
MR1691163 (2000m:11115)
Popescu, Cristian D. (1-TX)
On a refined Stark conjecture for function fields. (English summary)
Compositio Math. **116** (1999), *no. 3,* 321–367.
11R58 (11R27 11R29 11R42 19F27)

FEATURED REVIEW.

The zeta-function $\zeta_K(s)$ of a number field K has a zero of order $r(K)=r_1+r_2-1$ at $s=0$, r_1 and r_2 being respectively the number of real and complex places of K. In fact, Dedekind's evaluation of the residue of $\zeta_K(s)$ at $s=1$ together with the functional equation implies $\zeta_K(s)=-(h_KR_K/w_K)s^{r(K)}+O(s^{r_1+r_2})$, where h_K, R_K, and w_K are the class number, the regulator, and the number of roots of unity of K. The appearance of R_K in this equality is of particular interest because R_K is an $r(K)\times r(K)$ determinant of logarithms of absolute values of units $\varepsilon\in K$. The leading term in the Taylor expansion of $\zeta_K(s)$ at $s=0$ therefore encapsulates information about elements of K.

Let K/k be an abelian extension of number fields with Galois group G. By class field theory, K/k is determined by arithmetic invariants in the ring of integers of k. However, the proofs of class field theory do not provide efficient algorithms for constructing K out of these invariants. Hilbert asked in Problem #12 of his famous list if special values of analytic functions, defined by arithmetic information in completions of k, might generate K. If this is possible in the completion k_v at a place v of k, then we expect $K\subset k_v$, and so v should split in K/k.

The identity $\zeta_K(s)=\prod_{\chi\in\widehat{G}}L(s,\chi)$ factors the zeta-function of K as a product of L-functions that are defined over k by the same arithmetic invariants that determine K as a class field. In a series of papers [Advances in Math. **7** (1971), 301–343 (1971); MR0289429 (44 #6620); Advances in Math. **17** (1975), no. 1, 60–92; MR0382194 (52 #3082); Advances in Math. **22** (1976), no. 1, 64–84; MR0437501 (55 #10427); Adv. in Math. **35** (1980), no. 3, 197–235; MR0563924 (81f:10054)], H. Stark proposed conjectures detailing how the leading terms of the Taylor series of these L-functions at $s=0$ encapsulate information about S-units of K, S being any finite set of k-places containing the Archimedean places and all places that ramify in K/k. The conjectures apply most naturally to the imprimitive functions $\zeta_{K,S}(s)$ and $L_S(s,\chi)$ obtained by removing Euler factors over S. If $r(\chi)$ is the order of vanishing of $L_S(s,\chi)$ at $s=0$, then $L_S(s,\chi) = L(\chi)\cdot s^{r(\chi)} + O(s^{r(\chi)+1})$ defines a complex number $L(\chi)\neq 0$. Loosely speaking, Stark constructed an $r(\chi)\times r(\chi)$ determinant of $\mathbf{Q}$-linear forms in $\log|\varepsilon|_w$, the ε's being S-units of K and the w's in the set S_K of K-places dividing a place in S, and he conjectured that $L(\chi)/R(\chi)\in \mathbf{Q}(\chi)$, the subfield of $\mathbf{C}$ generated by the values of χ.

When the minimal value of the $r(\chi)$ is unity and S contains a distinguished split place v, Stark formulated a precise conjecture that identifies the linear form $R(\chi)$ in terms of an S-unit $\varepsilon_v\in K$ and its conjugates under G. Stark identified the denominator in $\mathbf{Q}(\chi)$ as w_K, and he conjectured that $K(\varepsilon^{1/w_K})/k$ is abelian. This "first order zero" conjecture implies an algorithm, currently implemented in PARI, for constructing K when k is totally real. J. Tate [J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 963–978 (1982); MR0656067 (83m:12018a)] referred to this conjecture as defined "over $\mathbf{Z}$" because it provides a precise denominator. Deligne [see J. T. Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s=0$*, Lecture notes edited by Dominique Bernardi and Norbert Schappacher, Progr. Math., 47, Birkhäuser Boston, Boston, MA, 1984; MR0782485 (86e:11112)(Chapter V)] applied the l-adic cohomological interpretation of L-functions to prove this conjecture in function fields.

Suppose $v_1,\cdots,v_r$ are split places in S and suppose T is an auxiliary set of finite places disjoint from S. Let $U_{S,T}$ be the group of S-units that are congruent to unity modulo the places in T. Inspired in part by work of B. H. Gross [J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 1, 177–197; MR0931448 (89h:11071)], K. Rubin [Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 33–62; MR1385509 (97d:11174)] formulated a general conjecture "over $\mathbf{Z}$" in terms of a Galois invariant regulator map $R\colon \Lambda_{S,T}\to \mathbf{R}[G]$, where $\Lambda_{S,T}$ is a lattice in $\mathbf{Q}\otimes\bigwedge^r_{\mathbf{Z}[G]}U_{S,T}$ containing $\bigwedge^r_{\mathbf{Z}[G]}U_{S,T}$ with $[\Lambda_{S,T}\colon \bigwedge^r_{\mathbf{Z}[G]}U_{S,T}]$ a divisor of $g=|G|$. Classical results on cyclotomic units and Gauss sums provide evidence for this conjecture when $k=\mathbf{Q}$, but only "up to primes dividing g", i.e., after tensoring with $\mathbf{Z}[1/g]$. The intervention of the lattice $\Lambda_{S,T}$ is a subtlety related to the fact that the Gras conjectures are not valid, in general, at primes dividing g.

In the paper under review, Popescu proves Rubin's conjecture in function fields over $\mathbf{F}_q$ up to primes dividing g. He invokes the l-adic étale cohomology for every prime l, as well as crystalline p-adic cohomology at the characteristic p of $\mathbf{F}_q$. Popescu proceeds by first proving a strong form of the conjecture over $\mathbf{Z}[1/g]$ for the case $r=0$. Using functorial properties, he then finds a unique element in $\mathbf{Z}[1/g]\mathrm{Fitt}_{\mathbf{Z}[G]}[A_{S,T}]\cdot\Lambda_{S,T}$ satisfying the conjecture for any $r\geq 0$. Here, $A_{S,T}$ is the S-class group of K trivialized along T. His techniques involve tensoring the l-adic [resp. crystalline] cohomology with $\mathbf{C}_l$ [resp. $\mathbf{C}_p$] and then decomposing into χ-components. When K/k is a constant field extension, Popescu proves a stronger form of the full conjecture.

These results provide the first extensive non-classical evidence for Rubin's conjecture for arbitrary r.

{See also the following review [MR1711315 (2000m:11116)].}

Reviewed by *David R. Hayes*

[References]

1. Berthelot, P.: *Cohomologie cristalline des schemas de characteristique* $p¿0$, Lecture Notes in Math. 407, Springer-Verlag, Berlin, 1974. MR0384804 (52 #5676)
2. Berthelot, P. and Ogus, A.: *Notes on Crystalline Cohomology*, Math. Notes 21, Princeton University Press, Princeton, NJ, 1978. MR0491705 (58 #10908)
3. Crew, R.: Geometric Iwasawa theory and a conjecture of Katz, *Canad. Math. Soc. Conf. Proc.* **7** (1987), 37–53. MR0894316 (88f:14016)
4. Gross, B.: On the values of Abelian L-functions at $s = 0$, *J. Fac. Sci. Univ. Tokyo* **59** (1988), 177–197. MR0931448 (89h:11071)
5. Hayes, D. R.: Explicit class field theory in global function fields, *Studies in Algebra and Number Theory*, Academic Press, New York, 1979, pp. 173–217. MR0535766 (81d:12011)
6. Hayes, D. R.: Stickelberger elements in function fields, *Compositio Math.* **55** (1985), 209–239. MR0795715 (87d:11091)
7. Hartshorne, R.: *Algebraic Geometry*, Springer-Verlag, Berlin, 1977. MR0463157 (57 #3116)
8. Katz, N. M. and Messing, W.: Some consequences of the Riemann hypothesis for varieties over finite fields, *Invent. Math.* **23** (1974), 73–77. MR0332791 (48 #11117)
9. Mazur, B. and Wiles, A.: Class fields of abelian extensions of Q, *Invent. Math.* **76** (1984), 179–330. MR0742853 (85m:11069)
10. Milne, J.: *Étale Cohomology*, Princeton Math. Series 33, Princeton University Press, Princeton, 1980. MR0559531 (81j:14002)
11. Moreno, C.: *Algebraic Curves over Finite Fields*, Cambridge University Press, 1991. MR1101140 (92d:11066)
12. Popescu, C. D.: On a refined stark conjecture for function fields, PhD Thesis, Ohio State University, 1996.
13. Popescu, C. D.: Gras-type conjectures for function fields, to appear in *Compositio Math.* MR1711315 (2000m:11116)
14. Rubin, K.: A Stark conjecture 'over **Z**' for Abelian L-functions with multiple zeros, *Ann. Inst. Fourier* **46** (1996). MR1385509 (97d:11174)
15. Stark, H.: L-functions at $s = 1$, I, II, III, IV, *Adv. in Math.* **7** (1971), 301–343; **17** (1975), 60–92; **22** (1976), 64–84; **35** (1980), 197–235. MR0289429 (44 #6620)
16. Tate, J.: On Stark's conjecture on the behaviour of $L(s, \chi)$ at $s = 0$, *J. Fac. Sci. Univ. Tokyo* **28** (1982), 963–978. MR0656067 (83m:12018a)
17. Tate, J.: *Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$*, Progr. in Math. 47, Birkhaüser, Boston, 1984. MR0782485 (86e:11112)