STARK'S QUESTION AND A STRONG FORM OF BRUMER'S CONJECTURE

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Abstract. We show that a question posed by Stark in 1980, regarding abelian L–functions of order of vanishing 2 at $s = 0$, has a negative answer in the case of characteristic p global fields. We provide links between various versions of Stark's question and a strong form of Brumer's Conjecture, in the general context of global fields of arbitrary characteristic. As a consequence, we show that the strong form of Brumer's Conjecture is in general false for characteristic p global fields.

INTRODUCTION

In [16], Stark developed a remarkable conjecture, interpreting the lowest non–vanishing derivatives at $s = 0$ of the (imprimitive) Artin L–functions $L_{K/k,S}(s, \chi)$ in terms of values of a Galois–equivariant regulator, defined on a finite dimensional $\mathbf{Q}$–vector space, constructed out of S–units in K, for any large enough set of primes S in k. In the 1970s and early 1980s, due to work of Chinburg, Stark and Tate, it became increasingly clear that, if one manages to replace the $\mathbf{Q}$–vector space in Stark's general conjecture by a Galois equivariant $\mathbf{Z}$–submodule (i.e. formulate a Stark Conjecture "over $\mathbf{Z}$"), the refined statement obtained this way would have far reaching applications to Hilbert's 12th problem and the theory of Multiplicative Galois Module Structure. In the early 1990s, Rubin showed that a Stark Conjecture "over $\mathbf{Z}$" for abelian L–functions would provide a new source of Euler Systems. In the last paper of [16], Stark formulates a conjecture "over $\mathbf{Z}$", covering the case of abelian L–functions of order of vanishing 1 at $s = 0$. As Tate shows in [18, IV.6], Brumer's Conjecture can be viewed as a weak form of a particular case of this statement. In the same year, Stark studied the case of L–functions of order of vanishing 2 at $s = 0$. Presumably, due to lack of compelling evidence, the refined statement at which he arrived in this case was formulated as a question rather than a conjecture (see [17], [15]).

In this paper, we answer Stark's Question and, as a consequence, we settle a strong form of Brumer's Conjecture, in the case of function fields. The paper is organized as follows. In §§1-2, we set the notations and state various forms of Stark's Question. In §3, we show that a weak form of Stark's Question, and consequently Stark's Question itself, has a negative answer in the case of function fields (see Theorem 3.3.2). In §4, we state a strong form of Brumer's Conjecture and provide links between this statement and various versions of Stark's Question, for abelian extensions of global fields of arbitrary characteristic (see Propositions 4.2.1, 4.2.2, 4.2.4). As a consequence, we show that the strong form of Brumer's Conjecture is in general false in characteristic $p > 0$ (see Corollary 4.2.2).

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It is worth mentioning that, recently, Rubin, the present author, as well as Burns and Flach have formulated versions "over **Z**" of Stark's general conjecture, in the case of abelian L–functions of arbitrary order of vanishing at $s = 0$ (see Conjecture B in [14], Conjecture C in [13], and The Equivariant Tamagawa Number Conjecture in [3], as well as the papers coauthored by Burns and Flach, cited in [3].) The Equivariant Tamagawa Number Conjecture was recently extended by Burns to nonabelian Artin L–functions as well (see [2]).

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1. NOTATION

Let K be a global field of arbitrary characteristic. Let μ_K be the group of roots of unity in K and $w_K := |\mu_K|$, the cardinality of μ_K. For a prime w of K, let K_w be the completion of K at w, and $|\cdot|_w : K_w \longrightarrow \mathbf{R}^+ \cup \{0\}$ the associated w–absolute value, normalized so that

$$|x|_w = \begin{cases} \pm x \text{ (the usual absolute value)}, & \text{if } K_w = \mathbf{R} \\ x\bar{x}, & \text{if } K_w = \mathbf{C} \\ (\mathbf{N}w)^{-\mathrm{ord}_w(x)}, & \text{if } K_w \text{ is nonarchimedian.} \end{cases}$$

Above, $\mathbf{N}w$ and $\mathrm{ord}_w(\cdot)$ denote respectively the cardinality of the residue field and the usual (discrete) valuation associated to the finite prime w.

Let us assume now that K/k is a finite, Galois extension of global fields, of Galois group $G := \mathrm{Gal}\,(K/k)$. Let S be a non–empty, finite set of primes in k, containing at least all the infinite primes and all the primes which ramify in K/k. Let S_K be the set of primes in K sitting above primes in S. We will denote by $O_{K,S}$ the ring of S_K–integers in K (i.e. the set of elements in K with non-negative valuations at all primes outside S_K.) $U_{K,S}$ and $A_{K,S}$ will denote the group of units and respectively the ideal–class group associated to the Dedekind domain $O_{K,S}$. Since the set S_K is closed under the natural action of G on primes in K, $O_{K,S}$, $U_{K,S}$, and $A_{K,S}$ are endowed with natural structures of left–modules over the integral group ring $\mathbf{Z}\,[G]$.

Assume now that K/k is in addition abelian. For every prime v of k, G_v will denote the decomposition group of w in K/k, for any prime w in K, sitting above v. If v is unramified in K/k, then G_v is cyclic, generated by the Frobenius automorphism associated to v in G, denoted by σ_v. Let $\widehat{G}$ denote the group of complex–valued, irreducible characters of G. For every $\chi \in \widehat{G}$, let $e_\chi = 1/|G| \sum_{\sigma\in G} \chi(\sigma)\cdot\sigma^{-1}$ be the corresponding idempotent in the complex group ring $\mathbf{C}\,[G]$. To every $\chi \in \widehat{G}$, and every set S as above, one can associate the Artin L–function of complex variable s, with Euler factors at primes in S removed, denoted in what follows by $L_{K/k,S}(s,\chi)$. For a given χ and S, $L_{K/k,S}(s,\chi)$ is the (unique) meromorphic continuation to $\mathbf{C}$ of the infinite Euler product $\prod_{v\notin S}(1 - \mathbf{N}v^{-s}\cdot\chi(\sigma_v))^{-1}$, which is uniformly and absolutely convergent on compact subsets of the half–plane $\mathrm{Re}(s) > 1$. It is well known that, if χ is different from the trivial character $\mathbf{1}_G$ of G, then $L_{K/k,S}(s,\chi)$ is holomorphic on the entire complex plane, whereas $L_{K/k,S}(s,\mathbf{1}_G)$ is holomorphic everywhere except for $s = 1$, where it has a pole of order 1. For fixed K/k, S and

χ as above, let $\operatorname{ord}_{s=0} L_{K/k,S}(s,\chi)$ denote the order of vanishing of $L_{K/k,S}(s,\chi)$ at $s=0$. As Tate shows in [18, I], one has

$$(1) \qquad \operatorname{ord}_{s=0} L_{K/k,S}(s,\chi) = \begin{cases} \operatorname{card}\{v \in S \mid \chi(G_v) = \{1\}\}, & \text{if } \chi \neq \mathbf{1}_G \\ \operatorname{card} S - 1, & \text{if } \chi = \mathbf{1}_G\,. \end{cases}$$

As in [18], we combine the abelian Artin L–functions into what we call the Stickelberger function, defined by

$$\Theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_{K/k,S}(s,\chi) \cdot e_{\chi^{-1}}\,.$$

$\Theta_{K/k,S}(s)$ is a meromorphic function in s, with values in the group ring $\mathbf{C}[G]$, holomorphic everywhere except for $s=1$, where it has a pole of order 1.

2. STARK'S QUESTION

Let K/k be an abelian extension of global fields and S a set of primes in k as in §1. Let us assume that the set of data $(K/k, S)$ satisfies the following set of hypotheses.

$$(\mathrm{H}) \qquad \begin{cases} S \text{ contains all the infinite primes.} \\ S \text{ contains all the primes which ramify in } K/k. \\ S \text{ contains at least 2 primes which split completely in } K/k. \\ \operatorname{card} S \geq 3. \end{cases}$$

Let us notice that, according to (1), hypotheses (H) imply that, for all $\chi \in \widehat{G}$, $\operatorname{ord}_{s=0} L_{K/k,S}(s,\chi) \geq 2$, and therefore

$$\Theta^{(2)}_{K/k,S}(0) := \lim_{s\to 0} s^{-2} \cdot \Theta_{K/k,S}(s)$$

makes sense as an element of $\mathbf{C}[G]$.

Let us choose a pair $V = (v_1, v_2)$ of distinct primes in S, which split completely in K/k, and let us fix $W = (w_1, w_2)$, where w_i is a prime in K sitting above v_i, for all $i = 1, 2$. For every $\mathbf{Z}[G]$–module M, let $\wedge^2_G M$ denote its 2–nd exterior power over $\mathbf{Z}[G]$. If R is a commutative ring, let $RM := R \otimes_{\mathbf{Z}} M$, viewed as an $R[G]$–module in the usual manner. As in [13], one can define a $\mathbf{C}[G]$-equivariant regulator map

$$\mathbf{C} \wedge^2_G U_{K,S} \xrightarrow{R_W} \mathbf{C}[G]\ ,$$

by letting $R_W(u_1 \wedge u_2) = \det_{1\leq i,j\leq 2}\left(-\sum_{\sigma\in G} \log|u_j|_{w_i^\sigma} \cdot \sigma\right)$, for $u_1, u_2 \in U_{K,S}$, and then extending to $\mathbf{C} \wedge^2_G U_{K,S}$ by $\mathbf{C}$–linearity.

Following Tangedal [19], Grant [9], and Sands [15], we will now state Stark's Question for abelian L–functions of order of vanishing 2 at $s=0$. We state Stark's Question in a Galois–equivariant form, in the spirit of the more general Conjectures B (see [14]) and C (see [13]) , which deal with abelian L–functions of arbitrary order of vanishing at $s=0$.

Question A (Stark, 1980). *Assume that the set of data $(K/k, S)$ satisfies hypotheses* (H)*. For V and W chosen as above, are there S-units u_1 and u_2 in $U_{K,S}$, such that the following conditions are simultaneously satisfied ?*

(1) $\Theta^{(2)}_{K/k,S}(0) = (1/w_K)^2 \cdot R_W(u_1 \wedge u_2)$.
(2) *The fields $K\left(u_1^{1/w_K}\right)$ and $K\left(u_2^{1/w_K}\right)$ are equal and are Galois, abelian extensions of k.*

Remarks. I. The reader will notice that our formulation of Stark's Question imposes fewer conditions on the S–units u_1 and u_2 than those appearing in [19], [9] and [15]. Namely, we are eliminating the following condition.

(3) *For each $\sigma \in G$, the conjugate u_1^σ generates the same fractional ideal in K as u_1, and u_2^σ generates the same fractional ideal as u_2.*

We chose to eliminate this condition, firstly, because this particular requirement on u_1 and u_2 has been subject to change over the years, secondly, because, as Sands notes in [15], at least as it stands, it is too strong to be expected to be true in general, and thirdly, because **in this paper we will only be concerned with the most important condition** (1) **in the statement of Stark's Question.**

II. In [17], Stark only formulates the above question in the particular case where k is a real quadratic number field and v_1 and v_2 are the two infinite primes in k. Later, Stark indicated that the original statement should be extended to the general situation described in this paper (see [9]), at least in the characteristic 0 situation. The extension to global fields of arbitrary characteristic is very natural and in line with the general philosophy of Stark's Conjectures displayed in [2],[3], [13], [14], and [18].

III. The main theoretical evidence in support of an affirmative answer to Question A comes from work of Tangedal [19], who shows that units u_1 and u_2 satisfying conditions (1)–(3) do exist indeed, if k is a real quadratic field, v_1 and v_2 are the infinite primes of k, K/k is a quadratic extension, and $\operatorname{card} S > 3$. A theoretical link between Question A and the more general Conjecture C was provided by Sands in [15, Theorem 5.2]. Numerical evidence in support of an affirmative answer to Question A comes mainly from work of Grant [9] and Sands [15].

One of the main goals of this paper is to show that, in the case of global function fields, the answer to a much weaker form of Question A, called Question B below, is in general "No". With notations as above, let $\widetilde{\wedge_G^2 U_{K,S}}$ be the image of $\wedge_G^2 U_{K,S}$ via the canonical (not necessarily injective) $\mathbf{Z}[G]$–morphism $\wedge_G^2 U_{K,S} \longrightarrow \mathbf{C} \wedge_G^2 U_{K,S}$.

Question B. *Assume that the set of data $(K/k, S)$ satisfies hypotheses* (H)*. For V and W chosen as above, is there an element ε_S in $(1/w_K)^2 \cdot \widetilde{\wedge_G^2 U_{K,S}}$, such that $\Theta^{(2)}_{K/k,S}(0) = R_W(\varepsilon_S)$?*

Obviously, for any u_1 and u_2 in $U_{K,S}$, we have $u_1 \wedge u_2 \in \widetilde{\wedge_G^2 U_{K,S}}$. However, in general, not every element in $\widetilde{\wedge_G^2 U_{K,S}}$ is of type $u_1 \wedge u_2$, with u_1 and u_2 in $U_{K,S}$. Therefore, if Question B is answered in the negative, then so is Question A.

For purposes which will become clear in §4, we will now state local versions of Questions A and B above. Let ℓ be a prime number and $\mathbf{Z}_{(\ell)} \subseteq \mathbf{Q}$ the localization of $\mathbf{Z}$ at the prime ideal $\ell\mathbf{Z}$.

Question $\mathbf{A}_{(\ell)}(K/k,S)$. *Let ℓ be a prime number. Assume that the set of data $(K/k,S)$ satisfies hypotheses* (H). *For V and W chosen as above, are there S-units $u_{1,\ell}$ and $u_{2,\ell}$ in $U_{K,S}$, and $n_\ell \in \mathbf{Z}_{(\ell)}$, such that the following conditions are simultaneously satisfied ?*

(1) $\Theta^{(2)}_{K/k,S}(0) = \left(n_\ell/w_K^2\right) \cdot R_W(u_{1,\ell} \wedge u_{2,\ell})$.
(2) *The fields $K\left(u_{1,\ell}^{1/w_K}\right)$ and $K\left(u_{2,\ell}^{1/w_K}\right)$ are equal and are Galois, abelian extensions of k.*

Question $\mathbf{B}_{(\ell)}(K/k,S)$. *Let ℓ be a prime number. Assume that the set of data $(K/k,S)$ satisfies hypotheses* (H). *For V and W chosen as above, is there an element $\varepsilon_{S,\ell}$ in $(1/w_K)^2 \cdot \widetilde{\mathbf{Z}_{(\ell)}\wedge^2_G U_{K,S}}$, such that $\Theta^{(2)}_{K/k,S}(0) = R_W(\varepsilon_{S,\ell})$?*

Obviously, if Question A (respectively Question B) is answered in the affirmative, then so is Question $\mathrm{A}_{(\ell)}$ (respectively Question $\mathrm{B}_{(\ell)}$), for all prime numbers ℓ.

3. STARK'S QUESTION FOR FUNCTION FIELDS.

In this section, we will construct a class of examples of sets of data $(K/k,S)$, with $\mathrm{char}\, k > 0$, satisfying hypotheses (H), and show that the answer to Question B (and therefore Question A) is negative in all these cases.

3.1. The extension K/k.

Let p be a prime number, $q := p^\nu$, for some strictly positive integer ν, and $\mathbf{F}_q$ the finite field of cardinality q. Let $k := \mathbf{F}_q(T)$ be the rational function field in one variable T, of exact field of constants $\mathbf{F}_q$. Let $\overline{k}_s$ be a fixed separable closure of k. For any prime v in k, we will denote by $\mathbf{F}_q(v)$ the residue field corresponding to the discrete valuation associated to v. The index $[\mathbf{F}_q(v) : \mathbf{F}_q]$ is called the $\mathbf{F}_q$–residual degree of v. Let v_∞ be the prime of k corresponding to the discrete valuation on k of uniformiser T^{-1}. Let $P_0 \in \mathbf{F}_q[T]$ be an irreducible polynomial of degree p, and v_0 the prime in k corresponding to the discrete valuation on k of uniformiser P_0. Obviously, the $\mathbf{F}_q$–residual degrees of v_∞ and v_0 are respectively 1 and p.

Definition 3.1.1. Let K_1 be the unique degree p constant field extension of k inside $\overline{k}_s$ (i.e. $K_1 := \mathbf{F}_{q^p}(T)$, in this case). Let $G_1 := \mathrm{Gal}(K_1/k)$.

Lemma 3.1.2. *The extension K_1/k defined above satisfies the following.*

(1) $G_1 \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}$.
(2) *K_1/k is unramified everywhere.*
(3) *v_0 splits completely in K_1/k.*

Proof. (1) is an easy consequences of the fact that the field extensions $\mathbf{F}_{q^p}/\mathbf{F}_q$ and $k/\mathbf{F}_q$ are linearly disjoint.

(2) is a consequence of the fact that finite fields have no inseparable extensions of finite degree.

(3) In general, if L is a constant field extension of degree n, of a general characteristic p function field M of exact field of constants $\mathbf{F}_q$, a prime v of M of $\mathbf{F}_q$–residual degree d_v splits in L/M into a product of exactly $\gcd(n, d_v)$ distinct primes (see [12], for example.) Therefore, statement (3) above is a direct consequence of the fact that the $\mathbf{F}_q$–residual degree of v_0 is p. □

Definition 3.1.3. Let K_0^* be the maximal abelian extension of k inside $\overline{k}_s$, of conductor dividing v_0^2, totally split at v_∞. Let $G_0^* := \mathrm{Gal}\,(K_0^*/k)$.

Lemma 3.1.4. *The field extension K_0^*/k defined above satisfies the following.*

(1) *The p–rank of the p–Sylow subgroup of G_0^* is at least* 2.
(2) *v_0 is totally ramified in K_0^*/k.*

Before we begin the proof of Lemma 3.1.4, we will need to make some notations and remind the reader certain general facts on the arithmetic of function fields. For the moment, let us assume that k is a general function field of exact field of constants $\mathbf{F}_q$. Let $\mathrm{Pic}^0\,(k)$ denote the Picard group of k (i.e. the quotient of the group of degree zero divisors on k by the subgroup of principal divisors.) Let J_k be the group of idèles associated to k. For any prime v of k, let U_v denote the group of units in the completion k_v of k with respect to the (normalized) discrete valuation ord_v associated to v. For all integers $i \geq 1$, let $U_v^{(i)}$ be the i–th term of the canonical filtration of U_v with respect to ord_v, explicitly given by

$$U_v^{(i)} := \{x \in U_v \mid \mathrm{ord}_v(x-1) \geq i\}\,.$$

One has an exact sequence of abelian groups

$$1 \longrightarrow J_k^0 \longrightarrow J_k \xrightarrow{\deg_k} \mathbf{Z} \longrightarrow 0\,,$$

where $\deg_k((x_v)_v) := \sum_v [\mathbf{F}_q(v) : \mathbf{F}_q]\cdot \mathrm{ord}_v(x_v)$, for all $(x_v)_v \in J_k$, and J_k^0 is defined to be the kernel of the map $\deg_k$. The group J_k^0 is linked to the Picard group by the following exact sequence

$$1 \longrightarrow k^\times \cdot \prod_v U_v \longrightarrow J_k^0 \xrightarrow{\widehat{\mathrm{div}_k}} \mathrm{Pic}^0(k) \longrightarrow 1\,,$$

where $\widehat{\mathrm{div}_k}((x_v)_v)$ is the class of $\mathrm{div}_k((x_v)_v) := \sum_v \mathrm{ord}_v(x_v)\cdot v$ in $\mathrm{Pic}^0(k)$, for all $(x_v)_v \in J_k^0$.

Proof of Lemma 3.1.4. (1) Due to the fact that, in the particular case under discussion, $\mathrm{Pic}^0(k) = \{0\}$ (as k is a genus 0 function field), and $[\mathbf{F}_q(v_\infty) : \mathbf{F}_q] = 1$, the above exact sequences lead to the equality

$$J_k = k^\times \cdot [k_{v_\infty}^\times \times \prod_{v \neq v_\infty} U_v]\,. \tag{2}$$

The definition of K_0^* and class-field theory, combined with equality (2) above, lead to the following group isomorphisms.

$$\begin{aligned}
G_0^* &\xrightarrow{\sim} J_k \,/\, k^\times \cdot [k_{v_\infty}^\times \times U_{v_0}^{(2)} \times \prod_{v\neq v_\infty, v_0} U_v] \\
&\xrightarrow{\sim} (k^\times \cdot [k_{v_\infty}^\times \times \prod_{v \neq v_\infty} U_v]) \,/\, (k^\times \cdot [k_{v_\infty}^\times \times U_{v_0}^{(2)} \times \prod_{v\neq v_\infty, v_0} U_v]) \\
&\xrightarrow{\sim} U_{v_0} \,/\, (U_{v_0} \cap k^\times \cdot [k_{v_\infty}^\times \times U_{v_0}^{(2)} \times \prod_{v\neq v_\infty, v_0} U_v]) \\
&\xrightarrow{\sim} U_{v_0} \,/\, \mathbf{F}_q^\times \cdot U_{v_0}^{(2)} \xrightarrow{\sim} (\mathbf{F}_{q^p}^\times/\mathbf{F}_q^\times) \times (U_{v_0}^{(1)}/U_{v_0}^{(2)}) \xrightarrow{\sim} (\mathbf{F}_{q^p}^\times/\mathbf{F}_q^\times) \times \mathbf{F}_{q^p}\,.
\end{aligned} \tag{3}$$

The first isomorphism above is the inverse of the Artin reciprocity map, the second is a direct consequence of equality (2), and the third and fourth come from abstract group theory. The fifth and sixth isomorphisms are direct consequences of the fact that one has a (topological) local field isomorphism $k_v \xrightarrow{\sim} \mathbf{F}_{q^n}((\pi_v))$, for any function field k of exact field of constants $\mathbf{F}_q$, and any prime v in k, of $\mathbf{F}_q$–residual degree n and uniformiser $\pi_v \in k$.

The isomorphisms above show that the p–Sylow subgroup of G_0^* is isomorphic to the additive group $\mathbf{F}_{q^p}$, and therefore is an elementary p–group of p–rank equal to $\log_p(q) \cdot p \geq p \geq 2$. This concludes the proof of Lemma 3.1.4 (1).

(2) Class–field theory, combined with the first isomorphism in (3), shows that, v_0 is totally ramified in K_0^*/k if and only if there exists a uniformiser π_{v_0} of v_0 in $(k^\times \cdot [k_{v_\infty}^\times \times U_{v_0}^{(2)} \times \prod_{v \neq v_\infty, v_0} U_v]) \cap k_{v_0}^\times$. This is obviously true for $\pi_{v_0} := P_0$, as the divisor of P_0 is $\operatorname{div}_k(P_0) = v_0 - p \cdot v_\infty$. □

Now, we will finalize the construction of the field extension K/k. Lemma 3.1.4 (1) implies that there exists at least a quotient G_0 of G_0^* such that $G_0 \xrightarrow{\sim} (\mathbf{Z}/p\mathbf{Z})^2$. Let us fix a group G_0 with this property. Galois theory associates G_0 to a unique field K_0, such that $k \subseteq K_0 \subseteq K_0^*$ and $\operatorname{Gal}(K_0/k) \xrightarrow{\sim} G_0$.

Definition 3.1.5. Let K be the compositum $K_1 \cdot K_0$ inside $\overline{k}_s$, where K_1 and K_0 are the subfields of $\overline{k}_s$ constructed above. Let $G := \operatorname{Gal}(K/k)$.

Lemma 3.1.6. *The field extension K/k defined above satisfies the following.*

1. *It is unramified away from v_0.*
2. *One has a group isomorphism $G \xrightarrow{\sim} (\mathbf{Z}/p\mathbf{Z})^3$.*
3. *The decomposition group G_{v_0} of v_0 in K/k is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^2$.*

Proof. (1) is a direct consequence of Lemma 3.1.2 (2) and Definition 3.1.3.

(2) Lemma 3.1.2 (2) and Lemma 3.1.4 (2) imply that the field extensions K_1/k and K_0/k are linearly disjoint inside $\overline{k}_s/k$. Therefore $G \xrightarrow{\sim} G_1 \times G_0 \xrightarrow{\sim} (\mathbf{Z}/p\mathbf{Z})^3$.

(3) Lemma 3.1.2 (2) and Lemma 3.1.4 (2) imply that $G_{v_0} = \operatorname{Gal}(K/K_1)$. Therefore, the linear disjointness of K_1/k and K_0/k gives $G_{v_0} \xrightarrow{\sim} G_0 \xrightarrow{\sim} (\mathbf{Z}/p\mathbf{Z})^2$. □

3.2. The set S.

For K/k defined in the previous section, we will construct a special finite set of primes S in k, such that the set of data $(K/k, S)$ satisfies hypotheses (H). With the same notations as in §3.1, let $S_0 := \{v_0\}$. The construction of S is based in an essential way on the following theorem, whose proof will be given in §3.4.

Theorem 3.2.1. *The p–Sylow subgroup $A_{K,S_0}^{(p)}$ of the S_0–ideal-class group A_{K,S_0} can be generated by two elements as a $\mathbf{Z}_p[G]$–module.*

Let ϖ_1 and ϖ_2 be ideal classes in $A_{K,S_0}^{(p)}$ which generate $A_{K,S_0}^{(p)}$ as a $\mathbf{Z}_p[G]$–module. Chebotarev's Density Theorem implies the existence of two primes w_1 and w_2 in K, simultaneously satisfying the following properties.

1. $w_1 \in \varpi_1$ and $w_2 \in \varpi_2$.
2. w_1 and w_2 do not lie above v_0 (i.e. w_1 , $w_2 \notin S_{0,K}$.)
3. If v_1 and v_2 are the primes in k below w_1 and w_2 respectively, then $v_1 \neq v_2$ and v_1 and v_2 split completely in K/k.

Let us fix two primes w_1 and w_2 in K which satisfy properties (1)–(3) above, and let v_1 and v_2 be the primes in k, sitting below w_1 and w_2 respectively.

Definition 3.2.1. Let $S := \{v_0, v_1, v_2\}$, with v_1 and v_2 defined above, and v_0 defined in §3.1.

Remarks. I. The definition of S implies right away that the set of data $(K/k, S)$ satisfies hypotheses (H).

II. The p–Sylow subgroup $A_{K,S}^{(p)}$ of the S–ideal–class group $A_{K,S}$ is trivial. Indeed, one has an exact sequence of $\mathbf{Z}_p\,[G]$–modules

$$1 \longrightarrow < \widehat{w_1}\,,\,\widehat{w_2} >_{\mathbf{Z}_p[G]} \longrightarrow A_{K,S_0}^{(p)} \longrightarrow A_{K,S}^{(p)} \longrightarrow 1\,,$$

where $< \widehat{w_1}\,,\,\widehat{w_2} >_{\mathbf{Z}_p[G]}$ is the $\mathbf{Z}_p\,[G]$–submodule of $A_{K,S_0}^{(p)}$, generated by the classes $\widehat{w_1}$ and $\widehat{w_2}$ of w_1 and w_2 respectively. However, w_1 and w_2 were chosen so that $< \widehat{w_1}\,,\,\widehat{w_2} >_{\mathbf{Z}_p[G]} = A_{K,S_0}^{(p)}$. Therefore, $A_{K,S}^{(p)} = \{1\}$.

3.3. Questions A and B for the set of data $(K/k, S)$.

The main goal of this section is to show that, for the set of data $(K/k, S)$ constructed in the previous two sections, Question B and, consequently, Question A, have negative answers.

In what follows, if Δ is a finite group and M is a $\mathbf{Z}\,[\Delta]$–module, then $\widehat{\mathrm{H}}^i\,(\Delta, M)$ $\mathrm{H}_i\,(\Delta, M)$ will denote the i–th Tate–cohomology group and respectively the i–th homology group of Δ with coefficients in M. For the moment, let us assume that K/k is an arbitrary, Galois extension of global fields (of arbitrary characteristic) of Galois group G (not necessarily abelian). Let Σ be a finite, non–empty set of primes in k, containing at least all the infinite primes as well as those which ramify in K/k. Y_Σ will denote the free abelian group generated by the set Σ_K of all primes in K sitting above primes in Σ. Since Σ_K is stable under the natural Galois action on primes in K, Y_Σ is endowed with a natural $\mathbf{Z}\,[G]$–module structure. Let X_Σ be the $\mathbf{Z}\,[G]$–submodule of Y_Σ defined by the exact sequence

$$0 \longrightarrow X_\Sigma \longrightarrow Y_\Sigma \xrightarrow{J_\Sigma} \mathbf{Z} \longrightarrow 0\,, \tag{4}$$

where J_Σ is the unique $\mathbf{Z}$–linear map sending every prime $w \in \Sigma_K$ to $1 \in \mathbf{Z}$. Since for every $v \in \Sigma$, and every $w \in \Sigma_K$ sitting above v, the G–stabilizer of w is the decomposition group $G_{w/v}$ of w in K/k, one clearly has $\mathbf{Z}\,[G]$–isomorphisms

$$Y_\Sigma \xrightarrow{\sim} \bigoplus_{v\in\Sigma} \mathbf{Z}[G/G_{w/v}] \xrightarrow{\sim} \bigoplus_{v\in\Sigma} (\mathbf{Z}[G] \otimes_{\mathbf{Z}[G_{w/v}]} \mathbf{Z})\,. \tag{5}$$

For the second and third module involved in the isomorphism above, one chooses a prime w above v, for each v in Σ. However, due to the fact that, for a given v, the groups $G_{w/v}$ are conjugate to one another, the $\mathbf{Z}\,[G]$–isomorphism class of the second and third module does not depend on this choice. For "large" sets Σ, the link between the $\mathbf{Z}\,[G]$–module structure of X_Σ and that of the group of Σ–units $U_{K,\Sigma}$ is given by a result of Tate (see [18, II.5]), whose ℓ–adic version we state below.

Theorem 3.3.1 (Tate). *Assume that K/k and Σ are as above, $G := \mathrm{Gal}(K/k)$, and ℓ is a prime number. Assume that $A^{(\ell)}_{K,\Sigma} = \{1\}$. Then, for all integers i, one has group isomorphisms*

$$\widehat{\mathrm{H}}^{i}\left(G, U_{K,\Sigma}\otimes \mathbf{Z}_\ell\right) \xrightarrow{\sim} \widehat{\mathrm{H}}^{i-2}\left(G, X_\Sigma \otimes \mathbf{Z}_\ell\right).$$

In the above statement, $\mathbf{Z}_\ell$ denotes the ring of ℓ–adic integers and $A^{(\ell)}_{K,\Sigma}$ denotes the ℓ–Sylow subgroup of the Σ–ideal–class group $A_{K,\Sigma}$ of K. Passing from Tate's original theorem "over $\mathbf{Z}$" to its version "over $\mathbf{Z}_\ell$" stated above can be done by tensoring all the exact sequences appearing in the proof of Tate's Theorem with $\mathbf{Z}_\ell$ over $\mathbf{Z}$. One then uses the fact that, since $\mathbf{Z}_\ell$ is a flat $\mathbf{Z}$–module, $*\otimes_{\mathbf{Z}}\mathbf{Z}_\ell$ is an exact functor from the category of $\mathbf{Z}[G]$–modules to the category of $\mathbf{Z}_\ell[G]$–modules, which commutes with the Tate–cohomology functors $\widehat{\mathrm{H}}^{i}(G,*)$.

Theorem 3.3.2. *For the set of data $(K/k, S)$ constructed in §§3.1–3.2, Questions $\mathrm{B}_{(p)}$ and, consequently, B and A, have negative answers.*

Proof. Let $V := (v_1, v_2)$, where v_1 and v_2 are the two totally split primes in S defined in §3.2. Let $W = (w_1, w_2)$, for fixed primes w_1, w_2 in K, sitting above v_1 and v_2 respectively. Let us assume that Question $\mathrm{B}_{(p)}$ has a positive answer. Let $\mathcal{U} = \{u_i^{(k)} \mid k = 1,\dots,n;\ i = 1,2\}$ be a subset of $U_{K,S}$, and $m \in \mathbf{Z}_{(p)}$, such that $\varepsilon_{S,p} := \left(m/w_K^2\right)\sum_{k=1}^n u_1^{(k)}\wedge u_2^{(k)}$, viewed as an element in $\mathbf{C}\wedge^2_G U_{K,S}$, satisfies the regulator condition $\Theta^{(2)}_{K/k,S}(0) = R_W(\varepsilon_{S,p})$ in $\mathbf{C}[G]$. We project this equality on the direct summand $\mathbf{C}[G]\,e_{\mathbf{1}_G} = \mathbf{C}e_{\mathbf{1}_G}$ of $\mathbf{C}[G]$, where $e_{\mathbf{1}_G} := 1/|G|\sum_{\sigma\in G}\sigma := 1/|G|\cdot \mathrm{N}_G$, and $N_G := \sum_{\sigma\in G}\sigma$ is the usual norm element in $\mathbf{C}[G]$. Since v_1 and v_2 are completely split in K/k, this projection leads to the following equality.

$$L^{(2)}_{K/k,S}(0,\mathbf{1}_G)\cdot e_{\mathbf{1}_G} = \left(\left(m/w_K^2\right)\sum_{k=1}^{n}\det_{1\le i,j\le 2}\left(-\log|N_G u_i^{(k)}|_{v_j}\right)\right)\cdot e_{\mathbf{1}_G}\,, \tag{6}$$

where $N_G u_i^{(k)}$ is the image of $u_i^{(k)}$ via the usual norm map $N_G : U_{K,S}\longrightarrow U_{k,S}$, for all i and k. However, equality (1) combined with the fact that $\mathrm{card}\, S = 3$, shows that $\mathrm{ord}_{s=0}L_{K/k,S}(s,\mathbf{1}_G) = 2$ in this case. Therefore, the left–hand side of equality (6) is the leading Taylor coefficient at $s=0$ of the ζ–function with S–Euler factors removed $\zeta_{k,S}(s)$, associated to k. The classical S–class–number formula gives

$$L^{(2)}_{K/k,S}(0,\mathbf{1}_G) = -\frac{1}{w_k}h_{k,S}\cdot R_{k,S}\,,$$

where $h_{k,S}$ is the cardinality of $A_{k,S}$ and $R_{k,S}$ is the classical S–regulator of the free, rank two $\mathbf{Z}$–module $U_{k,S}/\mu_k$. For a subset M of $U_{k,S}$, let $R_{k,S}(M)$ denote the $\mathbf{Z}$–submodule of $\mathbf{C}$ generated by

$$\det\begin{bmatrix}\log|\epsilon_1|_{v_1} & \log|\epsilon_1|_{v_2}\\ \log|\epsilon_2|_{v_1} & \log|\epsilon_2|_{v_2}\end{bmatrix},\ \text{for all } \epsilon_1,\epsilon_2\in M\,.$$

In particular, $R_{k,S}$ is the unique positive generator of $R_{k,S}(U_{k,S})$. The $\mathbf{Z}$–module $R_{k,S}(M)$ is always contained in $\mathbf{Z}\cdot R_{k,S}$, and it is non–zero if and only if the subgroup $\mathbf{Z}M$ of $U_{k,S}$ generated by M has finite index in $U_{k,S}$. Assuming that $\mathbf{Z}M$ has finite index in $U_{k,S}$, one has the following equality.

$$[\mathbf{Z}\cdot R_{k,S} : R_{k,S}(M)] = \frac{[U_{k,S} : \mathbf{Z}M]}{[\mu_k : \mu_k \cap \mathbf{Z}M]}\,.$$

Let $N_G\mathcal{U} := \{N_G u \mid u \in \mathcal{U}\}$. Equality (6) divided by $R_{k,S}$ and combined with the S–class–number formula, implies firstly that $R_{k,S}\,(N_G\mathcal{U}) \neq \{0\}$, and secondly that

$$\mathbf{Z}\cdot \frac{w_K^2}{w_k} h_{k,S} \subseteq \mathbf{Z}\cdot \frac{m\cdot [U_{k,S} : \mathbf{Z}\,(N_G\mathcal{U})]}{[\mu_k : \mu_k \cap \mathbf{Z}\,(N_G\mathcal{U})]} \subseteq \mathbf{Z}\cdot \frac{m\cdot |\widehat{\mathrm{H}}^0\,(G, U_{K,S})\,|}{[\mu_k : \mu_k \cap \mathbf{Z}\,(N_G\mathcal{U})]}\,.$$

The second inclusion above is a direct consequence of the equality $\widehat{\mathrm{H}}^0\,(G, U_{K,S}) = U_{k,S}/N_G U_{K,S}$ and inclusion $\mathbf{Z}\,(N_G\mathcal{U}) \subseteq N_G U_{K,S}$. Since $\operatorname{char} K = p$, we have $\mu_K \otimes \mathbf{Z}_p = \{1\}$. Therefore, if we tensor the double inclusion above by the ring of p–adic integers $\mathbf{Z}_p$, and keep in mind that $m \in \mathbf{Z}_{(p)}$, we obtain

$$h_{k,S}^{(p)} \geq |\widehat{\mathrm{H}}^0\,(G, U_{K,S}\otimes \mathbf{Z}_p)\,|\,, \tag{7}$$

where $h_{k,S}^{(p)} := |A_{k,S}^{(p)}|$. The main idea of the proof of Theorem 3.3.2 is to show that, under our working hypotheses, inequality (7) does not hold true.

We will first find an upper bound for $h_{k,S}^{(p)}$. Assume for the moment that k is any characteristic p global field, of exact field of constants $\mathbf{F}_q$. Also, assume that $S_0 = \{v_0\}$ and S are finite sets of primes in k, such that $S_0 \subseteq S$. Then, one has an exact sequence of groups

$$0 \longrightarrow \mathrm{Pic}^0(k) \longrightarrow A_{k,S_0} \xrightarrow{\ \deg_{v_0}\ } \mathbf{Z}/[\mathbf{F}_q(v_0) : \mathbf{F}_q]\cdot\mathbf{Z} \longrightarrow 0\,,$$

where $\deg_{v_0}$ is the $\mathbf{Z}$–linear map which takes the S_0–ideal-class $\widehat{v}$ of a prime v into $[\mathbf{F}_q(v) : \mathbf{F}_q] \bmod [\mathbf{F}_q(v_0) : \mathbf{F}_q]$ (see [12], for example). The inclusion $S_0 \subseteq S$ also induces a natural surjective group–morphism $A_{k,S_0} \twoheadrightarrow A_{k,S}$. However, in our case, $\mathrm{Pic}^0(k) = \{0\}$. Also, $[\mathbf{F}_q(v_0) : \mathbf{F}_q] = p$ (see §3.1). Therefore,

$$A_{k,S_0}^{(p)} \xrightarrow{\ \sim\ } \mathbf{Z}/p\mathbf{Z}\,, \text{ and } h_{k,S}^{(p)} \leq p\,. \tag{8}$$

Next, we will find a lower bound for $|\widehat{\mathrm{H}}^0\,(G, U_{K,S}\otimes\mathbf{Z}_p)\,|$. Remark II in §3.2 shows that, under our working hypotheses, Tate's Theorem 3.3.1 is applicable to $\Sigma := S$ and $\ell := p$. We therefore obtain group isomorphisms

$$\widehat{\mathrm{H}}^i\,(G, U_{K,S}\otimes\mathbf{Z}_p) \xrightarrow{\ \sim\ } \widehat{\mathrm{H}}^{i-2}\,(G, X_S\otimes\mathbf{Z}_p)\ ,\ \text{for all integers } i.$$

In particular, for $i = 0$, we have

$$|\widehat{\mathrm{H}}^0\,(G, U_{K,S}\otimes\mathbf{Z}_p)\,| = |\widehat{\mathrm{H}}^{-2}\,(G, X_S\otimes\mathbf{Z}_p)\,| = |\mathrm{H}_1\,(G, X_S\otimes\mathbf{Z}_p)\,|\,, \tag{9}$$

where the second equality above is a consequence of the definition of Tate cohomology groups at negative levels. Let us now recall that $S = \{v_0, v_1, v_2\}$, with v_1

and v_2 completely split in K/k, and therefore $G_{v_1} = G_{v_2} = \{1\}$. This shows that, if we tensor with $\mathbf{Z}_p$ the exact sequence (4), for $\Sigma := S$, and use isomorphism (5), we obtain the following exact sequence of $\mathbf{Z}_p[G]$–modules.

$$0 \longrightarrow X_S \otimes \mathbf{Z}_p \longrightarrow (\mathbf{Z}_p[G] \otimes_{\mathbf{Z}_p[G_{v_0}]} \mathbf{Z}_p) \oplus \mathbf{Z}_p[G]^2 \longrightarrow \mathbf{Z}_p \longrightarrow 0\,.$$

Next, we write the long exact sequence of homology groups corresponding to the above short exact sequence of G–modules. If we use Shapiro's Lemma for computing the homology groups of the middle term, we obtain an exact sequence

$$\cdots \longrightarrow \mathrm{H}_2(G_{v_0}, \mathbf{Z}_p) \longrightarrow \mathrm{H}_2(G, \mathbf{Z}_p) \longrightarrow \mathrm{H}_1(G, X_S \otimes \mathbf{Z}_p) \longrightarrow \cdots\,.$$

Theorem 6.4(iii) of [1] implies that, for any abelian group H, we have a canonical group isomorphism $\wedge^2_{\mathbf{Z}_p}(H \otimes \mathbf{Z}_p) \xrightarrow{\sim} \mathrm{H}_2(H, \mathbf{Z}_p)$. This result, combined with Lemma 3.1.6 (2) and (3), yields isomorphisms $\mathbf{Z}/p\mathbf{Z} \xrightarrow{\sim} \mathrm{H}_2(G_{v_0}, \mathbf{Z}_p)$ and respectively $(\mathbf{Z}/p\mathbf{Z})^3 \xrightarrow{\sim} \mathrm{H}_2(G, \mathbf{Z}_p)$. Therefore, the long exact sequence above becomes

$$\cdots \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow (\mathbf{Z}/p\mathbf{Z})^3 \longrightarrow \mathrm{H}_1(G, X_S \otimes \mathbf{Z}_p) \longrightarrow \cdots\,.$$

This sequence, combined with equality (9), shows that

$$|\widehat{\mathrm{H}}^0(G, U_{K,S} \otimes \mathbf{Z}_p)| = |\mathrm{H}_1(G, X_S \otimes \mathbf{Z}_p)| \geq p^2\,. \tag{10}$$

Inequalities (10), (8), and (7) lead to a contradiction ($p \geq p^2$). □

3.4. The proof of Theorem 3.2.1.

Let us assume for the moment that L/k is a finite, abelian extension of global fields (of arbitrary characteristic), of Galois group $G(L/k)$. Let Σ be a finite, non–empty set of primes in k, containing at least all the infinite primes and those which ramify in L/k. Let ℓ be a prime number. We will denote by $L_\Sigma^{(\ell)}$ the maximal abelian extension of L, of ℓ–power degree, unramified everywhere, and totally split at all the primes in Σ_L. The maximality of the field $L_\Sigma^{(\ell)}$ forces it to be a Galois extension of the base field k. This gives $G(L_\Sigma^{(\ell)}/L)$ a natural $\mathbf{Z}[G(L/k)]$–module structure, with $\sigma \in G(L/k)$ acting on $x \in G(L_\Sigma^{(\ell)}/L)$ by "lift and conjugation" (i.e. $\sigma * x := \overline{\sigma} x \overline{\sigma}^{-1}$, where $\overline{\sigma}$ is an arbitrary lift of σ to $G(L_\Sigma^{(\ell)}/k)$.) Class–field theory shows that the usual Artin map gives a group isomorphism

$$A_{L,\Sigma}^{(\ell)} \xrightarrow{\sim} G(L_\Sigma^{(\ell)}/L)\,, \tag{11}$$

which is also a $\mathbf{Z}[G(L/k)]$–isomorphism. In what follows, if H is a group, then $[H\,,\,H]$ denotes its commutator subgroup. Also, I_H denotes the usual augmentation ideal of $\mathbf{Z}[H]$. We will start by proving a group–theoretic lemma.

Lemma 3.4.1. *Let us assume that we have an exact sequence of groups*

$$1 \longrightarrow A \longrightarrow \mathcal{G} \xrightarrow{\pi} G \longrightarrow 1\,,$$

with A and G abelian and A normal in $\mathcal{G}$. Let us assume that π has a set–theoretic section $s : G \longrightarrow \mathcal{G}$, such that $s(x) \cdot s(y) = s(y) \cdot s(x)$, for all x, y in G. Then, if *one views A as a $\mathbf{Z}[G]$–module via the usual "lift–and–conjugation" G–action, we have an equality $[\mathcal{G}\,,\,\mathcal{G}] = I_G \cdot A$.*

Proof. In what follows, $[g,h] := ghg^{-1}h^{-1}$ denotes the usual commutator of two elements g and h in $\mathcal{G}$. It is very easy to show that $I_G \cdot A \subseteq [\mathcal{G}\,,\,\mathcal{G}]$. Indeed, let $g \in G$ and $a \in A$. Then, the definition of the G–action on A (call it $*$, say) implies that $(g-1)*a = (g*a)a^{-1} = s(g)as(g)^{-1}a^{-1} = [s(g),a]$, which is an element of $[\mathcal{G}\,,\,\mathcal{G}]$. Since I_G is generated over $\mathbf{Z}$ by elements of the type $g-1$, with $g \in G$, this concludes the proof of the above mentioned inclusion.

Let α, β be two elements in $\mathcal{G}$. Let $x, y \in G$, and $a, b \in A$, such that $\alpha = s(x)a$ and $\beta = s(y)b$. Since $s(y)^{-1}s(x)^{-1} = s(x)^{-1}s(y)^{-1}$, we have

$$\begin{aligned}[\alpha,\beta] =& s(x)as(y)ba^{-1}s(x)^{-1}b^{-1}s(y)^{-1} = \{s(x)as(x)^{-1}\}\cdot \\ &\cdot \{s(x)s(y)bs(y)^{-1}s(x)^{-1}\}\cdot\{s(y)s(x)a^{-1}s(x)^{-1}s(y)^{-1}\}\cdot\{s(y)b^{-1}s(y)^{-1}\}\,.\end{aligned}$$

Let us denote by m, n, p, and q respectively the elements appearing inside braces to the right of the second equality above. Since A is normal in $\mathcal{G}$, we have $m, n, p, q \in A$. Since A is assumed to be abelian and $a, b \in A$, the equalities above give

$$\begin{aligned}[\alpha,\beta] &= \{ma^{-1}\}\cdot\{nb^{-1}\}\cdot\{pa\}\cdot\{qb\} = \\ &= [s(x),a]\cdot[s(x)s(y),b]\cdot[s(y)s(x),a^{-1}]\cdot[s(y),b^{-1}]\,.\end{aligned}$$

Let us now recall that s is a section of π. Therefore, there exists an element $\mu \in A$, such that $s(x)s(y) = s(xy)\mu$. Since A is abelian, this implies that $[s(x)s(y),b] = [s(xy)\mu,b] = [s(xy),b]$ and $[s(y)s(x),a^{-1}] = [s(yx),a^{-1}]$. We obtain

$$\begin{aligned}[\alpha,\beta] &= [s(x),a]\cdot[s(xy),b]\cdot[s(yx),a^{-1}]\cdot[s(y),b] \\ &= \{(x-1)*a\}\cdot\{(xy-1)*b\}\cdot\{(yx-1)*a^{-1}\}\cdot\{(y-1)*b^{-1}\}\,.\end{aligned}$$

This shows that $[\alpha,\beta] \in I_G \cdot A$, which concludes the proof of Lemma 3.4.1. □

Corollary 3.4.2. *Assume that we have an exact sequence of groups*

$$1 \longrightarrow A \longrightarrow \mathcal{G} \xrightarrow{\pi} G \longrightarrow 1\,,$$

with A and G abelian and A normal in $\mathcal{G}$. Assume that (1) or (2) below are satisfied.

(1) *G is cyclic.*
(2) *The exact sequence above is split. (i.e. $\mathcal{G}$ is the semi–direct product $A \rtimes G$.)*

Then, we have an equality $[\mathcal{G}\,,\,\mathcal{G}] = I_G \cdot A$.

Proof. It is very easy to check that if either one of the conditions above is satisfied, one can construct a set–theoretic section s for π, such that $s(x)s(y) = s(y)s(x)$, for all $x, y \in G$. (Under condition (2), one can actually find a group morphism section s. Such a section satisfies the commutativity property automatically, since G is assumed to be abelian.) The corollary is then a consequence of Lemma 3.4.1. □

We now return to the notations and constructions described in §§3.1-3.2.

Proposition 3.4.3. *There is a group isomorphism*

$$\mathrm{H}_0(G_0, A^{(p)}_{K_0,S_0}) \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}\,.$$

Moreover, $A^{(p)}_{K_0,S_0}$ *is a cyclic* $\mathbf{Z}_p[G_0]$*–module.*

Proof. Firstly, since $k^{(p)}_{S_0}/k$ is split at v_0 and K/k is totally ramified at v_0, they are linearly disjoint extensions. This implies that

$$[G(K^{(p)}_{0,S_0}/k)\,,\,G(K^{(p)}_{0,S_0}/k)] = G(K^{(p)}_{0,S_0}/K_0\cdot k^{(p)}_{S_0})\,,\;G(K_0\cdot k^{(p)}_{S_0}/K_0) \xrightarrow{\sim} G(k^{(p)}_{S_0}/k)\,. \tag{12}$$

Secondly, since K_0/k is totally ramified at v_0 and unramified everywhere else, the following exact sequence is split.

$$1 \longrightarrow G(K^{(p)}_{0,S_0}/K_0) \longrightarrow G(K^{(p)}_{0,S_0}/k) \xrightarrow{\pi} G_0 \longrightarrow 1\,.$$

More precisely, if $\overline{w_0}$ is a prime above v_0 in $K^{(p)}_{0,S_0}$, then π induces an isomorphism between the inertia group of $\overline{w_0}$ in $K^{(p)}_{0,S_0}/k$ and G_0. The inverse of this isomorphism is a group–theoretic section of π, which makes the exact sequence above split. Corollary 3.4.2, applied to the exact sequence above, gives

$$[G(K^{(p)}_{0,S_0}/k)\,,\,G(K^{(p)}_{0,S_0}/k)] = I_{G_0}\cdot G(K^{(p)}_{0,S_0}/K_0)\,.$$

Now, if we combine the equality above with (12) and the $\mathbf{Z}_p[G_0]$–isomorphisms $A^{(p)}_{K_0,S_0} \xrightarrow{\sim} G(K^{(p)}_{0,S_0}/K_0)$, $A^{(p)}_{k,S_0} \xrightarrow{\sim} G(k^{(p)}_{S_0}/k)$, we obtain

$$\begin{aligned}&\mathrm{H}_0(G_0, A^{(p)}_{K_0,S_0}) := A^{(p)}_{K_0,S_0}/I_{G_0}A^{(p)}_{K_0,S_0} \xrightarrow{\sim} G(K^{(p)}_{0,S_0}/K_0)/I_{G_0}G(K^{(p)}_{0,S_0}/K_0) \xrightarrow{\sim}\\ &G(K^{(p)}_{0,S_0}/K_0)/G(K^{(p)}_{0,S_0}/K_0\cdot k^{(p)}_{S_0}) \xrightarrow{\sim} G(k^{(p)}_{S_0}/k) \xrightarrow{\sim} A^{(p)}_{k,S_0} \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}\,.\end{aligned}$$

The last isomorphism above is isomorphism (8), §3.3. This concludes the proof of the first part of Proposition 3.4.3.

Now, let us notice that, since G_0 is a p–group, $\mathbf{Z}_p[G_0]$ is a Noetherian, local ring of maximal ideal $\mathcal{M}_{p,G_0} := p\mathbf{Z}_p[G_0] + I_{G_0}\cdot\mathbf{Z}_p[G_0]$. Since $I_{G_0}\subseteq \mathcal{M}_{p,G_0}$, we have a canonical surjective $\mathbf{Z}_p[G_0]$–morphism

$$A^{(p)}_{K_0,S_0}/I_{G_0}\cdot A^{(p)}_{K_0,S_0} \twoheadrightarrow A^{(p)}_{K_0,S_0}/\mathcal{M}_{p,G_0}\cdot A^{(p)}_{K_0,S_0}\,.$$

Since the left–hand side of the above morphism is $\mathbf{Z}_p$–cyclic, the right–hand side is $\mathbf{Z}_p[G_0]$–cyclic. Nakayama's Lemma implies that $A^{(p)}_{K_0,S_0}$ is $\mathbf{Z}_p[G_0]$–cyclic. □

Proof of Theorem 3.2.1. Firstly, let us notice that K/K_0 is totally split at the unique prime $\widetilde{w_0}$ of K_0 which sits above v_0. This implies on one hand that $K\subseteq K^{(p)}_{0,S_0}$, and on the other hand that

$$[G(K^{(p)}_{S_0}/K)\,,\,G(K^{(p)}_{S_0}/K)] = G(K^{(p)}_{S_0}/K^{(p)}_{0,S_0})\,. \tag{13}$$

In order to simplify notations, we identify $G(K/K_0)$ with G_1, $G(K/K_1)$ with G_0, and G with $G_1\times G_0$, via the usual restriction isomorphisms. Since G_1 is cyclic, the exact sequence

$$1\longrightarrow G(K_{S_0}^{(p)}/K)\longrightarrow G(K_{S_0}^{(p)}/K_0)\longrightarrow G_1\longrightarrow 1$$

satisfies the hypotheses in Corollary 3.4.2. Therefore, the following holds.

$$[G(K_{S_0}^{(p)}/K)\,,\,G(K_{S_0}^{(p)}/K)]=I_{G_1}\cdot G(K_{S_0}^{(p)}/K)\,. \tag{14}$$

Now, we combine the $\mathbf{Z}_p[G]$–isomorphisms $A_{K_0,S_0}^{(p)}\xrightarrow{\sim}G(K_{0,S_0}^{(p)}/K_0)$ and $A_{K,S_0}^{(p)}\xrightarrow{\sim}G(K_{S_0}^{(p)}/K)$ with (13) and (14) above, and the exact sequence of Galois groups

$$1\longrightarrow G(K_{0,S_0}^{(p)}/K)\longrightarrow G(K_{0,S_0}^{(p)}/K_0)\longrightarrow G_1\longrightarrow 1\,.$$

We obtain the following exact sequence of $\mathbf{Z}_p[G_0]$–modules

$$1\longrightarrow A_{K,S_0}^{(p)}/I_{G_1}\cdot A_{K,S_0}^{(p)}\longrightarrow A_{K_0,S_0}^{(p)}\longrightarrow \mathbf{Z}/p\mathbf{Z}\longrightarrow 1\,.$$

Here, G_0 acts trivially on $\mathbf{Z}/p\mathbf{Z}$. Next, we write down the long exact G_0–homology sequence associated to the last short exact sequence.

$$\cdots\longrightarrow \mathrm{H}_1(G_0,\mathbf{Z}/p\mathbf{Z})\longrightarrow \mathrm{H}_0(G_0,A_{K,S_0}^{(p)}/I_{G_1}A_{K,S_0}^{(p)})\longrightarrow$$
$$\longrightarrow \mathrm{H}_0(G_0,A_{K_0,S_0}^{(p)})\longrightarrow \mathrm{H}_0(G_0,\mathbf{Z}/p\mathbf{Z})\longrightarrow 0\,.$$

However, since G_0 acts trivially on $\mathbf{Z}/p\mathbf{Z}$, we have $H_0(G_0,\mathbf{Z}/p\mathbf{Z})=\mathbf{Z}/p\mathbf{Z}$. Therefore, Proposition 3.4.3 shows that the last non–trivial morphism in the long exact sequence above is in fact an isomorphism. Now, if we take into account that $\mathrm{H}_1(G_0,\mathbf{Z}/p\mathbf{Z})\xrightarrow{\sim}G_0\otimes\mathbf{Z}/p\mathbf{Z}\xrightarrow{\sim}(\mathbf{Z}/p\mathbf{Z})^2$ (see [1]), the long exact sequence above yields the following surjective group–morphism.

$$(\mathbf{Z}/p\mathbf{Z})^2\twoheadrightarrow A_{K,S_0}^{(p)}/(I_{G_1},I_{G_0})A_{K,S_0}^{(p)}\,,$$

where $(I_{G_1},I_{G_0}):=I_{G_1}\mathbf{Z}_p[G]+I_{G_0}\mathbf{Z}_p[G]$. Now, since G is a p–group, $\mathbf{Z}_p[G]$ is a Noetherian, local ring, of maximal ideal $\mathcal{M}_{p,G}:=I_G\mathbf{Z}_p[G]+p\mathbf{Z}_p[G]$. Since $(I_{G_1},I_{G_0})\subseteq\mathcal{M}_{p,G}$, the surjective morphism above induces a surjective morphism

$$(\mathbf{Z}/p\mathbf{Z})^2\twoheadrightarrow A_{K,S_0}^{(p)}/\mathcal{M}_{p,G}A_{K,S_0}^{(p)}\,.$$

Therefore, $A_{K,S_0}^{(p)}/\mathcal{M}_{p,G}A_{K,S_0}^{(p)}$ can be generated by two elements as a $\mathbf{Z}_p[G]$–module. Nakayama's Lemma implies that $A_{K,S_0}^{(p)}$ is generated by two elements as a $\mathbf{Z}_p[G]$–module. □

4. A STRONG FORM OF BRUMER'S CONJECTURE.

This section has two goals. Firstly, we will provide links between Questions A and B and a strong form of Brumer's Conjecture, for arbitrary global fields. Secondly, as a consequence of the results proved in §3 above, we will show that the strong form of Brumer's Conjecture is in general false in characteristic $p>0$.

4.1. The statement.

Let K/k be an abelian extension of global fields of Galois group G. Let S_0 be a finite, nonempty set of primes in k, containing at least all the infinite primes, as well as all the primes which ramify in K/k. Let $\mathcal{A}(K/k) := \mathrm{Ann}_{\mathbf{Z}[G]}(\mu_K)$ be the annihilator of the $\mathbf{Z}[G]$–module μ_K of roots of unity in K. The following remarkable integrality result was proved independently by Deligne–Ribet [7] and Barsky–Cassou-Nogues [5] in the number field case, and Deligne (see [18, V]) and Hayes [11] in the function field case.

Theorem 4.1.1. *If $\alpha \in \mathcal{A}(K/k)$, then $\alpha \cdot \Theta_{K/k,S_0}(0) \in \mathbf{Z}[G]$.*

We are now ready to state Brumer's Conjecture, which is an attempt to generalize the classical theorem of Stickelberger.

Conjecture 4.1.2 (Brumer). *One has an inclusion of $\mathbf{Z}[G]$–ideals*

$$\mathcal{A}(K/k) \cdot \Theta_{K/k,S_0}(0) \subseteq \mathrm{Ann}_{\mathbf{Z}[G]}(A_{K,S_0}) \,.$$

Equivalently, for all primes ℓ, one has an inclusion of $\mathbf{Z}_{(\ell)}[G]$–ideals

$$\mathbf{Z}_{(\ell)}\mathcal{A}(K/k) \cdot \Theta_{K/k,S_0}(0) \subseteq \mathrm{Ann}_{\mathbf{Z}_{(\ell)}[G]}(A_{K,S_0}^{(\ell)}) \,.$$

In the case of function fields, Brumer's Conjecture was proved independently and with different methods by Deligne [18, V] and Hayes [11]. In the case of number fields, this conjecture is far from being proved. The statement has been known to hold true for a long time if $k = \mathbf{Q}$, as a result of the classical theorem of Stickelberger (see [20]). Wiles developed a series of results and techniques in [21], which lead to a proof of the conjecture above in the case where K is a CM–field, k is totally real, $\ell \nmid |G|$, and S_0 satisfies extra–hypotheses. Finally, by using the techniques developed in [21], Greither proves the conjecture above, for a very special class of CM extensions K of totally real fields k, under the assumption that $\ell \neq 2$ (see [10]).

If R is a Noetherian, commutative ring, and M is a finitely generated R–module, we will denote by $\mathrm{Fitt}_R(M)$ the R–Fitting ideal of M. For the definition and properties of Fitting ideals used in this paper, we refer the reader to [12]. We will only recall here the fact that we always have an inclusion $\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M)$, and the equality is very rare. (Equality happens if, for instance, M is a cyclic R–module.) We are now ready to state the strong form of Brumer's Conjecture.

Statement SBr$(K/k, S_0)$. *The following inclusion of $\mathbf{Z}[G]$–ideals holds true.*

$$\mathcal{A}(K/k) \cdot \Theta_{K/k,S_0}(0) \subseteq \mathrm{Fitt}_{\mathbf{Z}[G]}(A_{K,S_0}) \,.$$

For any prime ℓ, one can formulate the following ℓ–adic version of Statement SBr.

Statement SBr$_{(\ell)}(K/k, S_0)$. *The following inclusion of $\mathbf{Z}_{(\ell)}[G]$–ideals holds true.*

$$\mathbf{Z}_{(\ell)}\mathcal{A}(K/k) \cdot \Theta_{K/k,S_0}(0) \subseteq \mathrm{Fitt}_{\mathbf{Z}_{(\ell)}[G]}(A_{K,S_0}^{(\ell)}) \,.$$

Since the Fitting ideal behaves nicely with respect to extensions of scalars and direct sums (see [12]), and since $A_{K,S_0} = \bigoplus_\ell A_{K,S_0}^{(\ell)}$, Statement SBr is true if and only if Statement SBr$_{(\ell)}$ is true, for all prime numbers ℓ.

Remark. The motivation for introducing the stronger form SBr of Brumer's Conjecture is threefold. **Firstly,** evidence in support of Statement SBr has been found by various researchers over the years. In the case of function fields, we proved that a statement even stronger than SBr holds true for constant field extensions (see Theorem 4.2.9 in [12]). We also showed that statements $\mathrm{SBr}_{(\ell)}$ hold true for general function field extensions K/k, provided that $\ell \nmid |G|$ (see Theorem 3.1.1 in [12]). In the case of CM–extensions K of totally real number fields k, Greither showed in [10] that the "minus–part" of Statement $\mathrm{SBr}_{(\ell)}(K/k, S_0)$ holds true, provided that $\ell \neq 2$ and K/k satisfies extra–properties of cohomological type. Also, our close analysis of [21] revealed that Wiles's techniques lead to a proof of Statement $\mathrm{SBr}_{(\ell)}(K/k, S_0)$, for CM extensions K of totally real fields k, under his extra hypotheses on S_0, for primes $\ell \nmid |G|$.

Secondly, Statement SBr has very interesting links to the versions "over $\mathbf{Z}$" of Stark's Conjecture for abelian L–functions of arbitrary order of vanishing at $s = 0$, formulated by Rubin in [14] and the present author in [13]. We show in [12, §2] that, in the case of function fields for example, a slightly stronger version of statement $\mathrm{SBr}(K/k, S_0)$ implies Rubin's Conjecture B. A similar argument can be used to show that the same strong version of $\mathrm{SBr}(K/k, S_0)$ also implies our Conjecture C. Also, as the interested reader will see in §4.2 below, Statement SBr has connections to Questions A and B, formulated in §2 above.

Thirdly, the methods employed so far by various researchers in their attempts to prove Brumer's Conjecture seem to lead more naturally to Fitting ideals than annihilators of ideal–class groups. More precisely, the main techniques employed in [21] and [10] rely in an essential way on the Main Conjecture in Iwasawa Theory. The Main Conjecture is a statement linking values of L–functions to the Fitting ideal of a certain Λ–module, obtained as a projective limit of ideal–class groups, where Λ is the Iwasawa algebra associated to the cyclotomic $\mathbf{Z}_p$–extension. Wiles [21] and Greither [10] "project" the Main Conjecture to finite levels of the $\mathbf{Z}_p$–extension and obtain statements about Fitting ideals of ideal–class groups, in the spirit of Statement SBr. Finally, in the case of function fields, we use in [12] the Weil–Grothendieck interpretations of global L–functions in terms of characteristic polynomials of the action of a Frobenius morphism on various ℓ–adic étale and crystalline cohomology groups, to prove statements of type $\mathrm{SBr}_{(\ell)}$, in the cases mentioned above. Once again, the use of characteristic polynomials (and therefore determinants) leads naturally to Fitting ideals rather than annihilators.

4.2. Links between Questions A and B and Statement SBr.

In this section, we provide links between Statement $\mathrm{SBr}_{(\ell)}$ and Question $\mathrm{B}_{(\ell)}$ and respectively $\mathrm{A}_{(\ell)}$, for a given prime ℓ. As a consequence, we will show that statements $\mathrm{SBr}_{(p)}$ and, consequently, SBr are in general false in characteristic p. We are still working under the hypotheses and notations of §4.1. In particular, our global fields are of arbitrary characteristic.

Proposition 4.2.1. *Let ℓ be a prime number. Let $S = S_0 \cup \{v_1, v_2\}$, with v_1, v_2 distinct primes in k, split in K/k, and not belonging to S_0. Assume the following.*

1. *$A_{K,S}^{(\ell)}$ is trivial.*
2. *Statement $\mathrm{SBr}_{(\ell)}(K/k, S_0)$ is true.*

Then, Question $\mathrm{B}_\ell(K/k, S)$ has an affirmative answer.

Proof. Let $W = (w_1, w_2)$, with w_1 and w_2 primes in K, sitting above v_1 and v_2 respectively. For $i = 1, 2$, let $\widehat{w_i}^{(\ell)}$ be the projection of the class $\widehat{w_i}$ of w_i in A_{K,S_0}, onto the ℓ–Sylow component $A_{K,S_0}^{(\ell)}$ of A_{K,S_0}. Assumption (1) above shows that $\widehat{w_1}^{(\ell)}$ and $\widehat{w_2}^{(\ell)}$ generate the $\mathbf{Z}_{(\ell)}[G]$–module $A_{K,S_0}^{(\ell)}$ (see Remark II, §3.2 as well). Therefore, assumption (2) above, combined with the definition of the Fitting ideal, shows that there exist 2×2 matrices $A_k = (a_{ij}^k)_{1\leq i,j\leq 2}$, for $k = 1, \dots, n$, with entries a_{ij}^k in $\mathbf{Z}_{(\ell)}[G]$, such that the following hold true.

(*i*) $a_{i1}^k \cdot \widehat{w_1}^{(\ell)} + a_{i2}^k \cdot \widehat{w_2}^{(\ell)} = 0$ in $A_{K,S_0}^{(\ell)}$, for all $k = 1, \dots, n$, and $i = 1, 2$.

(*ii*) $w_K \cdot \Theta_{K/k,S_0}(0) = \sum_{k=1}^{n} \det(A_k)$.

It is easy to see that, since A_{K,S_0} is finite, (*i*) above implies that one can find $\beta \in \mathbf{Z}$, such that $\mathrm{ord}_\ell(\beta) = 0$ (i.e. $\beta^{-1} \in \mathbf{Z}_{(\ell)}$), $\beta \cdot a_{ij}^k \in \mathbf{Z}$, for all i, j, k, and

(*i'*) $\beta a_{i1}^k \cdot \widehat{w_1} + \beta a_{i2}^k \cdot \widehat{w_2} = 0$ in A_{K,S_0}, for all $k = 1, \dots, n$, and $i = 1, 2$.

Let us fix β satisfying the above properties. Then (*i'*) implies the existence of S–units $u_1^{(k)}, u_2^{(k)} \in U_{K,S}$, for $k = 1, \dots, n$, such that the following hold true.

$$\sum_{\sigma\in G} \mathrm{ord}_{w_j^\sigma}(u_i^{(k)}) \cdot \sigma = \beta \cdot a_{ij}^k\,, \text{ for all } i,\ j, \text{ and } k. \tag{15}$$

Since v_1 and v_2 are finite primes, completely split in K/k, we have $\Theta_{K/k,S}^{(2)}(s) = (1-\mathrm{N}v_1^{-s})(1-\mathrm{N}v_2^{-s})\Theta_{K/k,S_0}(s)$. Therefore, $\Theta_{K/k,S}^{(2)}(0) = \log(\mathrm{N}v_1)\log(\mathrm{N}v_2)\Theta_{K/k,S_0}(0)$. If we combine this equality, with (15), (*ii*) above, and the definition of the regulator R_W, we obtain:

$$\Theta_{K/k,S}^{(2)}(0) = R_W(\varepsilon_{S,\ell})\,,$$

where $\varepsilon_{S,\ell} := (\beta^{-1}/w_K)\sum_{k=1}^n u_1^{(k)} \wedge u_2^{(k)}$ is clearly in $(1/w_K)^2 \cdot \mathbf{Z}_{(\ell)}\widetilde{\wedge_G^2 U_{K,S}}$. This shows that Question $\mathrm{B}_\ell(K/k, S)$ indeed has an affirmative answer. □

Corollary 4.2.2. *Let K/k be the extension of global fields of char. p and S_0 the set of primes in k constructed in §§3.1-3.2. Then, statements $\mathrm{SBr}_{(p)}(K/k, S_0)$ and (consequently) $\mathrm{SBr}(K/k, S_0)$ are false.*

Proof. It is obviously sufficient to show that Statement $\mathrm{SBr}_{(p)}(K/k, S_0)$ is false. Let $S = S_0 \cup \{v_1, v_2\}$ be the set of primes in k defined in Definition 3.2.1. Then Remark II, §3.2, shows that the hypotheses in Proposition 4.2.1 are satisfied by K/k, S, and the prime $\ell := p$. On the other hand, Theorem 3.3.2 shows that Question $\mathrm{B}_p(K/k, S)$ has a negative answer. Therefore, Proposition 4.2.1 implies that Statement $\mathrm{SBr}_{(p)}(K/k, S_0)$ is false. □

Remark. At Victor Kolyvagin's suggestion, we also searched for counter–examples for the ℓ–part of the Strong Brumer Conjecture $\mathrm{SBr}_{(\ell)}(K/k, S_0)$, in the case where $\mathrm{char}(k) = p > 0$ and $\ell \neq p$. We would like to report here that we have found an infinite class of such counter–examples. As the techniques involved in dealing with such counter–examples are quite different from the ones developed in this paper, the detailed constructions will appear elsewhere.

Obviously, a proof almost identical to that of Proposition 4.2.1 leads to the following "global" link between Question B and Statement SBr.

Proposition 4.2.3. *Let $S := S_0 \cup \{v_1, v_2\}$, with v_1, v_2 distinct primes in k, split in K/k, and not belonging to S_0. Assume the following.*

(1) *$A_{K,S}$ is trivial.*
(2) *Statement $\mathrm{SBr}(K/k, S_0)$ is true.*

Then, Question $\mathrm{B}(K/k, S)$ has an affirmative answer.

Proof. Similar to the proof of Proposition 4.2.1. Left to the reader. □

Next, we establish links between Question A and Statement SBr. We remind the reader that, if G is a finite group and M is a $\mathbf{Z}[G]$–module, we say that M is G–cohomologically trivial if $\widehat{\mathrm{H}}^i(H, M) = 0$, for all subgroups H of G, and all $i \in \mathbf{Z}$.

Proposition 4.2.4. *Let ℓ be a prime number. Let $S = S_0 \cup \{v_1, v_2\}$, with v_1, v_2 distinct primes in k, split in K/k, and not belonging to S_0. Assume the following.*

(1) *$A_{K,S}^{(\ell)}$ is trivial.*
(2) *$A_{K,S_0}^{(\ell)}$ is G–cohomologically trivial.*
(3) *Statement $\mathrm{SBr}_{(\ell)}(K/k, S_0)$ is true.*

Then, Question $\mathrm{A}_{(\ell)}(K/k, S)$ has an affirmative answer.

We will need two lemmas of purely algebraic nature.

Lemma 4.2.5. *Let R be a commutative, semi–local, Noetherian ring, and P a finitely generated, projective R–module. Assume that the local ranks $\mathrm{rk}_{R_\mathfrak{m}}(P_\mathfrak{m})$ are independent of the maximal ideal $\mathfrak{m}$ of R. Then, P is a free R–module of rank equal to the local ranks.*

Proof. See Exercise 4.13* on page 137 of [8]. □

Lemma 4.2.6. (compare to Proposition 4 in [6]) *Let R denote a commutative, semi–local, Noetherian ring, and let $Q(R)$ be its total ring of fractions. Let M be a finitely generated R–module, such that $M \otimes_R Q(R) = 0$. Assume that the projective dimension of M over R is at most 1. Then, the following hold true.*

(1) *$\mathrm{Fitt}_R(M)$ is a principal ideal, generated by a non-zero divisor of R.*
(2) *Let $X = \{x_1, \dots, x_n\} \subseteq M$ be a fixed set of generators for the R–module M. Then, one can choose a generator for $\mathrm{Fitt}_R(M)$ of the form $\det(A)$, where $A = (a_{ij})_{1 \le i,j \le n}$ is an $n \times n$ matrix with entries in R, such that $\sum_{j=1}^n a_{ij} \cdot x_j = 0$ in M, for all $i = 1, \dots, n$.*

Proof. We will start by noting that, since R is Noetherian (and therefore the ideal (0) admits a primary decomposition in R), $Q(R)$ can be written as a direct sum of local Artin rings $Q(R) = \oplus_\mathfrak{p} R_\mathfrak{p}$. Here $R_\mathfrak{p}$ denotes the localization of R at the prime ideal $\mathfrak{p}$, and $\mathfrak{p}$ runs through the (finite) set of minimal prime ideals of R. This observation also shows that the condition $M \otimes_R Q(R) = 0$ is equivalent to $M_\mathfrak{p} = 0$ or, equivalently, $\mathrm{Ann}_R(M) \not\subseteq \mathfrak{p}$, for all minimal prime ideals $\mathfrak{p}$ of R.

We will now prove (1) and (2) simultaneously. For the fixed set of generators X of M, one has an exact sequence of R–modules

$$0 \longrightarrow K \xrightarrow{i} R^n \xrightarrow{\pi} M \longrightarrow 0\,,$$

where π sends the elements of an ordered canonical basis $\mathcal{E} := (e_1, \dots, e_n)$ of R^n into $x_1, x_2, \dots, x_n$ respectively, $K := \ker(\pi)$, and i is the inclusion map. Since

$\mathrm{pd}_R(M) \leq 1$, K is a projective R–module, and therefore locally free. Let $\mathfrak{m}$ be a maximal ideal of R, and $\alpha_{\mathfrak{m}} := \mathrm{rk}_{R_{\mathfrak{m}}}(K_{\mathfrak{m}})$. If we localize the exact sequence above at $\mathfrak{m}$, we obtain the following exact sequence of $R_{\mathfrak{m}}$–modules.

$$0 \longrightarrow R_{\mathfrak{m}}^{\alpha_{\mathfrak{m}}} \longrightarrow R_{\mathfrak{m}}^n \xrightarrow{\pi_{\mathfrak{m}}} M_{\mathfrak{m}} \longrightarrow 0\,.$$

However, there always exists a minimal prime ideal $\mathfrak{p}$ of R, such that $\mathfrak{p} \subseteq \mathfrak{m}$. If we localize the above exact sequence even further, at $\mathfrak{p}$, and take into account that $M_{\mathfrak{p}} = 0$, we obtain an isomorphism of $R_{\mathfrak{p}}$–modules $R_{\mathfrak{p}}^{\alpha_{\mathfrak{m}}} \xrightarrow{\sim} R_{\mathfrak{p}}^n$. This shows that $\alpha_{\mathfrak{m}} = n$ (see [8] Corollary 4.4(b)). We now apply Lemma 4.2.5 to the R–module K to conclude that K is a free R–module of rank n. Let us fix an ordered R–basis $\mathcal{K} := (k_1, \cdots, k_n)$ for K, and let $A = (a_{ij})_{1\leq i,j\leq n}$ be the matrix representation of the R–morphism i with respect to bases $\mathcal{K}$ and $\mathcal{E}$. Since i is injective, $\det(A)$ is a non–zero divisor in R. On the other hand, since $K \xrightarrow{\sim} R^n$, the definition of the Fitting ideal shows that $\mathrm{Fitt}_R(M) = \det(A)R$. □

Proof of Proposition 4.2.4. Let w_1, w_2, W, $\widehat{w_i}$, and $\widehat{w_i}^{(\ell)}$, for $i = 1, 2$, be as in the proof of Proposition 4.2.1. We will apply Lemma 4.2.6 to the semi–local ring $R := \mathbf{Z}_{(\ell)}[G]$, the finitely generated R–module $M := A_{K,S_0}^{(\ell)}$ and its set of R–generators $X := \{\widehat{w_1}^{(\ell)}, \widehat{w_2}^{(\ell)}\}$. Since M is finite and $Q(R) = \mathbf{Q}[G]$, we clearly have $M \otimes_R Q(R) = 0$. Also, since M is G–cohomologically trivial, we have $\mathrm{pd}_R(M) \leq 1$ (see [4], IV.9, Theorem 9). According to Lemma 4.2.6, we can choose a matrix $A = (a_{ij})_{1\leq i,j\leq 2}$ with entries in R, such that $\mathrm{Fitt}_R(M) = \det(A)\cdot R$, $\det(A)$ is a non–zero divisor in R, and $\sum_{j=1}^2 a_{ij}\cdot \widehat{w_j}^{(\ell)} = 0$ in M, for all $i = 1, 2$. Moreover, if we multiply the entries of A by a suitably chosen $\beta \in \mathbf{Z}_{(\ell)}^\times$, as in the proof of Proposition 4.2.2, we can further assume that A has in fact entries in $\mathbf{Z}[G]$, and $\sum_{j=1}^2 a_{ij}\cdot \widehat{w_j} = 0$ in A_{K,S_0}, for all $i = 1, 2$. As in the proof of Proposition 4.2.1, these equalities imply the existence of two S–units $u_1', u_2' \in U_{K,S}$, such that

$$R_W(u_1' \wedge u_2') = \log(\mathrm{N}v_1)\log(\mathrm{N}v_2)\det(A)\,. \tag{16}$$

Now, hypothesis (3) in Proposition 4.2.4, combined with the fact that $\det(A)$ is a non-zero divisor, shows that, for a suitably chosen $\beta' \in \mathbf{Z}_{(\ell)}^\times$, there exists a set $\{\xi_\alpha | \alpha \in \mathcal{A}(K/k)\} \subseteq \mathbf{Z}[G]$, satisfying

$$\beta'\cdot\alpha\cdot\Theta_{K/k,S_0}(0) = \xi_\alpha\cdot\det(A), \text{ and } \xi_\alpha = (\alpha/w_K)\cdot\xi_{w_K}, \text{ for all } \alpha\in\mathcal{A}(K/k)\,. \tag{17}$$

For a unit $u \in U_{K,S}$, let $\tilde{u}$ denote its image in $\mathbf{Q}U_{K,S}$ via the canonical $\mathbf{Z}[G]$–morphism $U_{K,S} \longrightarrow \mathbf{Q}U_{K,S}$. Let $\varepsilon := \widetilde{u_1'}^{(\xi_{w_K}/w_K)} \in \mathbf{Q}U_{K,S}$. Let $\varepsilon_\alpha := {u_1'}^{\xi_\alpha} \in U_{K,S}$, for all $\alpha \in \mathbf{Z}[G]$. Equalities (17) imply that the following hold true.

1. $\varepsilon^\alpha = \widetilde{\varepsilon_\alpha}$, for all $\alpha \in \mathcal{A}(K/k)$.
2. $\varepsilon_\alpha^\gamma = \varepsilon_\gamma^\alpha$, for all $\alpha, \gamma \in \mathcal{A}(K/k)$.

We combine the last equalities with Proposition 1.2 in [13], to conclude that there exists a unit $u_1 \in U_{K,S}$, such that $\varepsilon = \widetilde{u_1}^{1/w_K}$ in $\mathbf{Q}U_{K,S}$, and $K(u_1^{1/w_K})/k$ is a Galois abelian extension. Let $u_2 := {u_2'}^{w_K}\cdot u_1 \in U_{K,S}$. Then, clearly one has an equality of fields $K(u_1^{1/w_K}) = K(u_2^{1/w_K})$. Moreover, (16) and (17) show that

$$\Theta_{K/k,S}^{(2)} = \log(\mathrm{N}v_1)\log(\mathrm{N}v_2)\Theta_{K/k,S_0}(0) = ({\beta'}^{-1}/w_K^2)R_W(u_1\wedge u_2)\,.$$

This shows that, indeed, Question $\mathrm{A}_{(\ell)}(K/k, S)$ has an affirmative answer. □

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