Math 201a Fall 2023 Exercises Set 2

October 20, 2023

As usual, feedback on these optional problems is welcome.

1. Check that the following two ways of thinking about what it means for functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ to be adjoint are equivalent:

(i) There exists a family of isomorphisms $\Phi_{M,N} : \mathrm{Hom}_{\mathcal{D}}(FM, N) \to \mathrm{Hom}_{\mathcal{C}}(M, GN)$, natural in $M \in \mathcal{C}$ and $N \in \mathcal{D}$.

(ii) There are morphisms of functors $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ such that the compositions $F \xrightarrow{F(\eta_-)} F \circ G \circ F \xrightarrow{\epsilon_{F(-)}} F$ and $G \xrightarrow{\eta_{G(-)}} G \circ F \circ G \xrightarrow{G(\epsilon_-)} G$ are the identity morphisms of functors.

2. $(\mathcal{C}, \otimes, a, \mathbb{1}, l, r)$ be a monoidal category.

(a) Let $\mathrm{Skel}\,\mathcal{C}$ be any skeleton of $\mathcal{C}$, a full subcategory containing exactly one object from each isomorphism class of $\mathcal{C}$ (we can assume that $\mathbb{1} \in \mathrm{Skel}\,\mathcal{C}$.) Show that there is a monoidal structure $(\mathrm{Skel}\,\mathcal{C}, \otimes', a', \mathbb{1}, l, r)$ such that the inclusion functor $i : \mathrm{Skel}\,\mathcal{C} \to \mathcal{C}$ is a monoidal equivalence. Show that the unique quasi-inverse $j : \mathcal{C} \to \mathrm{Skel}\,\mathcal{C}$ is also naturally a monoidal functor.

(Hint: for objects $M, N \in \mathrm{Skel}\,\mathcal{C}$, define $M \otimes' N$ to be the unique object in $\mathrm{Skel}\,\mathcal{C}$ which is isomorphic to $M \otimes N$, and fix arbitrarily an isomorphism $J_{M,N} : M \otimes N \to M \otimes' N$ for each M, N. There is a unique way to define the action of $\otimes'$ on morphisms and the associativity constraints a', by using these isomorphisms J, so that $(i, J, 1_{\mathbb{1}})$ is a monoidal equivalence.

(b) Suppose we have an arbitrary monoidal equivalence

$$(F, J, \phi) : (\mathcal{C}, \otimes, a, \mathbb{1}, l, r) \to (\mathcal{D}, \otimes', a', \mathbb{1}', l', r').$$

By definition this means a monoidal functor which is an equivalence of categories. Show that there is a quasi-inverse G such that (G, J', ϕ') is also a monoidal equivalence.

(Hint: use part (a) to reduce to the case of a skeletal monoidal category).

3. Consider the category $\mathcal{D} = \mathrm{Fun}(\mathcal{C}, \mathcal{C})$, whose objects are functors and the morphisms are natural transformations. We saw that $(\mathcal{D}, \circ, a, \mathrm{id}_{\mathcal{C}}, l, r)$ is monoidal, where the product is the composition of functors, and a, l, r can be taken to be actual equalities.

(a) Show that a left dual to $G \in \mathcal{D}$ is a left adjoint F to G, and a right dual to G is a right adjoint H to G. Thus an object in $\mathcal{D}$ is rigid if and only if it has both left and right adjoints.

(b) Show that G is an invertible object in $\mathcal{C}$ if and only if G is an equivalence of categories from $\mathcal{C} \to \mathcal{C}$.

4. In class, for any monoidal category $(\mathcal{C}, \otimes, a, \mathbb{1}, l, r)$ we defined a functor $F : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ by $F(M) = M \otimes -$ and claimed that (F, J, ϕ) is monoidal, where $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ has the monoidal structure as in problem 3, and $J_{M,N} : F(M) \otimes F(N) \to F(M \otimes N)$ is given by the associativity constraint $a^{-1}_{M,N,-} : M \otimes (N \otimes -) \to (M \otimes N) \otimes -$, and ϕ is given by l^{-1}. Check the details carefully that this is a monoidal functor. What adjustments need to be made in order for $G(M) = - \otimes M$ to also naturally be a monoidal functor?

4. $(\mathcal{C}, \otimes, a, \mathbb{1}, l, r)$ be a monoidal category. Prove some of the following facts about duals we stated without proof in class:

(a) If X has a left dual X^*, then the functors $(- \otimes X, - \otimes X^*)$ form an adjoint pair.

(b) For any morphism $f : X \to Y$, where X and Y have left duals, we defined the dual morphism $f^* : Y^* \to X^*$. Show that if also $g : Y \to Z$ and X, Y, Z have left duals then $(g \circ f)^* = f^* \circ g^*$.

(c) If X and Y have left duals, then $X \otimes Y$ has a left dual and $(X \otimes Y)^* \cong Y^* \otimes X^*$.

5. Let R be a commutative ring. Consider $(\mathcal{C} = R\text{-Mod}, \otimes_R, a, \mathbb{1} = R, l, r)$ where a, l, r are the canonical isomorphisms.

(a) Suppose that $M \in \mathcal{C}$. Show that if M has a left dual M^* or a right dual ${}^*M$, then M is a projective module. (Hint: Hom/Tensor adjointness shows that $(- \otimes M, \mathrm{Hom}_R(M, -))$ is an adjoint pair of functors $\mathcal{C} \to \mathcal{C}$. If M has a left dual M^*, then $(- \otimes M, - \otimes M^*)$ is also an adjoint pair.)

(b) If M has a left dual M^*, then M is finitely generated.

(c) If M is a finitely generated projective module, then show that $\mathrm{Hom}_R(M, R)$ is both a right and left dual to M. (Hint: to define $\mathrm{coev} : R \to M \otimes_R \mathrm{Hom}_R(M, R)$, choose a surjective module map $f : R^n \to M$ for some n; since M is projective there is a splitting $g : M \to R^n$. Define $m_i = f(e_i)$ for the standard free basis $\{e_i\}$ of R^n and let $g_i = p_i \circ g : M \to R$ for each i, where $p_i : R^n \to R$ is the ith projection. Now define $\mathrm{coev}(r) = \sum_{i=1}^n r m_i \otimes g_i$.)

6. In problem set 1 #8, you showed that if $\mathcal{C}$ is a category with products and a terminal object T, then you can use the product $\prod$ to define a monoidal category $(\mathcal{C}, \prod, a, \mathbb{1} = T, l, r)$ for the canonical isomorphism a and where l and r are projections.

Show that the only object that has a left (or right) dual in this category is the unit object $\mathbb{1} = T$.

(Of course, an analogous result holds for the monoidal category with product given by coproduct and unit object being the initial object.)

7. let G be a group and k a field. Recall from lecture on 10/20 that a *projective representation* of G over k with *Schur multiplier* ψ is a k-vector space V and a function $\rho : G \to \mathrm{GL}_k(V)$ such that $\rho_g \circ \rho_h = \psi(g,h)\rho_{gh}$ for all g, h, for some scalars $\psi(g,h) \in k^\times$. We showed that this implies that $\psi : G \times G \to k^\times$ must be a 2-cocycle, i.e. $\psi(gh,k)\psi(g,h) = \psi(g,hk)\psi(h,k)$ for all g, h, k. That is, $\psi \in Z^2(G, k^\times)$ in the notation of group cohomlogy.

Now for fixed ψ let $\mathrm{Rep}_\psi(G)$ be the set of projective G-representations with Schur multiplier ψ. As mentioned in class, $\mathrm{Rep}_\psi(G)$ is naturally a left $\mathrm{Rep}(G)$-module category, where for $V \in \mathrm{Rep}(G)$ and $W \in \mathrm{Rep}_\psi(G)$, we define $V \otimes W = V \otimes_k W$ with diagonal action $g(v \otimes w) = gv \otimes gw$ as usual; it is easy to check that $V \otimes_k W \in \mathrm{Rep}_\psi(G)$ again.

Now consider two 2-cocycles ψ and ϕ. Show that if $\phi\psi^{-1} \in B^2(G, k^\times)$ is a 2-coboundary, that is there exists a function $\theta : G \to k^\times$ such that $\phi(g,h)\psi(g,h)^{-1} = \theta(g)\theta(gh)^{-1}\theta(h)$ for all g, h, then there is a $\mathrm{Rep}(G)$-module equivalence $F : \mathrm{Rep}_\psi(G) \to \mathrm{Rep}_\phi(G)$.