{Gal}(F'/F)$. Obviously, we have $\Delta \simeq \mathrm{Gal}(F'_m/F_m)$ and F'_m/F_m is unramified away from $S_p(F_m)$, for all $m \geq 0$. As a consequence, we have the following isomorphisms of $\mathbb{Z}_p[G]$–modules, for all m.

$$H^2_{et}(O_{F_m,S}[1/p], \mathbb{Z}_p(n)) \simeq H^2_{et}(O_{F'_m,S}[1/p], \mathbb{Z}_p(n))_\Delta \simeq H^2_{et}(O_{F'_m,S}[1/p], \mathbb{Z}_p(n))^\Delta\,.$$

$$H^2_{et}(O_{F_m}[1/p], \mathbb{Z}_p(n)) \simeq H^2_{et}(O_{F'_m}[1/p], \mathbb{Z}_p(n))_\Delta \simeq H^2_{et}(O_{F'_m}[1/p], \mathbb{Z}_p(n))^\Delta\,.$$

The two left-most isomorphisms above are a direct consequence of Proposition 2.10 in [13], while the two right-most isomorphisms are given by multiplication with the norm element $N_\Delta \in \mathbb{Z}_p[\Delta]$ (since $p \nmid |\Delta|$.) Obviously, the same $\mathbb{Z}_p[G]$–module isomorphisms hold if we replace the étale cohomology groups above with their Pontrjagin duals $Hom(*,\, \mathbb{Q}_p/\mathbb{Z}_p)$. The last ingredient needed is the observation that if we let $\pi_m : \mathbb{Z}_p[G'_m)] \twoheadrightarrow \mathbb{Z}_p[G_m]$ denote the usual group-ring morphisms given by Galois restrictions, then

$$\pi_m(\mathrm{Stick}_{F'_m/k,S\cup S_p,p}(1-n)) = \mathrm{Stick}_{F_m/k,S,p}(1-n), \quad \pi_m(\mathrm{Stick}_{F'_m/k,S\cup S_p,p}(1-n)) = \mathrm{Stick}_{F_m/k,S,p}(1-n)$$

for all m and n as above, where $G'_m := \mathrm{Gal}(F'_m/k)$. (Keep in mind that the extensions F'_m/F_m are unramified away from $S_p(F_m)$ and use the argument at the end of the proof of Proposition 5.1.) Now, Theorem 5.2 for the set of data $(F/k,\, p)$ follows immediately from Theorem 5.2 for the set of data $(F'/k,\, p)$, by applying the surjective ring morphisms π_m, for all m as above. Also, observe that if $\mathrm{pd}_{\mathbb{Z}_p[G'_m]}(M) \leq 1$ for some $\mathbb{Z}_p[G'_m]$–module M, then $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(M_\Delta) \leq 1$, which follows from the definitions, and that $\mathrm{pd}_{\mathbb{Z}_p[G_m]}(M^\Delta) \leq 1$, which follows from the isomorphism $M^\Delta \simeq M_\Delta$ (see above.)

□

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