# Examples of Generic Noncommutative Surfaces


by

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For my parents

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# CHAPTER I

# Introduction

## 1.1 Overview

The interaction between the fields of algebraic geometry and commutative ring theory has long been fruitful for both subjects. The very foundations of modern algebraic geometry as developed in Hartshorne's classic text [18], for example, rest heavily on commutative algebra. In the opposite direction, geometric techniques have often been indispensable to the proofs of algebraic theorems. The basic idea of noncommutative algebraic geometry is to try to generalize aspects of the correspondence between geometry and commutative rings to the noncommutative realm. This general notion is not particularly new, but only comparatively recently there has been great success in mimicking some of the more global methods of projective algebraic geometry to produce a geometric theory for noncommutative *graded* rings in particular.

This new subject is known as *noncommutative projective geometry*, and while of theoretical interest in its own right, it has also provided the solutions to many purely ring-theoretic questions. For example, the graded domains of dimension two, which correspond to noncommutative curves, have now been completely classified [3]. The noncommutative analogs of the projective plane $\mathbb{P}^2$ have been identified and classified as well. The generic noncommutative projective plane is called the *Sklyanin algebra*; despite its simple presentation by generators and relations, before the geometric approach of [5],[6] was developed it was not even known that this algebra was noetherian.

The classification theory of graded algebras of dimension three, or noncommutative projective surfaces, has also progressed substantially in recent years; see [29]. In this paper, we study a class of algebras of dimension at least three and show that they provide counterexamples to a number of open questions in the literature. In particular, these give new examples of noncommutative surfaces with much different behavior than any of those studied previously. Moreover, the rings $R$ that are the subject of our study have a simple and general construction: they are "generic" subalgebras of some very nice rings—*twisted homogeneous coordinate rings* of projective space—which have similar properties to commutative polynomial rings.

We shall prove that any such ring $R$ is noetherian, but that the noetherian property fails when one changes the base field to some larger commutative ring, answering a question in [2]. These rings also have unusual properties with respect to homological algebra. Even though $R$ is a *maximal order* in its ring of fractions, it satisfies some but not all of the $\chi$ *conditions*, a set of homological conditions on a ring which are hypotheses for a number of important theorems in noncommutative geometry. The existence of a ring with such behavior answers questions in [30] and [29]. Despite being a counterexample to all of these open questions, $R$ is not overly pathological and appears to have interesting geometric properties.

In the remainder of this introduction, we will give a more detailed and leisurely exposition of the subject of noncommutative geometry and state precisely our main results. We will try to keep technical definitions to a minimum, deferring them as much as possible to §2.1.

## 1.2 Noncommutative projective schemes

All rings in this introduction will be $\mathbb{N}$-graded algebras $A = \bigoplus_{i=0}^{\infty} A_i$ over an algebraically closed field $k$. We assume always that $A$ is *connected*, that is that $A_0 = k$, and finitely generated as a $k$-algebra. Let $A$-gr be the category of all noetherian $\mathbb{Z}$-graded left $A$-modules, with morphisms the homomorphisms preserving degree. Also, let $A$-qgr be the quotient category of $A$-gr by the full subcategory of modules with finite $k$-dimension (see §2.1 below for more details about quotient categories). For $M, N \in A$-gr, $\underline{\mathrm{Ext}}^i(M, N)$ will be notation for the Ext group calculated in the *ungraded* category.

We briefly review some commutative algebraic geometry. The necessary background on scheme theory may be found in [18, Chapter II]. Let $X$ be a projective scheme over a field $k$, with category of coherent sheaves $\operatorname{coh} X$. Let $\mathcal{L}$ be an invertible sheaf on $X$. From this data one may construct a graded "coordinate ring" $B = B(X, \mathcal{L}) = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})$, where $\mathrm{H}^0$ is the global sections functor. The multiplication is defined in the obvious way using the natural map

$$\mathrm{H}^0(\mathcal{L}^{\otimes m}) \otimes_k \mathrm{H}^0(\mathcal{L}^{\otimes n}) \longrightarrow \mathrm{H}^0(\mathcal{L}^{\otimes m} \otimes \mathcal{L}^{\otimes n}) = \mathrm{H}^0(\mathcal{L}^{\otimes m+n})$$

for all $m, n \geq 0$.

Recall that an invertible sheaf $\mathcal{L} \in \operatorname{coh} X$ is called *ample* if, given any $\mathcal{F} \in \operatorname{coh} X$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections for $n \gg 0$. The following result due to Serre is fundamental.

**Theorem 1.2.1.** *(Serre's Theorem) Let $X$ be a projective scheme over $k$ with $\mathcal{L}$ an ample invertible sheaf, and let $B = B(X, \mathcal{L})$. Then*

*(1) There is an equivalence of categories* $\operatorname{coh} X \sim B$-qgr.

*(2) If $X = \operatorname{proj} A$ for some connected $\mathbb{N}$-graded algebra $A$ which is finitely generated in degree 1, and $\mathcal{L} = \mathcal{O}(1)$ is the twisting sheaf on $X$, then $A$ and $B$ are isomorphic up to a finite dimensional vector space. In particular,* $\operatorname{coh}(\operatorname{proj} A) \sim A\text{-qgr}$. $\square$

There is a remarkable noncommutative generalization of the entire framework we have just summarized, the theory of *twisted homogeneous coordinate rings*. Suppose that $\varphi$ is an automorphism of the projective scheme $X$. For a sheaf $\mathcal{F} \in \operatorname{coh} X$, we use the notation $\mathcal{F}^{\varphi}$ for the pullback $\varphi^*\mathcal{F}$. For any invertible sheaf $\mathcal{L}$, we define

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^{\varphi} \otimes \mathcal{L}^{\varphi^2} \otimes \cdots \otimes \mathcal{L}^{\varphi^{n-1}}. \tag{1.2.2}$$

Then the *twisted homogeneous coordinate ring* $B = B(X, \mathcal{L}, \varphi)$ is given by $B = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}_n)$. To define the multiplication on $B$, note that $\mathcal{L}_m \otimes \mathcal{L}_n^{\varphi^m} \cong \mathcal{L}_{n+m}$, and that there is a natural map (in fact isomorphism) $\mathrm{H}^0(\mathcal{L}_n) \to \mathrm{H}^0(\mathcal{L}_n^{\varphi^m})$, so that altogether there is a multiplication map

$$B_m \otimes B_n = \mathrm{H}^0(\mathcal{L}_m) \otimes_k \mathrm{H}^0(\mathcal{L}_n) \to \mathrm{H}^0(\mathcal{L}_m) \otimes_k \mathrm{H}^0(\mathcal{L}_n^{\varphi^m}) \to \mathrm{H}^0(\mathcal{L}_{m+n}) = B_{m+n}$$

for every $m, n \geq 0$.

Of course, the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \varphi)$ is noncommutative in general. Let us give a simple example, which is known as the *Jordan quantum plane*. See also Proposition 2.3.4 below for a more general calculation of the multiplication in twisted homogeneous coordinate rings of projective space.

**Example 1.2.3.** [29, Example 3.4] Let $X = \mathbb{P}^1$, and define the automorphism $\varphi$ of $X$ by $\varphi(a_0 : a_1) = (a_0 : a_0 + a_1)$. The twisted homogeneous coordinate ring $B = B(X, \mathcal{O}(1), \varphi)$ has the explicit presentation $B = k\{x, y\}/(xy - yx - x^2)$.

Given the data for a twisted homogeneous coordinate ring $(X, \mathcal{L}, \varphi)$, we say that $\mathcal{L}$ is *$\varphi$-ample* if, given any $\mathcal{F} \in \operatorname{coh} X$, $\mathcal{F} \otimes \mathcal{L}_n$ is generated by its global sections for $n \gg 0$, where $\mathcal{L}_n$ is as in (1.2.2). Serre's Theorem then generalizes to twisted homogeneous coordinate rings in the following way.

**Theorem 1.2.4.** *[7], [20] Let $X$ be a projective scheme with automorphism $\varphi$ and $\varphi$-ample invertible sheaf $\mathcal{L}$. Then $B = B(X, \mathcal{L}, \varphi)$ is a noetherian ring, and $B\text{-qgr} \sim \operatorname{coh} X$.* □

The theory of twisted homogeneous coordinate rings produces a large class of noncommutative graded rings which are built from geometric data, and techniques from algebraic geometry may be used to study such rings.

Now given an arbitrary noncommutative $\mathbb{N}$-graded $k$-algebra $A$, one would like to associate some geometric object to $A$, but there are a number of problems with attempting to generalize the construction of "proj" naïvely to this setting. For one, noncommutative graded rings may have few graded prime ideals, so that the space of graded prime ideals with the Zariski topology is too small to be interesting. In addition, localization is only sometimes available for noncommutative rings, and thus it is unclear how one would define the structure sheaf.

Serre's Theorem and its generalization to twisted homogeneous coordinate rings (Theorem 1.2.4) suggest a different approach. If $A$ is a noncommutative graded ring, the quotient category $A\text{-qgr}$ makes perfect sense and this gives a natural way to define a noncommutative analog of coherent sheaves. This *category* is the intrinsic geometric object we associate to the ring $A$.

**Definition 1.2.5.** [8] Let $A$ be any $\mathbb{N}$-graded $k$-algebra. The *noncommutative projective scheme* $A\text{-proj}$ is defined to be the ordered pair $(A\text{-qgr}, \mathcal{A})$, where $\mathcal{A}$, the *distinguished object*, is the image of ${}_AA$ in $A\text{-qgr}$.

As we shall see in §1.5, there is a natural generalization of Serre's theorem for these noncommutative projective schemes as well, so that one achieves the same kind of useful interplay between rings and geometry as in the commutative case.

### 1.3 Curves and surfaces

In commutative algebraic geometry, curves and surfaces are the best-understood varieties. Much attention in noncommutative geometry has been similarly focused on graded rings of small dimension. The dimension function for rings which is most convenient here is the *GK-dimension*, which we will define in general in §2.1. For the purposes of this introduction, we just assume that all graded algebras $A$ of interest have a *Hilbert polynomial*, that is some $g \in \mathbb{Q}[x]$ such that $\dim_k A_n = g(n)$ for all $n \gg 0$; in this case $\mathrm{GK}(A) = \deg g + 1$. We define a noncommutative curve to be a category of the form $A$-qgr for some noetherian graded ring $A$ of GK-dimension 2, and a noncommutative surface to be $A$-qgr for a noetherian graded ring $A$ of dimension 3.

We begin by discussing the case of domains of dimension 2. Twisted homogeneous coordinate rings of projective curves turn out to be ubiquitous in this setting; a striking example of this is the following theorem of Artin and Stafford.

**Theorem 1.3.1.** *[3] Let $A$ be an $\mathbb{N}$-graded domain with $\mathrm{GK}(A) = 2$, which is finitely generated by elements of degree one. Then there is a projective curve $Y$, an automorphism $\sigma$ of $Y$, and a $\sigma$-ample invertible sheaf $\mathcal{L}$ on $Y$, such that $A$ is isomorphic in large degree to the twisted homogeneous coordinate ring $B(Y, \mathcal{L}, \sigma)$. The ring $A$ is noetherian and $A$-qgr $\sim \operatorname{coh} Y$.* $\square$

The case of domains of dimension 2 which are not generated in degree 1 is somewhat more subtle, but such rings may also be described in terms of geometric data—see [3] for the full details. These results, together with their extension to the case of semiprime rings [4], essentially settle the classification of noncommutative curves.

The theory of noncommutative surfaces, on the other hand, is more complicated. It is certainly not true, for example, that a graded domain of GK-dimension 3 which is generated in degree 1 is a twisted homogeneous coordinate ring of a commutative surface. An obvious first step in grappling with noncommutative surfaces is to identify the correct noncommutative analogs of $\mathbb{P}^2$, and there is good evidence that this problem has been solved—they are the Artin-Schelter regular algebras of dimension 3 and weight $(1,1,1)$.

**Definition 1.3.2.** Let $A$ be a connected, finitely generated $\mathbb{N}$-graded algebra, and let $k = A/A_{\geq 1}$. We say that $A$ is *Artin-Schelter regular* (or *AS-regular*) if $A$ has finite global dimension $d$, finite GK-dimension, and $A$ satisfies the *Gorenstein condition*: $\underline{\mathrm{Ext}}^i_A({}_Ak, A) = 0$ if $i \neq d$, and $\underline{\mathrm{Ext}}^d_A({}_Ak, A) \cong k_A$ (up to some shift of grading).

The preceding definition was formulated in an attempt to capture the essential homological properties of commutative polynomial rings. The AS-regular rings of dimension 3 have now been completely classified by results of Artin, Tate, Van den Bergh, and Stephenson [5], [6], [32], [33]. These rings are all noetherian domains, and they all have the Hilbert series of a weighted polynomial ring in three variables. It is thus natural to consider the AS-regular rings of dimension 3 with weight $(1,1,1)$ as the noncommutative coordinate rings of $\mathbb{P}^2$. In support of this notion, Bondal and Polishchuk [11] have developed a reasonable category-theoretic notion of a noncommutative $\mathbb{P}^2$, and the categories so identified are exactly the $A$-qgr for the AS-regular rings $A$ of dimension 3 and weight $(1,1,1)$. See [29, §11] for more details.

One of the crucial techniques underlying the proof of the classification theorem for AS-regular rings is the study of point modules. For any connected $\mathbb{N}$-graded ring $A$ which is finitely generated in degree 1, we say that $M \in A$-gr is a *point module* if it is cyclic, generated in degree 0, and $\dim_k M_n = 1$ for $n \geq 0$. If $A$ is commutative, then its point modules naturally correspond to the (closed) points of the scheme $\operatorname{proj} A$. If $A$ is an AS-regular algebra of dimension 3, one can prove that the set of point modules for $A$ is also parameterized by a commutative projective scheme $X$. For example, consider the *Sklyanin algebras*, which have explicit presentation

$$S = Skl_3(a,b,c) = k\{x,y,z\}/(axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2)$$

for generic $(a : b : c) \in \mathbb{P}^2$. For such an algebra $S$, the set of point modules is parameterized by an elliptic curve $E$. There is a natural way to construct a surjective homomorphism from $S$ to a twisted homogeneous coordinate ring $B = B(E, \mathcal{L}, \varphi)$. The properties of the twisted homogeneous coordinate ring $B$ follow by studying the geometry of the elliptic curve $E$, and then information may be pulled back to the ring $S$. This geometric method was the first way that many of the Sklyanin algebras' most basic properties, such as the noetherian property and the Hilbert function, were proven (a more recent method, also geometric, is given in [34].)

The classification of noncommutative surfaces in general is still a very active area of current research. In the commutative case, it is known that every projective surface is birational to a minimal nonsingular surface, and there is Zariski's theorem that any two birational surfaces may be transformed into each other by repeatedly blowing up or blowing down. Part of this framework has been introduced with some success to the noncommutative case. For example, Van den Bergh has developed a theory of noncommutative blowing up and down [37]. For a graded domain $A$ of finite GK-dimension, one may localize all of the nonzero homogeneous elements to obtain the *graded quotient ring* $D$ of $A$, and then the zeroeth graded piece $D_0$ is a division ring which is an analog of the function field in the commutative case. Thus it is natural to define two noncommutative schemes $A$-proj and $B$-proj to be *birationally equivalent* if the associated division rings of $A$ and $B$ are isomorphic. There is as yet no noncommutative analog of Zariski's theorem, however.

In this thesis, we shall study a class of graded algebras which provide new examples of noncommutative surfaces (and higher dimensional noncommutative varieties) with very different properties from the known examples. The existence of these algebras answers several open foundational questions in the literature, as we shall describe in the next sections. Moreover, these rings demonstrate just how different the surface case can be from curves, and add a new wrinkle to the classification project for surfaces.

## 1.4 Extension of base rings

In our discussion of the AS-regular algebras of dimension 3 above, we noted that for such rings the point modules are parameterized by a commutative variety, and that studying the geometry of this variety gives information about the ring itself. This is a very useful technique in general, and so it is natural to wonder for which rings the point modules form a nice geometric object. Artin and Zhang have proven a very general result in this vein. We note the following definition.

**Definition 1.4.1.** A $k$-algebra $A$ is called *strongly (left) noetherian* if $A \otimes_k B$ is a left noetherian ring for all commutative noetherian $k$-algebras $B$.

The following is the special case of Artin and Zhang's theorem of greatest interest to us:

**Theorem 1.4.2.** *[9, Corollary E4.11, Corollary E4.12]. Let $A$ be a connected $\mathbb{N}$-graded strongly noetherian algebra over an algebraically closed field $k$.*

*(1) The point modules over $A$ are naturally parameterized by a commutative projective scheme over $k$.*

*(2) There is some $d \geq 0$ such that every point module $M$ for $A$ is uniquely determined by its truncation $M/M_{\geq d}$.* □

The natural notion of parameterization intended here is defined formally in §3.4. The strong noetherian property holds for many standard examples of noncommutative rings, including all finitely generated commutative $k$-algebras, all twisted homogeneous coordinate rings of projective $k$-schemes, and the AS-regular algebras of dimension 3 [2, Section 4]. On the other hand, somewhat earlier then the work of Artin and Zhang on Theorem 1.4.2, Resco and Small [26] had given an example of a noetherian finitely generated algebra over a field which is not strongly noetherian. This algebra is not graded, however, nor is the base field algebraically closed, and so the example falls outside the paradigm of noncommutative projective geometry. This prompts the following question.

**Question 1.4.3.** *[2, page 580] Is every finitely generated $\mathbb{N}$-graded noetherian $k$-algebra strongly noetherian?*

We will produce a whole class of algebras $R$, obtained by the following construction, which will answer this question in the negative.

**Definition 1.4.4.** Let $t \geq 2$, and let $S$ be a twisted homogeneous coordinate ring $B(\mathbb{P}^t, \mathcal{O}(1), \varphi)$ for a generic choice of $\varphi$. Then let $R$ be a subalgebra of $S$ generated by a generic codimension one subspace of $S_1$.

Let us give an explicit example for $t = 2$. Let $\varphi$ be the automorphism of $\mathbb{P}^2$ given by $\varphi(a : b : c) = (pa : qb : c)$ for $p, q \in k$ which are algebraically independent over the prime subfield of $k$. Then $S = B(\mathbb{P}^2, \mathcal{O}(1), \varphi)$ has the following presentation:

$$S = k\{x, y, z\}/(pxz - zx, qyz - zy, pq^{-1}xy - yx).$$

Now let $R$ be the subalgebra of $S$ generated by any two independent elements $r_1, r_2 \in S_1$ such that $kr_1 + kr_2$ does not contain $x$, $y$, or $z$. A large number of other examples are given in Chapter V.

Our precise result will be the following.

**Theorem 1.4.5.** *(Theorem 3.4.9) The ring $R$ of Definition 1.4.4 is a connected $\mathbb{N}$-graded $k$-algebra, finitely generated in degree 1, which is noetherian but not strongly noetherian.* $\square$

Some of the most basic properties of the rings $R$ of Definition 1.4.4 are quite non-trivial to prove. Indeed, in [19] Jordan studied the special case of these algebras where $t = 2$, but did not settle the noetherian property. We will need to expend a good deal of effort to prove the noetherian property for $R$.

We will offer two different proofs, in fact, that $R$ is not strongly noetherian. First, we will classify the set of point modules for $R$, from which we can see that $R$ fails to satisfy part (2) of Theorem 1.4.2. For the second proof we construct an explicit commutative noetherian ring $B$ such that $R \otimes B$ is not noetherian. The ring $B$ that works is an *infinite affine blowup* of affine space, which was defined in [2] and is an interesting construction in itself.

## 1.5 The $\chi$ conditions and cohomological dimension

As we mentioned earlier, one of the driving problems in noncommutative geometry is the classification of noncommutative surfaces. In order to make this project feasible, it has become clear that one needs some restrictions on which rings or categories should be considered. For example, in [1] Artin studies noncommutative surfaces of the form $A$-proj for graded rings $A$ of dimension 3, assuming some basic "niceness" properties on the rings, and makes the provocative conjecture that every such surface is birationally equivalent to $A'$-proj for $A'$ from a short list of known rings. Naturally, Artin assumes that $A$ is noetherian, finitely generated in degree 1, and that GK-dimension behaves well for $A$; the more important assumption is that $A$ has a *balanced dualizing complex*, which provides information similar to Serre duality in the commutative case. By results of Van den Bergh, Yekutieli, and Zhang [36, Theorem 6.3] [42, Theorem 4.2], the existence of of a balanced dualizing complex for a graded ring $A$ is equivalent to the hypothesis that $A$ and its opposite ring $A^{op}$ satisfy the $\chi$ *conditions* and have finite *cohomological dimension*. Let us define these notions.

**Definition 1.5.1.** Let $A$ be a connected $\mathbb{N}$-graded $k$-algebra. For $i \geq 1$, we say that $A$ satisfies $\chi_i$ (on the left) if $\dim_k \underline{\mathrm{Ext}}^j(A/A_{\geq 1}, M) < \infty$ for all $M \in A\text{-gr}$ and all $j \leq i$. If $A$ satisfies $\chi_i$ for all $i > 0$, then we say that $A$ satisfies $\chi$.

We define cohomology groups for a noncommutative scheme $A\text{-proj} = (A\text{-qgr}, \mathcal{A})$ by setting $\mathrm{H}^i(\mathcal{M}) = \mathrm{Ext}^i_{A\text{-qgr}}(\mathcal{A}, \mathcal{M})$.

**Definition 1.5.2.** Let $A$ be a connected $\mathbb{N}$-graded $k$-algebra. The *cohomological dimension* of $A$-proj (or $A$) is defined to be

$$\mathrm{cd}(A\text{-proj}) = \min\{i \geq 0 \mid \mathrm{H}^i(\mathcal{M}) = 0 \text{ for all } \mathcal{M} \in A\text{-qgr}\}.$$

It is trivial that if $A$ is commutative then it satisfies $\chi$, and it is well known that commutative projective schemes have finite cohomological dimension. Understanding which noncommutative rings satisfy these conditions, and thus how stringent assumptions such as Artin's are in the noncommutative case, is a very important problem.

The $\chi$ conditions actually appear in a number of fundamental results in the theory of noncommutative projective schemes. First and foremost there is the following noncommutative analog of Serre's theorem (Theorem 1.2.1) due to Artin and Zhang.

**Theorem 1.5.3.** *[8, Theorem 4.5]. Let $A$ be a noetherian $\mathbb{N}$-graded algebra, and let $A\text{-proj} = (A\text{-qgr}, \mathcal{A})$. Then $B = \bigoplus_{i \geq 0} \mathrm{H}^0(\mathcal{A}[i])$ is a graded ring and there is a canonical homomorphism $\psi : A \to B$. If $A$ satisfies $\chi_1$ then $\psi$ is an isomorphism in large degree, and $A\text{-qgr} \cong B\text{-qgr}$.* □

In other words, if $A$ satisfies $\chi_1$ then $A$ can be more or less reconstructed from $A$-proj as a "coordinate ring".

There is also the following noncommutative version of Serre's finiteness theorem.

**Theorem 1.5.4.** *[8, Theorem 7.4] Let $A$ be a left noetherian finitely $\mathbb{N}$-graded algebra. If $A$ satisfies $\chi$, then the following conditions hold:*

*(1) $\dim_k \mathrm{H}^j(\mathcal{N}) < \infty$ for all $j \geq 0$ and all $\mathcal{N} \in A\text{-qgr}$.*

*(2) For any $\mathcal{N} \in A\text{-qgr}$ and $j \geq 0$, one has $\mathrm{H}^j(\mathcal{N}[n]) = 0$ for $n \gg 0$.* □

The $\chi$ condition is not always satisfied by noncommutative graded rings. The basic counterexamples were given by Stafford and Zhang in [30]. Let $S = k\{x, y\}/(xy - yx - x^2)$ be the twisted homogeneous coordinate ring of $\mathbb{P}^1$ given in Example 1.2.3 above, and let $T = k + Sy \subseteq S$. If $\operatorname{char} k = 0$, then $T$ is a noetherian ring. However $T$ fails to satisfy $\chi_i$ on the left (and on the right) for all $i \geq 1$.

The noncommutative Serre's theorem (Theorem 1.5.3) fails for the ring $T$; writing $T\text{-proj} = (T\text{-qgr}, \mathcal{T})$, the coordinate ring $\bigoplus_{i \geq 0} \mathrm{H}^0(\mathcal{T}[i])$ is isomorphic to $S$, which is not equal to $T$ in large degree. It is nonetheless still true that $T$-proj is very nice; in fact $\operatorname{coh} \mathbb{P}^1 \sim S\text{-qgr} \sim T\text{-qgr}$. By taking the polynomial extension ring $T[z]$, however, one obtains a ring for which $T[z]$-proj is also badly behaved [30].

Because the nice relationship between $A$ and $A$-proj guaranteed by Theorem 1.5.3 is so fundamental, it is natural to restrict one's attention to rings which satisfy

$\chi_1$. The full $\chi$ condition is needed, though, for other important theorems such as the existence of dualizing complexes. Stafford and Zhang have asked the following natural question.

**Question 1.5.5.** *[30, Section 4] Does there exist a noetherian ring which satisfies $\chi_1$ but not $\chi$?*

We will give such an example below.

**Theorem 1.5.6.** *(Theorem 4.4.3) The ring $R$ of Definition 1.4.4 is a noetherian connected finitely $\mathbb{N}$-graded $k$-algebra, finitely generated in degree 1, for which $\chi_1$ holds but $\chi_i$ fails for all $i \geq 2$.* $\square$

Thus $R$ satisfies the noncommutative Serre's theorem (Theorem 1.5.3). The noncommutative Serre's finiteness theorem (Theorem 1.5.4) fails for $R$, however; in fact, $\dim_k \mathrm{H}^1(\mathcal{R}) = \infty$, where $\mathcal{R}$ is the distinguished object of $R$-proj. One consequence we will draw is that the category $R$-qgr is something quite different from the standard examples of noncommutative schemes.

**Theorem 1.5.7.** *(Theorem 4.4.8) The category $R$-qgr is not equivalent to the category $\operatorname{coh} X$ of coherent sheaves on $X$ for any commutative projective scheme $X$. More generally, $R$-qgr $\not\simeq$ $S$-qgr for every graded ring $S$ which satisfies $\chi_2$.* $\square$

The kind of pathology introduced by idealizer rings such as the examples $T$ and $T[z]$ of Stafford and Zhang is easily removed, and so is not very relevant to the understanding of noncommutative surfaces. For example, these rings can be avoided by requiring any ring of interest to be a *maximal order*, which is the analog in noncommutative ring theory of a commutative integrally closed ring. This is a natural assumption, akin to working with normal schemes in the commutative case. Stafford and Van den Bergh ask this question:

**Question 1.5.8.** *[29, Page 194] Does the $\chi$ condition hold for all graded rings which are maximal orders?*

We show to the contrary the following result.

**Theorem 1.5.9.** *(Theorem 4.3.4) The ring $R$ of Definition 1.4.4 is a noetherian $\mathbb{N}$-graded maximal order for which $\chi$ fails.* □

This result suggests that the ring $R$ is less pathological than the previous examples of rings failing $\chi$, and one expects $R$-proj to be quite interesting geometrically. As further evidence that this noncommutative scheme has reasonable properties, we will show that the notion of cohomological dimension behaves well for $R$-proj.

**Theorem 1.5.10.** *(Theorem 4.5.11) $R$-proj has finite cohomological dimension. In particular,* $\mathrm{cd}(R\text{-proj}) \leq t = \mathrm{GK}(R) - 1$. □

One hope is that cohomological dimension could be used as a good measure of dimension for noncommutative schemes in general, but it is unknown at present if it is always a finite number. The fact that $\mathrm{cd}(R\text{-proj}) < \infty$, despite the odd behavior of cohomology in $R$-proj in other ways, suggests that perhaps cohomological dimension is indeed always finite.

## 1.6 Structure of the thesis

Let us briefly describe the organization of the thesis. In Chapter II we will collect some basic definitions and background material for the results to follow. Chapter III is dedicated to the study of the noetherian and strongly noetherian properties for the rings $R$ of Definition 1.4.4, in particular Theorem 1.4.5. Chapter IV contains more homological considerations, including the proofs of Theorems 1.5.6—1.5.10. Finally, in Chapter V we present some examples, and show the connection between our rings and the special cases considered by Jordan in [19].

# CHAPTER II

# Background Material

In this chapter, we lay some groundwork for our main results in Chapters III and IV. In the first two sections, we collect some basic facts and definitions about graded rings and about the GK-dimension for modules. Next, we study the notion of a Zhang twist, which is a general way to produce many graded rings from a given one, all of which have equivalent module categories. We will be particularly interested in Zhang twists of commutative polynomial rings, because they give a completely algebraic formulation of twisted homogeneous coordinate rings of projective space, as we show below. In the last two sections of the chapter we discuss some simple results from commutative algebra. We define Castelnuovo-Mumford regularity for modules over a commutative polynomial ring, and use this theory to prove some lemmas about graded ideals of points in projective space. Because these final two sections are technical and we shall need the results in them only sporadically, the reader may wish to skip them and move directly to Chapter III, referring back to these sections when necessary.

## 2.1 Basic definitions

We wish to fix from the outset some terminology and definitions concerning graded rings and abelian categories. We make the convention that 0 is a natural number, so that $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Throughout this section, let $A = \bigoplus_{i=0}^{\infty} A_i$ be an $\mathbb{N}$-graded algebra over an algebraically closed field $k$. Assume also that $A$ is *connected*, that is that $A_0 = k$, and finitely generated as an algebra by $A_1$. Let $A$-Gr be the abelian category whose objects are the $\mathbb{Z}$-graded left $A$-modules $M = \bigoplus_{i=-\infty}^{\infty} M_i$, and where the morphisms $\operatorname{Hom}(M, N)$ are the module homomorphisms $\phi$ satisfying $\phi(M_n) \subseteq N_n$ for all $n$. We shall follow a standard convention for category names, where if $A$-Xyz is some abelian category then $A$-xyz is the full subcategory consisting of the noetherian objects. For example, we let $A$-gr be the full subcategory of the noetherian objects of $A$-Gr. For $M \in A$-Gr and $n \in \mathbb{Z}$, the *shift* of $M$ by $n$, denoted $M[n]$, is the module with the same ungraded module structure as $M$ but with the grading shifted so that $(M[n])_m = M_{n+m}$. Then for $M, N \in A$-Gr we may define

$$\underline{\operatorname{Hom}}_A(M, N) = \bigoplus_{i=-\infty}^{\infty} \operatorname{Hom}_A(M, N[i]).$$

The group $\underline{\operatorname{Hom}}_A(M, N)$ is a $\mathbb{Z}$-graded vector space and we also write $\underline{\operatorname{Hom}}_A(M, N)_i$ for the $i$th graded piece $\operatorname{Hom}_A(M, N[i])$. Under mild hypotheses, for example if $M$ is finitely generated, the group $\underline{\operatorname{Hom}}_A(M, N)$ may be identified with the set of ungraded module homomorphisms from $M$ to $N$. It is a standard result that the category $A$-Gr has enough injectives, so we may define right derived functors $\operatorname{Ext}^i_A(M, -)$ of $\operatorname{Hom}_A(M, -)$ for any $M$. The definition of $\underline{\operatorname{Hom}}$ generalizes to

$$\underline{\operatorname{Ext}}^i_A(M, N) = \bigoplus_{i=-\infty}^{\infty} \operatorname{Ext}^i_A(M, N[i]).$$

See [8, Section 3] for a discussion of the basic properties of $\underline{\operatorname{Ext}}$.

For a module $M \in A$-Gr, a *tail* of $M$ is any submodule of the form $M_{\geq n} = \bigoplus_{i=n}^{\infty} M_i$, and a *truncation* of $M$ is any factor module of the form $M_{\leq n} = M/M_{\geq n+1}$. A *subfactor* of $M$ is any module of the form $N/N'$ for graded submodules $N' \subseteq N$ of $M$. For the purposes of this paper, $M \in A$-Gr is called *torsion* if for all $m \in M$

there exists some $n \geq 0$ such that $(A_{\geq n})m = 0$. Note that if $M \in A$-gr, then $M$ is torsion if and only if $\dim_k M < \infty$. We say that $M \in A$-Gr is *left bounded* if $M_i = 0$ for $i \ll 0$, and *right bounded* if $M_i = 0$ for $i \gg 0$. $M$ is *bounded* if it is both left and right bounded. If we want to be more specific, we say that $M \in R$-Gr is *bounded in* $[l, r]$ for $l \leq r \in \mathbb{Z} \cup \{-\infty, \infty\}$ if $M_i = 0$ unless $l \leq i \leq r$. A *(finite) filtration* of $M \in A$-Gr is a sequence of graded submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$; we call the modules $M_i/M_{i-1}$ for $0 < i \leq n$ the *factors* of the filtration.

A *point module* over $A$ is a graded module $M$ such that $M$ is cyclic, generated in degree 0, and $\dim_k M_n = 1$ for all $n \geq 0$. Note that a tail of a point module is a shift of some other point module. A *point ideal* is a left ideal $I$ of $A$ such that $A/I$ is a point module, or equivalently such that $\dim_k I_n = \dim_k A_n - 1$ for all $n \geq 0$. Since $A$ is generated in degree 1, the point ideals of $A$ are in one-to-one correspondence with isomorphism classes of point modules over $A$.

Suppose that $A$ is a domain. A (left and right) *Ore set* in $A$ is a multiplicatively closed subset $Z$ of $A$ which satisfies an additional condition ensuring that localization at the set $Z$ makes sense (see [17, Chapter 9]); in this case we write $AZ^{-1}$ for the localized ring. In case the set $X$ of all nonzero homogeneous elements of $A$ is an Ore set, the localization $D = AX^{-1}$ is called the *graded quotient ring* of $A$. The ring $D$ is a $\mathbb{Z}$-graded algebra, which it is not hard to see can be written as a skew-Laurent ring $D \cong F[z, z^{-1}; \sigma]$ for some division ring $F$ and automorphism $\sigma$ of $F$ (explicitly, $F[z, z^{-1}; \sigma]$ is the $\mathbb{Z}$-graded ring with basis $\{z^i\}_{i \in \mathbb{Z}}$ as a left $F$-space subject to the relations $z^i f = \sigma^i(f) z^i$ for all $f \in F$ and all $i \in \mathbb{Z}$). If the set of all nonzero elements $Y$ of $A$ forms an Ore set then the localization $Q = AY^{-1}$ is a division ring called the *Goldie quotient ring* of $A$. If $A$ is a graded domain of finite GK-dimension (see §2.2) then both $X$ and $Y$ are Ore sets and both quotient rings exist [24, C.I.1.6], [21, 4.12].

We recall some notions from the theory of abelian categories. For more details, see [25, Sections 4.3–4.5]. Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories. We use the notation $\mathcal{C} \sim \mathcal{C}'$ to mean that the categories $\mathcal{C}$ and $\mathcal{C}'$ are equivalent. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called *Serre* if for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}$, both $M' \in \mathcal{D}$ and $M'' \in \mathcal{D}$ if and only if $M \in \mathcal{D}$. Suppose that $\mathcal{D}$ is a Serre subcategory of $\mathcal{C}$. Then there is a *quotient category* $\mathcal{C}/\mathcal{D}$ and an exact *quotient functor* $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ which are defined as follows. The objects of $\mathcal{C}/\mathcal{D}$ are the objects of $\mathcal{C}$, and $\pi$ is the identity map on objects. For $X, Y \in \mathcal{C}$, we define the morphisms in $\mathcal{C}/\mathcal{D}$ by

$$\operatorname{Hom}_{\mathcal{C}/\mathcal{D}}(\pi X, \pi Y) = \varinjlim \operatorname{Hom}_{\mathcal{C}}(X', Y/Y')$$

where the limit is taken over all pairs $(X', Y')$ such that $X/X' \in \mathcal{D}$ and $Y' \in \mathcal{D}$. If the functor $\pi$ has a right adjoint $\omega : \mathcal{C}/\mathcal{D} \to \mathcal{C}$, then $\omega$ is called the *section functor*, and $\mathcal{D}$ is called a *localizing subcategory* of $\mathcal{C}$. The functor $\omega$ is left exact if it exists.

We shall only need the following special case of the quotient category construction. For $A$ a noetherian $\mathbb{N}$-graded ring, let $A$-Tors be the full subcategory of torsion objects in $A$-Gr. This is a Serre subcategory and we may define the quotient category $A$-Qgr $= A$-Gr$/A$-Tors. Then, as usual, let $A$-qgr be the full subcategory of $A$-Qgr consisting of the noetherian objects. Alternatively, $A$-qgr may be obtained as the quotient category $A$-gr$/A$-tors, where $A$-tors $= A$-Tors $\cap A$-gr. One may also form the category $A$-Qgr by putting an equivalence relation on morphisms in $A$-Gr, where two maps $\psi_1, \psi_2 : M \to N$ in $A$-Gr are set equivalent precisely when $(\psi_1 - \psi_2)$ has kernel and cokernel in $A$-Tors, and then defining the morphisms in $A$-Qgr to be equivalence classes of morphisms in $A$-Gr. It is a fact that $A$-Tors is always a localizing subcategory of $A$-Gr, that is to say the section functor $\omega$ always exists. In fact, for torsionfree $M \in A$-Gr we may describe $\omega\pi(M)$ explicitly as the unique largest essential extension $M'$ of $M$ such that $M'/M$ is torsion. For all $\mathcal{M} \in A\text{-qgr}$, $\omega(\mathcal{M})$ is torsionfree and $\pi\omega(\mathcal{M}) \cong \mathcal{M}$.

## 2.2 GK-dimension

If $B$ is a commutative ring and $M$ is any $B$-module, then the Krull dimension of $M$ is defined to be the length of a maximal chain of prime ideals in the ring $(B/\operatorname{ann} M)$. Many noncommutative rings, including those of interest in this thesis, tend to have relatively few prime ideals, and so some different definitions of dimension are more useful in the noncommutative setting. The basic dimension function we shall use below is the *Gelfand-Kirillov dimension*, or GK-dimension for short.

GK-dimension is only defined for algebras over a field, and we further restrict our attention to finitely generated algebras. We will mention only a few of its basic properties here; a detailed treatment may be found in [21].

**Definition 2.2.1.** Let $A$ be any finitely generated $k$-algebra, and let $V$ be a finite dimensional vector subspace of $A$ which contains a generating set for $A$ and the element 1. Then we define

$$\mathrm{GK}(A) = \inf\{\alpha \in \mathbb{R} \mid \dim_k V^n \leq n^\alpha \text{ for } n \gg 0\}.$$

This definition is independent of the choice of $V$.

Let $M = \sum_{i=1}^m Am_i$ be a finitely generated left $A$-module. Put $W_n = \sum_{i=1}^m V^n m_i$ for all $n \geq 0$; we define

$$\mathrm{GK}(M) = \inf\{\alpha \in \mathbb{R} \mid \dim_k W_n \leq n^\alpha \text{ for } n \gg 0\}.$$

Again, this is independent of the choice of generating set for $M$ and the choice of $V$.

For an arbitrary left $A$-module $N$, we let

$$\mathrm{GK}(N) = \sup\{\mathrm{GK}(M) \mid M \text{ is a finitely generated submodule of } N\}.$$

The GK-dimension of algebras and modules need not be an integer. In fact there are examples of algebras with GK-dimension $d$ for any real number $d \geq 2$. In cases of interest, however, GK-dimension is usually integral. For instance, if $A$ is a finitely generated commutative $k$-algebra then the GK-dimension for $A$-modules agrees with the usual commutative Krull dimension. A very interesting open question is whether $\mathrm{GK}(A)$ must be an integer for every $\mathbb{N}$-graded noetherian domain $A$.

Now let $A$ be an $\mathbb{N}$-graded noetherian $k$-algebra. Given $M \in A$-gr, the *Hilbert function* of $M$ is the function $H(n) = \dim_k M_n$ for $n \in \mathbb{Z}$. If $A$ is a finitely generated $k$-algebra and $M \in A$-gr, then $\mathrm{GK}(M)$ depends only on the Hilbert function of $M$ [21, 6.1]; explicitly, $\mathrm{GK}(M) = \inf\{\alpha \in \mathbb{R} \mid \dim_k(M_{\leq n}) \leq n^\alpha \text{ for } n \gg 0\}$. In particular, if $M$ has a *Hilbert polynomial*, that is $\dim_k M_n = f(n)$ for $n \gg 0$ and some polynomial $f \in \mathbb{Q}[n]$, then $\mathrm{GK}(M) = \deg f + 1$ (with the convention $\deg(0) = -1$). Unlike the commutative case, a finitely generated graded module over a noncommutative graded ring may not have a Hilbert polynomial. We shall see, however, that Hilbert polynomials do exist for all modules over the rings of interest to us below. In practice, then, we shall measure GK-dimension from the Hilbert polynomial and will not need to work directly from the definition.

A dimension function on modules $d : \{\text{left } A\text{-modules}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is called *exact* if given any exact sequence $0 \to M' \to M \to M'' \to 0$ in $A$-gr, $d(M) = \max(d(M'), d(M''))$. The GK-dimension is exact for modules over a noetherian $\mathbb{N}$-graded ring [23, 4.9]. For more general rings the GK-dimension does sometimes fail to be exact; this is an inconvenience with which we shall not have to trouble ourselves.

We mention one last definition. We say that $M \in A$-Gr is *(graded) critical* if $\mathrm{GK}(M/N) < \mathrm{GK}(M)$ for all nonzero graded submodules $N$ of $M$.

## 2.3 Zhang twists

We now introduce a fundamental algebraic construction which produces new graded rings from a given one.

**Definition 2.3.1.** Let $A$ be a $\mathbb{N}$-graded $k$-algebra, and let $\phi$ be any graded automorphism of $A$. Then we may form a new graded ring $B$ with the same underlying $k$-space as $A$, but with new multiplication $*$ given by $f * g = \phi^n(f)g$ for all $f \in A_m, g \in A_n$. The ring $B$ is called the *left Zhang twist* of $A$ by the *twisting system* $\{\phi^i\}_{i \in \mathbb{N}}$. Similarly, the multiplication $\star$ given by $f \star g = f\phi^m(g)$ for all $f \in A_m, g \in A_n$ gives a new graded ring $C$ we call the *right Zhang twist* of $A$ by the twisting system $\{\phi^i\}_{i \in \mathbb{N}}$.

It is straightforward to check the ring axioms for Zhang twists. For more details, see also [43], which is the basic reference for this subject.

One nice property of the Zhang twist construction is that it actually preserves the properties of the graded module categories over the two rings. Suppose that $A$ is an $\mathbb{N}$-graded algebra with automorphism $\phi$, and that $B$ is the left Zhang twist of $A$ by the twisting system $\{\phi^i\}$. Given $M \in A\text{-Gr}$, we construct a module $\theta(M) \in B\text{-Gr}$ which has the same underlying vector space as $M$ but has a new module action given by $a \star x = \phi^n(a)x$ for $a \in A_m$, $x \in M_n$. This gives a functor $\theta : A\text{-Gr} \to B\text{-Gr}$ which is in fact an equivalence of categories [43, Theorem 3.1].

As an immediate consequence of the equivalence of categories, we note that $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_B(\theta(M), \theta(N))$ (as vector spaces) holds for all $M, N \in A\text{-Gr}$. In fact, since we can define the derived functors $\operatorname{Ext}^i$ of Hom by using injective resolutions, and graded injective objects correspond under the equivalence of categories, we have more generally that $\operatorname{Ext}^i_A(M, N) \cong \operatorname{Ext}^i_B(\theta(M), \theta(N))$ for all $M, N \in U\text{-Gr}$ and all $i \geq 0$.

Given $B$ a left Zhang twist of $A$, we would also like to understand the relationship between the $\underline{\mathrm{Ext}}$ groups over $A$ and $B$. This turns out to be a bit more complicated, since the shift functors in the categories $A$-Gr and $B$-Gr do not necessarily correspond under the equivalence of categories. The answer involves the following standard notion of twisting a module by an automorphism.

**Definition 2.3.2.** Let $A$ be any ring with a ring automorphism $\psi$. For any left $A$-module $M$ the *twist of $M$ by $\psi$*, denoted ${}_{\psi}M$, is a module with the same underlying space as $M$ but with new action $*$ given by $a * m = \psi(a)m$, for all $a \in A$, $m \in M$.

**Lemma 2.3.3.** *Let $B$ be the left Zhang twist of $A$ by the twisting system $\{\phi^i\}$, and let $\theta : A\text{-Gr} \to B\text{-Gr}$ be the equivalence of categories.*

*(1) For any $N \in A\text{-Gr}$ and $n \in \mathbb{Z}$, we have*

$$(\theta(N))[n] \cong \theta({}_{\phi^n}N[n]).$$

*(2) Let $M, N \in A\text{-Gr}$. For any $i \geq 0$ and $n \in \mathbb{Z}$ there are $k$-space isomorphisms*

$$\underline{\mathrm{Ext}}^i_B(\theta(M), \theta(N))_n \cong \underline{\mathrm{Ext}}^i_A(M, {}_{\phi^n}N)_n \cong \underline{\mathrm{Ext}}^i_A({}_{\phi^{-n}}M, N)_n.$$

*(3) Let $I, J$ be graded left ideals of $A$. Under our identification of the underlying vector spaces of $A$ and $B$, we may identify $\theta(I)$ with $I$ and $\theta(J)$ with $J$. For any $i \geq 0$ and $n \in \mathbb{Z}$ there are $k$-space isomorphisms*

$$\underline{\mathrm{Ext}}^i_B(B/I, B/J))_n \cong \underline{\mathrm{Ext}}^i_A(A/I, A/\phi^{-n}(J))_n \cong \underline{\mathrm{Ext}}^i_A(A/\phi^n(I), A/J)_n.$$

*Proof.* Part (1) follows directly from the various definitions, and is left to the reader. Part (2) is a consequence of (1) since $\mathrm{Ext}^i$ commutes with the equivalence of categories. Finally, part (3) is just the special case of (2) with $M = A/I, N = A/J$, once one calculates that ${}_{\phi^n}(A/J) \cong A/\phi^{-n}(J)$. □

Our motivation for introducing the Zhang twist construction is the following proposition, which shows that twisted homogeneous coordinate rings of projective space may be formulated entirely in the language of Zhang twists. Although this result is well known, we include a full proof for lack of a reference that works in this generality.

**Proposition 2.3.4.** *(Twisted Homogeneous coordinate rings of projective space) Let* $U = k[x_0, x_1, \ldots, x_t]$ *be a polynomial ring, and let* $X = \operatorname{proj} U = \mathbb{P}^t$*. Let* $\phi$ *be a graded automorphism of* $U$*, and let* $\varphi$ *be the induced automorphism of* $X$*. Let* $\mathcal{L} = \mathcal{O}(1)$ *be the twisting sheaf of Serre on* $X$*. Then the twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \varphi)$ *is isomorphic to the right Zhang twist of* $U$ *by the twisting system* $\{\phi^i\}_{i \geq 0}$*.*

*Proof.* Given any module $M \in U$-gr, we let $\widetilde{M}$ represent the corresponding coherent sheaf on $\mathbb{P}^t$. By definition, we set $\mathcal{O}(m) = \widetilde{U[m]}$ for $m \in \mathbb{Z}$. Recall our notation $\mathcal{F}^\psi$ for the pullback $\psi^*(\mathcal{F})$. Now, by [18, Proposition 5.12], for any $M \in U$-gr we have

$$(\widetilde{M})^{\varphi^n} = (\varphi^n)^*(\widetilde{M}) \cong \widetilde{({}_{\phi^n}U) \otimes_U M} \cong \widetilde{{}_{\phi^n}M}$$

for all $n \in \mathbb{Z}$, where ${}_{\phi^n}M$ is the twist of $M$ by the automorphism $\phi^n$, as in Definition 2.3.2. For each $n \geq 0$ there is an isomorphism of $U$-modules

$$U[n] \longrightarrow {}_{\phi^m}U[n],$$

given by the formula $u \mapsto \phi^m(u)$, which induces the standard isomorphism of sheaves

$$\mathcal{O}(n) \longrightarrow \mathcal{O}(n)^{\varphi^m}. \tag{2.3.5}$$

In particular, setting $\mathcal{L} = \mathcal{O}(1)$, it follows that $\mathcal{L}^{\varphi^i} \cong \mathcal{L}$ for all $i \geq 0$, and so

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\varphi \otimes \cdots \otimes \mathcal{L}^{\varphi^{n-1}} \cong \mathcal{L}^{\otimes n} = \mathcal{O}(n)$$

for all $n$. Now taking global sections of (2.3.5), we get a map

$$\mathrm{H}^0(\mathcal{L}_n) \cong \mathrm{H}^0(\mathcal{O}(n)) \longrightarrow \mathrm{H}^0(\mathcal{O}(n)^{\varphi^m}) \cong \mathrm{H}^0(\mathcal{L}_n^{\varphi^m}). \tag{2.3.6}$$

By [18, Proposition 5.13], we may identify $\mathrm{H}^0(\mathcal{O}(n))$ as a vector space with $U_n$ for all $n \geq 0$, and under this identification the map (2.3.6) is simply the map $U_n \xrightarrow{\phi^m} U_n$. Now the multiplication map in the twisted homogeneous coordinate ring is by definition

$$\mathrm{H}^0(\mathcal{L}_m) \otimes_k \mathrm{H}^0(\mathcal{L}_n) \to \mathrm{H}^0(\mathcal{L}_m) \otimes_k \mathrm{H}^0(\mathcal{L}_n^{\varphi^m}) \to \mathrm{H}^0(\mathcal{L}_{m+n}),$$

which may be identified with the map

$$\begin{aligned} U_m \otimes U_n &\to U_{m+n} \\ f \otimes g &\mapsto f\phi^m(g). \end{aligned}$$

This is exactly the multiplication in the right Zhang twist of $U$ by the twisting system $\{\phi^i\}$. $\square$

The class of rings appearing in the preceding proposition will play an important role below. From now on, we will work exclusively with the Zhang twist formulation, and will not use the twisted homogeneous coordinate ring construction further; this will allow us to make all of our considerations algebraic. It will also be more convenient for us to work with *left* Zhang twists of $U$ below, and we do so from now on. This does not make much difference, since if $S$ is the left Zhang twist of $U$ by $\{\phi^i\}_{i\in\mathbb{N}}$, then $S$ is isomorphic to the right Zhang twist by $\{\phi^{-i}\}_{i\in\mathbb{N}}$ [43, Theorem 4.3]. Then by Example 2.3.4, $S \cong B(\mathbb{P}^t, \mathcal{O}(1), \varphi^{-1})$, where $\varphi$ is the automorphism of $\mathbb{P}^t$ induced by $\phi$.

## 2.4 Castenuovo-Mumford regularity

In this section and the next, we discuss the notion of Castelnuovo-Mumford regularity for modules over commutative polynomial rings. In particular, we will use some recent results in this subject to provide quick proofs of some technical lemmas about the products of graded ideals of points with multiplicities in projective space. The reader may prefer to skim these two sections and skip directly to Chapter III, referring back to the results here when they are needed. The particular results which we will reference in later chapters are Corollary 2.4.8 and Lemmas 2.5.6—2.5.9.

Let $U = k[x_0, x_1, \dots x_t]$ be a polynomial ring over an algebraically closed field $k$, graded as usual with $\deg(x_i) = 1$ for all $i$. Generally speaking, the notion of regularity for a $U$-module $M$ is a convenient way of encapsulating information about the degrees of the generators of all of the syzygies of $M$.

**Definition 2.4.1.** [16, page 509] Let $M \in U$-gr. Take a minimal graded free resolution of $M$:

$$0 \to \bigoplus_{i=1}^{r_{(t+1)}} U[-e_{i,t+1}] \to \dots \to \bigoplus_{i=1}^{r_0} U[-e_{i,0}] \to M \to 0.$$

If $e_{i,j} \leq m + j$ for all $i, j$ then we say that $M$ is *$m$-regular*. The *regularity* of $M$, $\operatorname{reg} M$, is the smallest integer $m$ for which $M$ is $m$-regular (if $M = 0$ then we set $\operatorname{reg} M = -\infty$).

There are other equivalent characterizations of regularity, with different advantages. In the special case of ideals, we have the following criterion using sheaf cohomology. For $M \in U$-Gr we have the associated quasi-coherent sheaf $\mathcal{M}$ of $\mathbb{P}^t$, and we write $\mathcal{M}(j)$ for the sheaf associated to $M[j]$.

**Lemma 2.4.2.** *[10, Definition 3.2] Let $I$ be an ideal of $U$. Let $\mathcal{I}$ be the corresponding sheaf of ideals on $\mathbb{P}^t$. Then $I$ is $m$-regular if and only if the following conditions hold:*

*(1) the natural maps* $I_j \to \mathrm{H}^0(\mathcal{I}(j))$ *are isomorphisms for all* $j \geq m$.

*(2)* $\mathrm{H}^i(\mathcal{I}(j)) = 0$ *for all* $i, j$ *with* $j + i \geq m$ *and* $i \geq 1$. □

The next trivial lemmas show that regularity behaves well with respect to exact sequences.

**Lemma 2.4.3.** *[16, Corollary 20.19] let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence in* $U$-gr*. Then*

(1) $\operatorname{reg} M' \leq \max(\operatorname{reg} M, \operatorname{reg} M'' + 1)$

(2) $\operatorname{reg} M \leq \max(\operatorname{reg} M', \operatorname{reg} M'')$

(3) $\operatorname{reg} M'' \leq \max(\operatorname{reg} M' - 1, \operatorname{reg} M)$. □

**Lemma 2.4.4.** *Let* $I, J$ *be graded ideals of* $U$*. Then*

(1) $\operatorname{reg}(I + J) \leq \max(\operatorname{reg}(I \cap J) - 1, \operatorname{reg} I, \operatorname{reg} J)$.

(2) $\operatorname{reg}(I \cap J) \leq \max(\operatorname{reg}(I + J) + 1, \operatorname{reg} I, \operatorname{reg} J)$.

*Proof.* From the exact sequence

$$0 \to (I \cap J) \to (I \oplus J) \to (I + J) \to 0$$

and the obvious fact that $\operatorname{reg}(I \oplus J) = \max(\operatorname{reg} I, \operatorname{reg} J)$, both statements are easy consequences of Lemma 2.4.3. □

Let us define some related notions. For $I$ a graded ideal of $U$, we define the *saturation* of $I$ to be

$$I^{sat} = \{x \in U | (U_{\geq n})x \subseteq I \text{ for some } n\}.$$

The ideal $I^{sat}$ is the unique largest extension of $I$ inside $U$ by a torsion module. In addition, the saturation of $I$ is also the largest ideal which defines the same ideal sheaf as $I$ on $\mathbb{P}^t$. If $\mathcal{I}$ is the ideal sheaf on $\mathbb{P}^t$ corresponding to $I$, then we have

$(I^{sat})_m \cong \mathrm{H}^0(\mathcal{I}(m))$ [18, Exercise 5.10]. If $I^{sat} = I$ then we say that $I$ is *saturated.* The *saturation degree* of $I$ is defined to be

$$\operatorname{sat} I = \min\{m \in \mathbb{N} | (I^{sat})_{\geq m} = I_{\geq m}\}.$$

Finally, for a module $M \in U\text{-gr}$, let $d(M)$ be the largest degree of an element in a minimal generating set for $M$.

A module or ideal that is $m$-regular stabilizes in degree $m$ in a number of important ways.

**Lemma 2.4.5.**

*(1) If $M \in U\text{-gr}$ is $m$-regular, then $d(M) \leq \operatorname{reg} M$.*

*(2) If $I$ is a graded ideal of $U$ then $\operatorname{sat} I \leq \operatorname{reg} I$.*

*(3) If $I$ is a graded ideal of $U$, then $I$ is $m$-regular if and only if $I_{\geq m}$ is $m$-regular.*

*Proof.* (1) This is immediate from Definition 2.4.1.

(2) Let $\mathcal{I}$ be the sheaf of ideals associated to $I$. Then since $(I^{sat})_m \cong \mathrm{H}^0(\mathcal{I}(m))$, the statement follows from Lemma 2.4.2.

(3) This is also clear by Lemma 2.4.2, since $I$ and $I_{\geq m}$ have the same associated sheaf of ideals $\mathcal{I}$. $\square$

The regularity of an ideal $I \subseteq U$ might be much greater than the minimal generating degree $d(I)$, but at least there is the following bound.

**Lemma 2.4.6.** *[10, Proposition 3.8] Let $I$ be a homogeneous ideal of $U$, and let $d = d(I)$. Then $\operatorname{reg} I \leq (2d(I))^{t!}$.* $\square$

We close this section with a simple application of regularity to the analysis of bounds for $\underline{\mathrm{Ext}}$ groups.

**Lemma 2.4.7.** *Let $I, J$ be homogeneous ideals of $U$. There is a constant $d \in \mathbb{Z}$, depending only on* $\operatorname{reg} I$ *and* $\operatorname{reg} J$*, such that* $\operatorname{reg}(\underline{\operatorname{Ext}}^i_U(U/I, U/J)) < d$ *for all* $i \geq 0$.

*Proof.* Note first that $\operatorname{reg} U/I \leq \operatorname{reg} I - 1$ and $\operatorname{reg} U/J \leq \operatorname{reg} J - 1$, by Lemma 2.4.3(3). Take a minimal graded free resolution of $U/I$:

$$0 \to \bigoplus_{i=1}^{r_{(t+1)}} U[-e_{i,t+1}] \to \ldots \to \bigoplus_{i=1}^{r_0} U[-e_{i,0}] \to U/I \to 0.$$

By the definition of regularity, $e_{i,j} \leq \operatorname{reg} U/I + (t+1) \leq \operatorname{reg} I + t$ for all $i, j \geq 0$. Now apply $\underline{\operatorname{Hom}}(-, U/J)$ to the complex with the $U/I$ term deleted, producing a complex

$$0 \to L_0 \xrightarrow{\psi^0} L_1 \xrightarrow{\psi^1} \ldots \xrightarrow{\psi^t} L_{t+1} \to 0$$

where $L_j = \bigoplus_{i=1}^{r_j} U/J[-e_{i,j}]$. Then $\operatorname{reg} L_j \leq (\operatorname{reg} I + \operatorname{reg} J + t - 1)$ for all $j \geq 0$.

Now consider the map $\psi^i : L_i \to L_{i+1}$ for some $i \geq 0$. Certainly $L_i$ is generated in degrees less than or equal to $\operatorname{reg} L_i$, by Lemma 2.4.5(1). Then $\operatorname{Im} \psi^i$ is also generated in degrees less than or equal to $\operatorname{reg} L_i$. By Lemma 2.4.6, $\operatorname{reg}(\operatorname{Im} \psi^i) \leq f(\operatorname{reg} L_i)$ where $f(x) = (2x)^{t!}$. By Lemma 2.4.3(1),

$$\operatorname{reg}(\ker \psi^i) \leq \max(\operatorname{reg} L_i, \operatorname{reg}(\operatorname{Im} \psi^i) + 1) \leq f(\operatorname{reg} L_i) + 1.$$

Finally, $\underline{\operatorname{Ext}}^i(U/I, U/J) \cong \ker \psi^i / \operatorname{Im} \psi^{i-1}$ and so by 2.4.3(3),

$$\begin{aligned} \operatorname{reg}(\underline{\operatorname{Ext}}^i(U/I, U/J)) &\leq \max(f(\operatorname{reg} L_i) + 1, f(\operatorname{reg} L_{i-1})) \\ &\leq f(\operatorname{reg} I + \operatorname{reg} J + t - 1) + 1 \end{aligned}$$

and thus we may take $d = f(\operatorname{reg} I + \operatorname{reg} J + t - 1) + 1$. □

**Corollary 2.4.8.** *Let $I, J$ be any homogeneous ideals of $U$, and let $\phi$ be an automorphism of $U$. Then there is some fixed $d \geq 0$ such that for all $n \in \mathbb{Z}$ such that $U/(I + \phi^n(J))$ is bounded, $\underline{\operatorname{Ext}}^i_U(U/I, U/\phi^n(J))$ is also bounded with $d$ as a right bound.*

*Proof.* If $U/(I+\phi^n(J))$ is bounded, then $E^n = \underline{\mathrm{Ext}}^i_U(U/I, U/\phi^n(J))$ is certainly also bounded, since it is killed by $I + \phi^n(J)$. It is clear from the definition of regularity that the ideals $\{\phi^n(J)\}_{n\in\mathbb{Z}}$ all have the same regularity, and so by Lemma 2.4.7 there is some bound $d \geq 0$ such that $\operatorname{reg} E^n \leq d$ for all $n \in \mathbb{Z}$. Then if $E^n$ is bounded, $d$ is a right bound for it by Lemma 2.4.5(2). □

## 2.5 Ideals of point sets

In Chapters II and III we shall need certain detailed lemmas about the ideals of points with multiplicities, especially about the relationship between products and intersections of such ideals. We will use the theory of Castelnuovo-Mumford regularity which we introduced in the previous section as a tool to give brief proofs of the facts we require.

As before, let $U = k[x_0, \ldots x_t]$ be a polynomial ring, with corresponding projective space $\operatorname{Proj} U = \mathbb{P}^t$. Throughout this thesis, the word *point* will always refer to a closed point of an (irreducible) variety. We often use the notation $\mathfrak{m}_d$ for the homogeneous ideal of polynomials in $U$ which vanish at a point $d$ of $\mathbb{P}^t$. We shall use the notation $f \in \mathfrak{m}_d$ and $f(d) = 0$ interchangeably.

The only nontrivial ingredients that we need are the following theorems of Conca and Herzog concerning the regularity of products.

**Theorem 2.5.1.** *[14, Theorem 2.5]*

*If $I$ is a graded ideal of $U$ with $\dim U/I \leq 1$, then for any $M \in U$-gr we have $\operatorname{reg} IM \leq \operatorname{reg} I + \operatorname{reg} M$.* □

**Theorem 2.5.2.** *[14, Theorem 3.1]*

*Let $I_1, I_2, \ldots, I_e$ be (not necessarily distinct) nonzero ideals of $U$ generated by linear forms. Then $\operatorname{reg}(I_1 I_2 \ldots I_e) = e$.* □

We fix some notation for the following string of lemmas. Let $d_1, d_2, \dots, d_n$ be distinct points in $\mathbb{P}^t$, and $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ the corresponding prime ideals of $U$. Let $0 < e_i \in \mathbb{N}$ for all $i$ and set $e = \sum_{i=1}^n e_i$. Let $I = (\mathfrak{m}_1^{e_1}\mathfrak{m}_2^{e_2} \dots \mathfrak{m}_n^{e_n})$ and $J = (\mathfrak{m}_1^{e_1} \cap \mathfrak{m}_2^{e_2} \cap \dots \cap \mathfrak{m}_n^{e_n})$.

**Lemma 2.5.3.**

*(1) $J$ is saturated.*

*(2) $I^{sat} = J$.*

*Proof.* (1) It is easy to see that a proper graded ideal is saturated if and only if the irrelevant ideal $U_{\geq 1}$ is not one of its associated primes. Each $\mathfrak{m}_i^{e_i}$ is already a primary ideal with a single associated prime $\mathfrak{m}_i$, so $J = \bigcap_{i=1}^n \mathfrak{m}_i^{e_i}$ is a primary decomposition for $J$, and the primes associated to $J$ are just the $\mathfrak{m}_i$.

(2) $J/I$ is killed by the ideal $K = \sum_i \prod_{j \neq i} \mathfrak{m}_j^{e_j}$. Now $K$ is contained in no point ideal of $U$, so $U_{\geq m} \subseteq K$ for some $m$ and thus $J/I$ is torsion. By part (1) we must have $I^{sat} = J$. $\square$

**Lemma 2.5.4.**

*(1)* $\operatorname{reg} I = e$.

*(2)* $I_{\geq e} = J_{\geq e}$.

*(3)* $\operatorname{reg} J \leq e$.

*Proof.* (1) This is an immediate consequence of Theorem 2.5.2.

(2) By Lemma 2.5.3(2), we know that $I^{sat} = J$. Since $I$ is $e$-regular by the first part, $\operatorname{sat} I \leq e$ by Lemma 2.4.5(2). Thus $I_{\geq e} = (I^{sat})_{\geq e} = J_{\geq e}$.

(3) This follows from Lemma 2.4.5(3) and the first two parts. $\square$

When the points $\{d_i\}_{i=1}^n$ do not all lie on a line, we can get better degree bounds.

**Lemma 2.5.5.** *Assume that the points $d_1, d_2, \ldots, d_n$ do not all lie on a line. Then* $\operatorname{reg} J \leq e - 1$.

*Proof.* The hypothesis on the points forces some three of the points $\{d_i\}$ to be non-collinear (in particular $n \geq 3$); by relabeling we may assume that $d_1, d_2, d_3$ do not lie on a line. Let $K = (\mathfrak{m}_1 + \mathfrak{m}_2 \cap \mathfrak{m}_3)$. Then $K_1 = U_1$, and so clearly $K = U_{\geq 1}$. Thus $\operatorname{reg} K = 1$ (for instance by Theorem 2.5.2). By Lemma 2.4.4 and Lemma 2.5.4(3) we have that $(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3)$ is 2-regular.

Now $L = (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3)(\mathfrak{m}_1^{e_1-1} \cap \mathfrak{m}_2^{e_2-1} \cap \mathfrak{m}_3^{e_3-1} \cap \mathfrak{m}_4^{e_4} \cap \cdots \cap \mathfrak{m}_n^{e_n})$ is $(e-1)$-regular, using Lemma 2.5.4 and Theorem 2.5.1. Since $I = (\mathfrak{m}_1^{e_1}\mathfrak{m}_2^{e_2} \ldots \mathfrak{m}_n^{e_n}) \subseteq L$, clearly $L^{sat} = J$ by Lemma 2.5.3(2). Since $L$ is $(e-1)$-regular, $\operatorname{sat} L \leq e-1$ by Lemma 2.4.5(2) and so $L_{\geq e-1} = J_{\geq e-1}$. Thus $J$ is $(e-1)$-regular by Lemma 2.4.5(3).

$\square$

Now we state the series of results which we shall use later in the paper. The first is a simple Hilbert function calculation, which is presumably well-known. Because this lemma will be so fundamental below, we include a brief proof.

**Lemma 2.5.6.** *Let $e_i > 0$ for all $1 \leq i \leq n$, and let $e = \sum e_i$. Set $J = \bigcap_{i=1}^n \mathfrak{m}_i^{e_i}$ for some distinct point ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$. Then $\dim_k J_m = \binom{m+t}{t} - \sum_i \binom{e_i+t-1}{t}$ for all $m \geq e-1$.*

*In particular, if $J = \bigcap_{i=1}^n \mathfrak{m}_i$ then $\dim_k J_m = \binom{m+t}{t} - n$ for $m \geq n-1$.*

*Proof.* Suppose that $n = 1$ and $\mathfrak{m}_1 = (x_0, x_1, \ldots, x_{t-1})U$. Then $(\mathfrak{m}_1^{e_1})_m$ has as a basis all monomials of degree $m$ containing $x_t$ to the power at most $m - e_1$, and a combinatorial count gives that $\dim_k(U/\mathfrak{m}_1^{e_1})_m = \binom{e_1+t-1}{t}$ for all $m \geq e_1 - 1$. By symmetry, the same result holds for an arbitrary point ideal $\mathfrak{m}_i$. Then

$$\dim_k(U/J)_m \leq \sum_{i=1}^n \dim_k(U/\mathfrak{m}_i^{e_i})_m = \textstyle\sum_{i=1}^n \binom{e_i+t-1}{t}$$

holds for $m \geq \max(e_i - 1)$ so certainly for $m \geq e - 1$.

Assume now that $m \geq e-1$. Since the base field $k$ is infinite and $m-e_i+1 \geq n-1$, it is easy to see that for each $i$ we may choose a polynomial $f_i \in U_{m-e_i+1}$ such that $f_i \in \mathfrak{m}_j$ for for all $j \neq i$ but $f_i \notin \mathfrak{m}_i$. Then one may check that the vector space $V = \sum_{i=1}^n U_{e_i-1} f_i \subseteq U_m$ satisfies $V \cap J_m = 0$, and furthermore $\dim_k V = \sum_{i=1}^n \binom{e_i+t-1}{t}$. We conclude that $\dim_k (U/J)_m \geq \sum_{i=1}^n \binom{e_i+t-1}{t}$. Thus $\dim_k (U/J)_m = \sum_{i=1}^n \binom{e_i+t-1}{t}$ and so $\dim_k J_m = \binom{m+t}{t} - \sum_i \binom{e_i+t-1}{t}$ as required. □

The next lemma, a "Chinese remainder theorem" type result, follows easily from the preceding lemma; we leave the proof to the reader.

**Lemma 2.5.7.** *Let $\{d_i\}_{i=1}^n$ be distinct points in $\mathbb{P}^t$. Fix some particular choice of homogeneous coordinates for the points $d_i$, so that $f(d_i)$ is defined for $f \in U$. Then given scalars $a_1, a_2, \dots, a_n \in k$, there is a polynomial $f \in U_m$ for any $m \geq n-1$ with $f(d_i) = a_i$ for all $1 \leq i \leq n$.* □

We restate for easy reference below the special case of Lemma 2.5.4(2) above where all $e_i = 1$.

**Lemma 2.5.8.** *Let $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$ be the ideals of $U$ corresponding to distinct points $d_1, \dots d_n$ in $\mathbb{P}^t$. Then $(\prod_{i=1}^n \mathfrak{m}_i)_{\geq n} = (\bigcap_{i=1}^n \mathfrak{m}_i)_{\geq n}$.* □

The last lemma is quite technical and will be required only in §3.4.

**Lemma 2.5.9.** *Let the points $d_1, d_2, \dots, d_n, d_{n+1}$ be distinct, and assume that the points $d_1, \dots, d_n$ do not all lie on a line. Let $\mathfrak{m}_i \subseteq U$ be the homogeneous ideal corresponding to $d_i$.*

*(1)* $(\bigcap_{i=1}^n \mathfrak{m}_i)_{n-1} (\mathfrak{m}_{n+1})_1 = (\bigcap_{i=1}^{n+1} \mathfrak{m}_i)_n$.

*(2)* $(\bigcap_{i=1}^n \mathfrak{m}_i)_{n-1} (\mathfrak{m}_1)_1 = (\bigcap_{i=2}^n \mathfrak{m}_i \cap \mathfrak{m}_1^2)_n$.

*(3)* $(\bigcap_{i=2}^n \mathfrak{m}_i \cap \mathfrak{m}_1^2)_n (\mathfrak{m}_{n+1})_1 = (\bigcap_{i=2}^{n+1} \mathfrak{m}_i \cap \mathfrak{m}_1^2)_{n+1}$.

*(4) Let $b_1, b_2 \in \mathbb{P}^t$, with corresponding ideals $\mathfrak{n}_1, \mathfrak{n}_2$, be such that $b_j \neq d_i$ for $j = 1, 2$ and $1 \leq i \leq n$. Then $(\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_1)_n = (\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_2)_n$ implies $b_1 = b_2$.*

*Proof.* (1) Set $K = \bigcap_{i=1}^n \mathfrak{m}_i$, $L = \mathfrak{m}_{n+1}$, and $M = \bigcap_{i=1}^{n+1} \mathfrak{m}_i$. By Lemmas 2.5.4 and 2.5.5, we have that $\operatorname{reg} K \leq n-1$ and $\operatorname{reg} L \leq 1$. Then by Theorem 2.5.1, $\operatorname{reg}(KL) \leq n$. Since clearly $M = (KL)^{sat}$, by Lemma 2.4.5(2) it follows that $(KL)_n = M_n$. Finally, by Lemma 2.4.5(1), $K$ is generated in degrees $\leq n-1$ and $L$ is generated in degree 1. Thus $K_{n-1}L_1 = (KL)_n = M_n$.

(2)-(3) The proofs of these parts are very similar to the proof of (1) and are omitted.

(4) The ideals $K = (\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_1)$ and $L = (\bigcap_{i=1}^n \mathfrak{m}_i \cap \mathfrak{n}_2)$ are each $n$-regular by Lemma 2.5.5, so both are generated in degrees $\leq n$. Now since $b_1 \neq d_i$ for all $i$, if $b_1 \neq b_2$ then the ideals $K$ and $L$ must differ in large degree, so they must differ in degree $n$. $\square$

# CHAPTER III

## The Noetherian and Strong Noetherian Properties

In the present chapter, we will define the classes of rings which will be of interest to us for the rest of the thesis. For every automorphism $\varphi$ of $\mathbb{P}^t$ we define a Zhang twist $S = S(\varphi)$ of a commutative polynomial ring, and for each choice of a point $c \in \mathbb{P}^t$ we define a subalgebra $R = R(\varphi, c)$ of $S$. In the first two sections, we discuss some generalities about such rings; most important, we prove a simple characterization of the elements of $R$ among those of $S$ (Theorem 3.2.3). This will allow us to understand contractions and extensions of left ideals between the rings $R$ and $S$, and in §3.3 we will use this information to show that the noetherian property for $R$ is equivalent to a geometric condition on the set of points $\{\varphi^i(c)\}_{i \in \mathbb{Z}}$, namely that this set should be *critically dense* in $\mathbb{P}^t$ (see Definition 3.3.11). Later on, in Chapter V, we will show that this condition holds for generic choices of $\varphi$ and $c$.

Next, we completely classify all of the point modules for $R$. It will follow from the classification that for any $n \geq 0$, there is a whole $\mathbb{P}^{t-1}$-parameterized family of different $R$-point modules $\{P(e)\}_{e \in \mathbb{P}^{t-1}}$ such that all of the truncations $P(e)_{\leq n}$ are isomorphic. By a theorem of Artin and Zhang (Theorem 1.4.2), such behavior implies that the strong noetherian property fails for $R$. In the final section, we actually construct a commutative ring $B$ for which $R \otimes_k B$ is not noetherian, and thus demonstrate explicitly the failure of the strong noetherian property for $R$. The ring $B$ is obtained by an interesting geometric construction—it is the coordinate ring of an infinite affine blowup of affine space at a countable point set.

### 3.1 The algebras $S(\varphi)$

Fix a polynomial ring $U = k[x_0, x_1, \dots, x_t]$, where from now on we always assume that $t \geq 2$. For any graded automorphism $\phi$ of $U$ let $S$ be the left Zhang twist of $U$ by the twisting system $\{\phi^i\}_{i \in \mathbb{N}}$ (as defined in §2.3). The automorphism $\phi$ of $U$ induces an automorphism $\varphi$ of $\mathbb{P}^t$, and as we saw in Example 2.3.4 and the following commentary, $S$ is isomorphic to the twisted homogeneous coordinate ring $B(\mathbb{P}^t, \mathcal{O}(1), \varphi^{-1})$. Thus $S$ is determined up to automorphism by the geometric data $\varphi$ and we write $S = S(\varphi)$. An alternative way of seeing that $\varphi$ determines the particular twist $S$ is to note that automorphisms $\phi_1, \phi_2$ of $U$ give the same automorphism $\varphi$ of $\mathbb{P}^t$ if and only if $\phi_1 = a\phi_2$ for some $a \in k^\times$ [18, II 7.1.1]. Then automorphisms of $U$ which are scalar multiples give isomorphic Zhang twists $S$ [43, Proposition 5.13].

We identify the underlying vector spaces of $S$ and $U$. Since $S$ and its subalgebras are our main interest, our notational convention from now on (except in the appendix) will be to let juxtaposition indicate multiplication in $S$ and to use the symbol $\circ$ for multiplication in $U$. Many basic properties of the ring $S = S(\varphi)$ are immediate since they are obvious for $U$ and invariant under Zhang twists. In particular, $S$ is a noetherian domain of GK dimension $t + 1$ [43, Propositions 5.1,5.2,5.7]. As in §2.3, we also have an equivalence of the graded module categories $\theta : U\text{-Gr} \sim S\text{-Gr}$. For any graded ideal $I$ of $U$, $\theta(I)$ is a graded left ideal of $S$. We often simply identify these left ideals and call both $I$.

Let $\mathfrak{m}_d$ stand for the ideal in $U$ of a point $d \in \mathbb{P}^t$. Then $\mathfrak{m}_{\varphi(d)} = \phi^{-1}(\mathfrak{m}_d)$ for all $d \in \mathbb{P}^t$. Since the equivalence of categories preserves Hilbert functions and the property of being cyclic, it is clear that the point modules over $S$ are the modules of the form $\theta(U/\mathfrak{m}_d) = S/\mathfrak{m}_d$ for $d \in \mathbb{P}^t$. We will use the following notation:

**Notation 3.1.1.** Given a point $d \in \mathbb{P}^t$, let $P(d)$ be the left point module $\theta(U/\mathfrak{m}_d) = S/\mathfrak{m}_d$ of $S$.

We may also describe point modules over $S$ by their *point sequences*. If $M$ is a point module over $S$, then the annihilator of $M_n$ in $S_1 = U_1$ is some codimension 1 subspace which corresponds to a point $d_n \in \mathbb{P}^t$. The point sequence of $M$ is defined to be the sequence $(d_0, d_1, d_2, \dots)$ of points of $\mathbb{P}^t$. Clearly two $S$-point modules are isomorphic if and only if they have the same point sequence.

**Lemma 3.1.2.** *Let $d$ be an arbitrary point of $\mathbb{P}^t$.*

*(1) $P(d)$ has point sequence $(d, \varphi(d), \varphi^2(d), \dots)$.*

*(2) $(P(d))_{\geq n} \cong P(\varphi^n(d))[-n]$ as $S$-modules.*

*Proof.* (1) By definition $P(d) = S/\mathfrak{m}_d$. If $f \in S_1$, then $fS_n = \phi^n(f) \circ U_n \subseteq \mathfrak{m}_d$ if and only if $\phi^n(f) \in \mathfrak{m}_d$, in other words $f \in \phi^{-n}(\mathfrak{m}_d) = \mathfrak{m}_{\varphi^n(d)}$.

(2) By part (1), $(P(d))_{\geq n}$ is the shift by $(-n)$ of the point module with point sequence $(\varphi^n(d), \varphi^{n+1}(d), \dots)$. □

Finally, we record the following simple facts which we shall use frequently.

**Lemma 3.1.3.** *(1) If $M$ is a cyclic graded 1-critical $S$-module, then $M \cong P(d)[i]$ for some $d \in \mathbb{P}^t$ and $i \in \mathbb{Z}$.*

*(2) If $M \in S$-gr, then $M$ has a finite filtration with factors which are graded cyclic critical $S$-modules.*

*Proof.* (1) The equivalence of categories $U$-Gr $\sim$ $S$-Gr preserves the GK-dimension of finitely generated modules, since it preserves Hilbert functions, and so it also preserves the property of being GK-critical. It is standard that the cyclic 1-critical graded $U$-modules are just the point modules over $U$ and their shifts. Under the equivalence of categories, the corresponding $S$-modules are the $S$-point modules and their shifts.

(2) Each module $N \in U$-gr has a finite filtration composed of graded cyclic critical $U$-modules, so the same holds for $S$-modules by the equivalence of categories. $\square$

## 3.2 The algebras $R(\varphi, c)$

Let $S = S(\varphi)$ for some $\varphi \in \operatorname{Aut} \mathbb{P}^t$. For any codimension-1 subspace $V$ of $S_1 = U_1$, we let $R = k\langle V \rangle \subseteq S$ be the subalgebra of $S$ generated by $V$. The vector subspace $V$ of $U_1$ corresponds to a unique point $c \in \mathbb{P}^t$. Then $R$ is determined up to isomorphism by the geometric data $(\varphi, c)$ and we write $R = R(\varphi, c)$. We emphasize again that we always assume that $t \geq 2$ from now on; for smaller $t$ the ring $R$ is not very interesting.

We shall see that the basic properties of $R(\varphi, c)$ depend closely on properties of the iterates of the point $c$ under $\varphi$. It is convenient to let $c_i = \varphi^{-i}(c)$ for all $i \in \mathbb{Z}$. Then the ideal of the point $c_i$ is $\phi^i(\mathfrak{m}_c)$. In case $c$ has finite order under $\varphi$, that is $\phi^n(c) = c$ for some $n > 0$, the algebra $R(\varphi, c)$ behaves quite differently from the case where $c$ has infinite order. The finite order case turns out to have none of the interesting properties of the infinite order case (see Lemma 5.2.5), and so we will exclude it.

**Standing Hypothesis 3.2.1.** *Assume that $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$ is given such that $c$ has infinite order under $\varphi$, or equivalently the points $\{c_i\}_{i \in \mathbb{Z}}$ are distinct.*

We note some relationships among the various $R(\varphi, c)$. In particular, part (1) of the next lemma will allow us to transfer our left sided results to the right.

**Lemma 3.2.2.** *Let* $(\varphi, c) \in (\operatorname{Aut}\mathbb{P}^t) \times \mathbb{P}^t$*, and let* $\psi$ *be any automorphism of* $\mathbb{P}^t$*. Then*

*(1)* $R(\varphi, c)^{op} \cong R(\varphi^{-1}, \varphi(c))$*.*

*(2)* $R(\varphi, c) \cong R(\psi\varphi\psi^{-1}, \psi(c))$*.*

*Proof.* (1) Set $S = S(\varphi)$ and $S' = S(\varphi^{-1})$, identifying the underlying spaces of each with that of $U$. Let $\phi$ be an automorphism of $U$ corresponding to $\varphi$. Then it is straightforward to check that the vector space map defined on the graded pieces of $U$ by sending $f \in U_m$ to $\phi^{-m}(f) \in U_m$ is a graded algebra isomorphism from $S^{op}$ to $S'$. The isomorphism maps $(\mathfrak{m}_c)_1$ to $(\mathfrak{m}_{\varphi(c)})_1$ and so it restricts to an isomorphism $R(\varphi, c)^{op} \cong R(\varphi^{-1}, \varphi(c))$.

(2) Similarly, let $\sigma$ be an automorphism of $U$ corresponding to $\psi$. One checks that the vector space map of $U$ defined by $f \mapsto \sigma^{-1}(f)$ is an isomorphism of $S(\varphi)$ onto $S(\psi\varphi\psi^{-1})$ which maps $(\mathfrak{m}_c)_1$ to $\sigma^{-1}(\mathfrak{m}_c)_1 = (\mathfrak{m}_{\psi(c)})_1$, and so restricts to an isomorphism $R(\varphi, c) \cong R(\psi\varphi\psi^{-1}, \psi(c))$. $\square$

We now prove an important characterization of the elements of $R = R(\varphi, c)$ which is foundational for all that follows.

**Theorem 3.2.3.** *Let* $R = R(\varphi, c)$*, and assume Hypothesis 3.2.1. Then for all* $n \geq 0$*,*

$$R_n = \{f \in U_n \mid f(c_i) = 0 \textit{ for } 0 \leq i \leq n-1\}.$$

*Proof.* By definition $R = k\langle V\rangle \subseteq S$, where $V = (\mathfrak{m}_c)_1$ considered as a subset of $U$. For $n = 0$ the statement of the theorem is $R_0 = U_0 = k$, which is clearly correct, so assume that $n \geq 1$. Then

$$R_n = V^n = \phi^{n-1}(V) \circ \phi^{n-2}(V) \circ \ldots \circ \phi(V) \circ V.$$

Now $\phi^i(V) = (\mathfrak{m}_{c_i})_1$, and the points $c_i$ are distinct by Hypothesis 3.2.1. Thus by Lemma 2.5.8 we get that

$$R_n = (\mathfrak{m}_{c_{n-1}})_1 \circ \ldots \circ (\mathfrak{m}_{c_1})_1 \circ (\mathfrak{m}_{c_0})_1 = [(\mathfrak{m}_{c_{n-1}}) \circ \ldots \circ (\mathfrak{m}_{c_1}) \circ (\mathfrak{m}_{c_0})]_n$$
$$= [(\mathfrak{m}_{c_{n-1}}) \cap \cdots \cap (\mathfrak{m}_{c_1}) \cap (\mathfrak{m}_{c_0})]_n.$$

The statement of the theorem in degree $n$ follows. □

The theorem has a number of easy consequences.

**Lemma 3.2.4.** *Let* $R = R(\varphi, c)$*. Then* $\dim_k R_n = \binom{n+t}{t} - n$ *for all* $n \geq 0$*. In particular,* $\mathrm{GK}(R) = t + 1$.

*Proof.* The Hilbert function of $R$ follows from Theorem 3.2.3 and Lemma 2.5.6. Since we always assume that $t \geq 2$, it is clear that the Hilbert polynomial of $R$ has degree $t$ and so $\mathrm{GK}(R) = t + 1$. □

**Lemma 3.2.5.** *The rings* $R = R(\varphi, c)$ *and* $S = S(\varphi)$ *have the same graded quotient ring* $D$ *and Goldie quotient ring* $Q$*. The inclusion* $R \hookrightarrow S$ *is a essential extension of left (or right)* $R$*-modules.*

*Proof.* Since both $R$ and $S$ are domains of finite GK-dimension, they both have graded quotient rings and Goldie quotient rings (see §2.1), and clearly the graded quotient ring $D'$ of $R$ is contained in the graded quotient ring $D$ of $S$. Since we assume always that $t \geq 2$, we may choose a nonzero polynomial $g \in S_1$ with $g \in \mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1}$. Then Theorem 3.2.3 implies that $g \in R_1$ and $S_1 g \subseteq R_2$. Thus $S_1 \subseteq R_2(R_1)^{-1} \subseteq D'$ and consequently $D' = D$. Then $Q$, the Goldie quotient ring of the domain $D$, is also the Goldie quotient ring for both $R$ and $S$. The last statement of the proposition is now clear. □

## 3.3 The noetherian property for $R$

Let $S = S(\varphi)$ and $R = R(\varphi, c)$. In this section we will characterize those choices of $\varphi$ and $c$ satisfying Hypothesis 3.2.1 for which the ring $R$ is noetherian. To do this, we will first analyze the structure of the factor module ${}_R(S/R)$ in detail, and then use this information to understand contractions and extensions of left ideals between $R$ and $S$.

The following notation will be convenient in this section.

**Notation 3.3.1.**

(1) $A_n = \{0, 1, \dots n-1\}$ for $n > 0$ and $A_n = \emptyset$ for $n \leq 0$.

(2) For $B \subseteq \mathbb{Z}$, set $B + m = \{b + m \mid b \in B\}$.

**Definition 3.3.2.** Let $B \subseteq \mathbb{N}$. We define a left $R$-module $T^B \subseteq S$ by specifying its graded pieces as follows:

$$(T^B)_n = \{f \in S_n \mid f(c_i) = 0 \text{ for } i \in A_n \setminus B\}$$

We then define the left $R$-module $M^B = T^B/R \subseteq (S/R)$.

We should check that $T^B$ really is closed under left multiplication by $R$. If $g \in R_m$ and $f \in (T^B)_n$, then $gf = \phi^n(g) \circ f$. Now $g(c_i) = 0$ for $i \in A_m$ by Theorem 3.2.3 and $f(c_i) = 0$ for $i \in A_n \setminus B$ by definition. Thus $[\phi^n(g) \circ f](c_i) = 0$ for $i \in (A_m + n) \cup (A_n \setminus B) \supseteq (A_{n+m} \setminus B)$, and so $gf \in (T^B)_{n+m}$ as required. Also, by Theorem 3.2.3 the extreme cases are $R = T^{\emptyset}$ and $S = T^{\mathbb{N}}$. In particular, $R \subseteq T^B$ always holds, and so $M^B$ is well defined.

**Lemma 3.3.3.** *The Hilbert function of $M^B$ is given by*

$$\dim_k(M^B)_n = |A_n \cap B|.$$

*Proof.* Immediate from Lemma 2.5.6. □

In the special case of Definition 3.3.2 where $B$ is a singleton set, $M^B$ is just a shifted $R$-point module.

**Lemma 3.3.4.** *Let $j \in \mathbb{N}$. Then $M = M^{\{j\}}$ is an $R$-point module shifted by $j+1$. In fact, $M \cong {}_R P(c_{-1})[-j-1]$.*

*Proof.* By Lemma 3.3.3 the Hilbert function of $M$ is

$$\dim_k M_n = \begin{cases} 0 & 0 \leq n \leq j \\ \\ 1 & j+1 \leq n \end{cases}$$

so that $M$ does have the Hilbert function of a point module shifted by $j+1$. For convenience of notation set $m = j+1$, and let us calculate $\operatorname{ann}_R(M_m)$. Now $fM_m = 0$ for $f \in R_n$ if and only if $f(T^{\{j\}})_m \subseteq R_{m+n}$. Since $M_m \neq 0$, we may choose $g \in (T^{\{j\}})_m$ such that $g \notin R$; then $g(c_j) \neq 0$. Also, because $\dim_k M_m = 1$ it is clear that $(T^{\{j\}})_m = R_m + kg$, and so $f(T^{\{j\}})_m \subseteq R$ if and only if $fg \in R$. Now $fg = \phi^m(f) \circ g$, and so by Theorem 3.2.3 we have that $fg \in R$ if and only if $\phi^m(f)(c_j) = 0$, equivalently $f(c_{-1}) = 0$, since $m = j+1$. In conclusion, $\operatorname{ann}_R(M_m) = \mathfrak{m}_{c_{-1}} \cap R$.

Thus we have an injection of $R$-modules given by right multiplication by $g$:

$$\psi : (R/(\mathfrak{m}_{c_{-1}} \cap R))[-m] \xrightarrow{g} T^{\{j\}}/R = M.$$

By Lemma 2.5.6, $R/(\mathfrak{m}_{c_{-1}} \cap R)$ has the Hilbert function of a point module and so both sides have the same Hilbert function. Thus $\psi$ is actually an isomorphism. In particular, $M$ is cyclic and so is a shifted $R$-point module.

We also have the injection $R/(\mathfrak{m}_{c_{-1}} \cap R) \to S/(\mathfrak{m}_{c_{-1}}) = P(c_{-1})$, and since both sides have the Hilbert function of a point module this is also an isomorphism of $R$-modules. So $M \cong {}_R(P(c_{-1}))[-j-1]$. □

We may now understand the structure of ${}_R(S/R)$ completely.

**Proposition 3.3.5.** *The modules* $\{M^{\{j\}}\}_{j\in\mathbb{N}}$ *are independent submodules of* $S/R$*. Also, for* $B \subseteq \mathbb{N}$,

$$M^B = \bigoplus_{j\in B} M^{\{j\}}.$$

*Proof.* We first show the independence of the $M^{\{j\}}$. It is enough to work with homogeneous elements; fix $n \geq 0$ and let $\sum_{j\in\mathbb{N}} f_j = 0$ for some $f_j \in (M^{\{j\}})_n$. Let $f_j = g_j + R$ for some elements $g_j \in (T^{\{j\}})_n \subseteq S_n$. Thus $\sum g_j \in R_n$. Suppose that some $f_j \neq 0$, and let $k = \min\{j|f_j \neq 0\}$. Now $(M^{\{j\}})_{\leq j} = 0$ for all $j$ by Lemma 3.3.3, and so we must have $k \leq n-1$. Then $k \in A_n \setminus \{j\}$ for any $j \neq k$, so that $g_j(c_k) = 0$ for all $j \neq k$, by the definition of $T^{\{j\}}$. Since $\sum g_j \in R_n$, Theorem 3.2.3 implies that $(\sum g_j)(c_k) = 0$ also holds and so $g_k(c_k) = 0$. But then $g_k \in (T^{\{k\}})_n \cap (\mathfrak{m}_{c_k}) = R_n$, and thus $f_k = 0$, a contradiction. We conclude that all $f_j = 0$, and so the $M^{\{j\}}$ are independent.

For the second statement of the proposition, by Lemma 3.3.3 the Hilbert function of $M^B$ is $\dim_k(M^B)_n = |A_n \cap B|$, while the Hilbert function of $\bigoplus_{j\in B} M^{\{j\}}$ is

$$\dim_k \big[\bigoplus_{j\in B} M^{\{j\}}\big]_n = \#\{j \in B | j \leq n-1\} = |A_n \cap B|.$$

Thus the Hilbert functions are the same on both sides of our claimed equality. Since $\sum_{j\in B} M^{\{j\}} \subseteq M^B$ is clear and we know that the $M^{\{j\}}$ are independent by the first part of the proposition, the equality follows. □

**Corollary 3.3.6.** *(1) Given* $B \subseteq \mathbb{N}$, $M^B$ *is a noetherian* $R$*-module if and only if the set* $B$ *has finite cardinality.*

*(2)* ${}_R(S/R) \cong \bigoplus_{j=0}^{\infty} {}_RP(c_{-1})[-j-1]$*. In particular,* ${}_R(S/R)$ *is not finitely generated.*

*Proof.* (1) is clear since a point module is noetherian. (2) follows by taking $B = \mathbb{N}$ in the proposition and using also Lemma 3.3.4. □

Next, we analyze the noetherian property for some special types of $R$-modules which may be realized as subfactors of $S/R$.

**Proposition 3.3.7.** *For $f \in R_n$, let $N = (Sf \cap R)/Rf \in R$-Gr. Set $D = \{i \in \mathbb{N} \mid f(c_i) = 0\}$ and $B = (D - n) \cap \mathbb{N}$. Then*

*(1) $N \cong M^B[-n]$.*

*(2) $N$ is noetherian if and only if $|D| < \infty$.*

*Proof.* First, if we set $T = \{g \in S \mid gf \in R\}$ and $M = T/R$, then $N \cong M[-n]$. So it is enough for (1) to show that $T = T^B$.

Let $g \in S_m$ be arbitrary. Note that $A_n \subseteq D$ since $f \in R_n$. Then

$$
\begin{aligned}
& gf = \phi^n(g) \circ f \in R \\
\Longleftrightarrow\ & [\phi^n(g) \circ f](c_i) = 0 && \text{for all } i \in A_{n+m} \\
\Longleftrightarrow\ & \phi^n(g)(c_i) = 0 && \text{for all } i \in A_{n+m} \setminus D \\
\Longleftrightarrow\ & g(c_i) = 0 && \text{for all } i \in (A_{n+m} \setminus D) - n \\
\Longleftrightarrow\ & g(c_i) = 0 && \text{for all } i \in A_m \setminus (D - n) \ \ (\text{since } A_n \subseteq D) \\
\Longleftrightarrow\ & g \in T^B && \text{by Definition 3.3.2}
\end{aligned}
$$

Thus $T = T^B$ and (1) holds.

For (2), note that $D$ has finite cardinality if and only if $B$ does, and apply Corollary 3.3.6(1). $\square$

**Proposition 3.3.8.** *For $f \in R_n$, let $M = S/(R + Sf) \in R$-Gr. Set $D = \{i \in \mathbb{N} \mid f(c_i) = 0\}$. Then*

*(1) $M \cong M^D$.*

*(2) $M$ is noetherian if and only if $|D| < \infty$.*

*Proof.* Set $B = \mathbb{N} \setminus D$. We will show that $R + Sf = T^B$. Then we will have that $S/(R + Sf) = S/T^B \cong (M^{\mathbb{N}}/M^B) \cong M^D$ by Proposition 3.3.5.

Suppose that $h \in (R+Sf)_m$; then $h = g_1 + g_2 f = g_1 + \phi^n(g_2) \circ f$ for some $g_1 \in R_m$ and $g_2 \in S_{m-n}$. Now $g_1(c_i) = 0$ for $i \in A_m$, and $f(c_i) = 0$ for $i \in D$, so that $h(c_i) = 0$ for $i \in A_m \cap D$. So we have $(R + Sf) \subseteq T^B$.

Note that $(R + Sf)_m = R_m = (T^B)_m$ for $m < n$, so assume that $m \geq n$ and let $h \in (T^B)_m$. Since $\{0, 1, \dots n-1\} \subseteq D$, we have $|A_m \setminus D| \leq m - n$. Fix some particular choices of homogeneous coordinates for the $c_i$, so that $g(c_i)$ is defined for homogeneous $g \in U$. We may choose $g \in U_{m-n}$ such that $[\phi^n(g)](c_i) = h(c_i)/f(c_i)$ for all $i \in A_m \setminus D$, by Lemma 2.5.7. Then $[h - \phi^n(g) \circ f](c_i) = 0$ for all $i \in A_m$ and so $h - gf \in R$. Thus $h \in R + Sf$. We conclude that $(R + Sf) = T^B$, as we wished, and (1) is proved.

Part (2) is then immediate from Corollary 3.3.6(1). □

Given a left ideal $I$ of $R$, we may extend to a left ideal $SI$ of $S$, and then contract back down to get the left ideal $SI \cap R$ of $R$. The factor $(SI \cap R)/I$ is built up out of the 2 types of modules we considered in Propositions 3.3.7 and 3.3.8.

**Lemma 3.3.9.** *Let $I$ be a finitely generated nonzero graded left ideal of $R$, and set $M = (SI \cap R)/I$. Then $M$ has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ such that each factor $M_{i+1}/M_i$ is isomorphic with shift to a subfactor of either $(Ss_i \cap R)/Rs_i$ or $S/(R + Ss_i)$ for some nonzero homogeneous $s_i \in R$.*

*Proof.* Let $I = \sum_{i=1}^{n} Rr_i$ for some homogeneous $r_i \in R$. If $n = 1$ the result is obvious, so assume that $n \geq 2$.

Set $J = \sum_{i=1}^{n-1} Rr_i$. By induction on $n$, $(SJ \cap R)/J$ and hence also $(SJ \cap R) + I/I$

have filtrations of the required type. It is enough then to show that

$$N = (SI \cap R)/((SJ \cap R) + I) = (SI \cap R)/((SJ + Rr_n) \cap R)$$

has the required filtration. But $N$ injects into $L = SI/(SJ + Rr_n)$. Now $R$ is an Ore domain by Lemma 3.2.5, so we may choose $0 \neq r \in R$ such that $rr_n \in J$. Then $L$ is a surjective image (with shift) of $S/(R + Sr)$, so $N$ is a subfactor of $S/(R + Sr)$. $\square$

In certain circumstances the noetherian property passes to subrings. The following lemma is just a slight variant of a number of similar results in the literature (for example, see [2, Lemma 4.2]).

**Lemma 3.3.10.** *Let $A \hookrightarrow B$ be any extension of $\mathbb{N}$-graded rings. Suppose that $B$ is left noetherian, and that $(BI \cap A)/I$ is a noetherian left $A$-module for all finitely generated homogeneous left ideals $I$ of $A$. Then $A$ is left noetherian.*

*Proof.* It is enough to prove that $A$ is graded left noetherian, that is that all homogeneous left ideals are finitely generated. Let $I$ be a homogeneous left ideal of $A$. Then $BI$ is a homogeneous left ideal of $B$, which is finitely generated since $B$ is noetherian, and so we may pick a finite set of homogeneous generators $r_1, r_2, \dots r_n \in I$ such that $BI = \sum_{i=1}^{n} Br_i$. Let $J = \sum_{i=1}^{n} Ar_i$. Then $BI = BJ$, and since $J$ is finitely generated over $A$ we may apply the hypothesis to conclude that $(BJ \cap A)/J = (BI \cap A)/J$ is a noetherian $A$-module. The submodule $I/J$ of $(BI \cap A)/J$ is then noetherian over $A$, in particular finitely generated over $A$. Finally, since $J$ is finitely generated over $A$, so is $I$. $\square$

We note the definition of an unusual geometric condition on a set of points of a variety, which appeared in [2, p. 582].

**Definition 3.3.11.** Let $\mathcal{S}$ be an infinite set of (closed) points of a variety $X$. We say $\mathcal{S}$ is *critically dense* in $X$ if every proper Zariski-closed subset $Y \subsetneq X$ contains only finitely many points of $\mathcal{S}$.

We may now prove our main result characterizing the noetherian property for $R$.

**Theorem 3.3.12.** *Let $R = R(\varphi, c)$ for some $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$ such that Hypothesis 3.2.1 holds. As always, set $c_i = \varphi^{-i}(c)$. Then*

*(1) $R(\varphi, c)$ is left noetherian if and only the set $\{c_i\}_{i \geq 0}$ is critically dense in $\mathbb{P}^t$.*

*(2) $R(\varphi, c)$ is right noetherian if and only the set $\{c_i\}_{i \leq -1}$ is critically dense in $\mathbb{P}^t$.*

*(3) $R(\varphi, c)$ is noetherian if and only the set $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$.*

*Proof.* (1) Set $\mathcal{C} = \{c_i\}_{i \geq 0}$ and suppose that $\mathcal{C}$ is critically dense. Then for any nonzero homogeneous polynomial $f \in R$, the set $D = \{i \in \mathbb{N} | f(c_i) = 0\}$ is finite, so by Propositions 3.3.7 and 3.3.8 the left $R$ modules $(Sf \cap R)/Rf$ and $S/(R + Sf)$ are noetherian. By Lemma 3.3.9, for any finitely generated homogeneous left ideal $I$ of $R$, the left $R$-module $(SI \cap R)/I$ is noetherian. By Lemma 3.3.10, $R$ is a left noetherian ring.

Conversely, if $\mathcal{C}$ fails to be critically dense, then we may choose a nonzero homogeneous polynomial $h \in S$ which vanishes at infinitely many points of $\mathcal{C}$. Since by Lemma 3.2.5 we know that $R \hookrightarrow S$ is an essential extension of $R$-modules, there exists a homogeneous $g \in R$ such that $0 \neq f = gh \in R$. Then $f$ also vanishes at infinitely many points of $\mathcal{C}$, and so by Proposition 3.3.7 $(Sf \cap R)/Rf$ is not a noetherian left $R$-module. Since this module is a subfactor of $R$, we conclude that $R$ is not a left noetherian ring.

(2) Using Lemma 3.2.2(1), this part follows immediately from part (1).

(3) This follows from the fact that for any infinite sets $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{P}^t$, $\mathcal{C}_1 \cup \mathcal{C}_2$ is

critically dense if and only if both $\mathcal{C}_1$ and $\mathcal{C}_2$ are. $\square$

In Section 5.1 we will examine the critical density condition appearing in Theorem 3.3.12 more closely. In particular, we shall prove that there exist many choices of $\varphi$ and $c$ for which $R(\varphi, c)$ is noetherian:

**Proposition 3.3.13.** *(See Theorem 5.1.5 below) Let $\varphi$ be the automorphism of $\mathbb{P}^t$ defined by $(a_0 : a_1 : \cdots : a_t) \mapsto (a_0 : p_1 a_1 : p_2 a_2 : \cdots : p_t a_t)$, and let $c$ be the point $(1 : 1 : \cdots : 1) \in \mathbb{P}^t$. If the scalars $\{p_1, p_2, \ldots p_t\}$ are algebraically independent over the prime subfield of $k$, then $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense and $R(\varphi, c)$ is noetherian.* $\square$

The noetherian case is our main interest, so in the remainder of Chapter III and throughout Chapter IV we will assume the following hypothesis.

**Standing Hypothesis 3.3.14.** *Let $c_i = \varphi^{-i}(c)$. Assume that the point set $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$, so that $R(\varphi, c)$ is noetherian. We will refer to this as the* critical density condition*.*

Below, we will frequently use the following exact sequence to study an arbitrary cyclic left $R$-module $R/I$:

$$0 \to (SI \cap R)/I \to R/I \to S/SI \to S/(R + SI) \to 0. \tag{3.3.15}$$

We note what the results of this section tell us about the terms of this sequence.

**Lemma 3.3.16.** *Assume the critical density condition, and let $0 \neq I$ be a graded left ideal of $R$.*

*(1) As left $R$-modules, $(SI \cap R)/I$ and $S/(R+SI)$ have finite filtrations with factors which are either torsion or a tail of the shifted $R$-point module ${}_R(P(c_{-1}))[-i]$ for some $i \geq 0$. In particular, $S/(R + SI)$ is a noetherian left $R$-module.*

*(2) ${}_R(S/J)$ is a noetherian module for all nonzero left ideals $J$ of $S$.*

*Proof.* (1) Let $0 \neq f \in R$ be arbitrary. Since $\{c_i\}_{i\in\mathbb{Z}}$ is a critically dense set, $f(c_i) = 0$ holds for only finitely many $i \in \mathbb{Z}$. Then by the results 3.3.4—3.3.8, the left $R$-modules $(Sf \cap R)/Rf$ and $S/(R + Sf)$ are isomorphic to finite direct sums of shifted point modules of the form ${}_R(P(c_{-1}))[-i]$ for various $i \geq 0$. Now using Lemma 3.3.9, it is clear that $(SI \cap R)/I$ has a filtration of the right kind. Similarly, $S/(R+SI)$ is a homomorphic image of $S/(R+Sf)$ for any $0 \neq f \in I$, so it also has the required filtration and is clearly noetherian.

(2) It is immediate from the exact sequence (3.3.15) for $I = Rr$ and part (1) that ${}_R(S/Sr)$ is noetherian for any homogeneous $0 \neq r \in R$. It is enough to show that ${}_R(S/Sx)$ is noetherian for an arbitrary homogeneous $0 \neq x \in S$. There is some nonzero homogeneous $y \in R$ such that $yx \in R$, by Lemma 3.2.5. Then since $(S/Syx)$ is a noetherian $R$-module, so is $S/Sx$. □

## 3.4 Point modules and the strong noetherian property

Let $S = S(\varphi)$ and $R = R(\varphi, c)$ for $(\varphi, c)$ satisfying the critical density condition, so that $R$ is noetherian. Recall from §2.1 the definition of point modules and point ideals for an $\mathbb{N}$-graded algebra. Using an explicit presentation for the ring, Jordan [19] classified the point modules for $R$ in a special case. We classify the point modules for the rings $R(\varphi, c)$ in general and get a similar result, using a different method which does not rely on relations. From the classification it will follow that $R$ is not strongly noetherian, using a Theorem of Artin and Zhang (see §1.4).

We have already seen that the point modules over $S$ are easily classified up to isomorphism—they are simply the $\{P(d) \mid d \in \mathbb{P}^t\}$ (recall Notation 3.1.1). There is a close relationship between the point modules over the rings $S$ and $R$, as we begin to see in the next proposition.

**Proposition 3.4.1.** *Let $M$ be a point module over $R$. Then $M_{\geq n} \cong {}_R(P_{\geq n})$ for some $S$-point module $P$ and some $n \geq 0$.*

*Proof.* We have $M = R/I$ for a unique point ideal $I$ of $R$. We will use the exact sequence (3.3.15); there are two cases.

Suppose first that $(SI \cap R)/I = 0$. Then we have an injection $R/I \to S/SI$. By Lemma 3.3.16(1) we know that $\mathrm{GK}_R(S/(R + SI)) \leq 1$, and clearly $\mathrm{GK}_R(R/I) = 1$, so that $\mathrm{GK}_R(S/SI) = 1$ since GK-dimension is exact for modules over the graded noetherian ring $R$. By Lemma 3.3.16(2), ${}_R(S/SI)$ is finitely generated, and so $\mathrm{GK}_S(S/SI) = \mathrm{GK}_R(S/SI) = 1$ since for finitely generated modules the GK-dimension depends only on the Hilbert function. Now choose a filtration of $S/SI$ composed of cyclic critical $S$-modules (Lemma 3.1.3(2)); the factors must be shifts of $S$-point modules and ${}_Sk$. Since $M$ is a $R$-submodule of $S/SI$, this forces some tail of $M$ to agree with a tail of an $S$-point module.

Suppose instead that $N = (SI \cap R)/I \neq 0$. Then $N$ is a nonzero submodule of the point module $M$, so it is equal to a tail of $M$. By Lemma 3.3.16(1), some tail of $N$, and thus a tail of $M$, must be isomorphic as an $R$-module to a tail of some $P(c_{-1})[-i] \cong P(c_{-1-i})_{\geq i}$ (using also Lemma 3.1.2). □

The next two results are technical preparation for the $R$-point module classification.

**Lemma 3.4.2.** *Let $I$, $J$ be point ideals for the ring $R$. If $I_{\geq m} = J_{\geq m}$ for some $m \geq 0$ then $I = J$.*

*Proof.* Suppose that $I \neq J$. Let $n$ be minimal such that $I_n \neq J_n$. Then $R_1 I_n \subseteq I_{n+1}$ and $R_1 J_n \subseteq J_{n+1} = I_{n+1}$ and thus $R_1(J_n + I_n) \subseteq I_{n+1}$. Since $\dim_k I_n = \dim_k J_n = \dim_k R_n - 1$ and $I_n \neq J_n$, $I_n + J_n = R_n$ and so $R_{n+1} = R_1 R_n \subseteq I_{n+1}$, a contradiction. $\square$

For the rest of this section we will make frequent use of the criterion for $R$-membership given in Theorem 3.2.3 without comment. Also, recall that $\circ$ indicates multiplication in the polynomial ring $U$, and juxtaposition indicates multiplication in $S$ (or $R$).

**Lemma 3.4.3.** *(1) Suppose that $d$ is a point of $\mathbb{P}^t$ with $d \notin \{c_i\}_{i\geq 0}$. Then*

$$(R \cap \mathfrak{m}_d)_m R_1 = (R \cap \mathfrak{m}_{\varphi^{-1}(d)})_{m+1} \text{ for } m \gg 0.$$

*(2) $(R \cap \mathfrak{m}_{c_{-1}})_m R_1 = (R \cap \mathfrak{m}_{c_0}^2)_{m+1}$ for $m \gg 0$.*

*(3) Fix some $i \geq 0$. Then $(R \cap \mathfrak{m}_{c_i}^2)_m R_1 = (R \cap \mathfrak{m}_{c_{i+1}}^2)_{m+1}$ for $m \gg 0$.*

*(4) If ${}_R P(d_1)_{\geq n} \cong {}_R P(d_2)_{\geq n}$ for $d_1, d_2 \in \mathbb{P}^t$ and some $n \geq 0$, then $d_1 = d_2$.*

*Proof.* (1) Note that the critical density condition implies that for $m \gg 0$ the points $c_1, \dots, c_{m-1}$ do not all lie on a single line of $\mathbb{P}^t$. By Lemma 2.5.9(1),

$$(R \cap \mathfrak{m}_d)_m R_1 = (\mathfrak{m}_{c_m} \cap \mathfrak{m}_{c_{m-1}} \cap \dots \cap \mathfrak{m}_{c_1} \cap \mathfrak{m}_{\varphi^{-1}(d)})_m \circ (\mathfrak{m}_{c_0})_1$$

$$= (\mathfrak{m}_{c_m} \cap \mathfrak{m}_{c_{m-1}} \cap \dots \cap \mathfrak{m}_{c_1} \cap \mathfrak{m}_{c_0} \cap \mathfrak{m}_{\varphi^{-1}(d)})_{m+1} = (R \cap \mathfrak{m}_{\varphi^{-1}(d)})_{m+1}$$

for $m \gg 0$.

(2), (3) Similar to the argument for (1), using Lemma 2.5.9 parts (2) and (3) respectively.

(4) By Lemma 3.1.2, we have for any $d \in \mathbb{P}^t$ that $P(d)_{\geq n} \cong P(\varphi^n(d))[-n]$ as $S$-modules. Thus we may reduce to the case where $n = 0$.

Since $\operatorname{ann}_S P(d_i)_0 = \mathfrak{m}_{d_i}$, we must have $\mathfrak{m}_{d_1} \cap R = \mathfrak{m}_{d_2} \cap R$. In degree $m$ this means

$$(\mathfrak{m}_{c_0} \cap \dots \cap \mathfrak{m}_{c_{m-1}} \cap \mathfrak{m}_{d_1})_m = (\mathfrak{m}_{c_0} \cap \dots \cap \mathfrak{m}_{c_{m-1}} \cap \mathfrak{m}_{d_2})_m. \tag{3.4.4}$$

Suppose first that $d_1, d_2 \notin \{c_i\}_{i \in \mathbb{N}}$. Since the point set $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense, it follows that for $m \gg 0$ the points $\{c_i\}_{i=0}^{m-1}$ do not all lie on a line. Then by Lemma 2.5.9(4), the equation (3.4.4) for $m \gg 0$ implies that $d_1 = d_2$.

Otherwise we may assume without loss of generality that $d_1 = c_j$ for some $j \geq 0$ and that $d_2 \notin \{c_i\}_{i=0}^{j-1}$. Then the equation (3.4.4) for $n = j+1$ violates Lemma 2.5.6 unless $d_1 = c_j = d_2$. $\square$

We may now classify the point modules over the ring $R(\varphi, c)$.

**Theorem 3.4.5.** *Assume the critical density condition (Hypothesis 3.3.14).*

*(1) For any point $d \in \mathbb{P}^t \setminus \{c_i\}_{i \geq 0}$, the $S$-point module $P(d)$ is an $R$-point module, with point ideal $(R \cap \mathfrak{m}_d)$.*

*(2) For each $i \geq 0$, the $S$-module $P(c_i)_{\geq i+1}$ is a shifted $R$-point module. There is a $\mathbb{P}^{t-1}$-parameterized family of non-isomorphic $R$-point modules $\{P(c_i, e) \mid e \in \mathbb{P}^{t-1}\}$ with $P(c_i, e)_{\geq i+1} \cong {}_R P(c_i)_{\geq i+1}$ and $P(c_i, e)_{\leq i} \cong {}_R P(c_i)_{\leq i}$ for any $e \in \mathbb{P}^{t-1}$. These are exactly the point modules whose point ideals contain the left ideal $(R \cap \mathfrak{m}_{c_i}^2)$ of $R$.*

*(3) All of the point modules given in parts (1) and (2) above are non-isomorphic, and every point module over $R(\varphi, c)$ is isomorphic to one of these.*

*Proof.* Suppose that $d \in \mathbb{P}^t$, so $P(d) = S/\mathfrak{m}_d$ by definition. For any $i \geq 0$,

$$R_1(P(d))_i = 0 \Longleftrightarrow R_1 S_i \subseteq \mathfrak{m}_d \Longleftrightarrow (\mathfrak{m}_{c_i})_1 \circ U_i \subseteq \mathfrak{m}_d \Longleftrightarrow d = c_i. \tag{3.4.6}$$

(1) Let $d \notin \{c_i\}_{i \geq 0}$. In this case it is clear from (3.4.6) that $P(d)$ is already an $R$-point module. Also, the corresponding point ideal is $\operatorname{ann}_R P(d)_0 = R \cap \mathfrak{m}_d$.

(2) Fix some $i \geq 0$. From (3.4.6) it is clear that $M = {}_R P(c_i) = M_{\leq i} \oplus M_{\geq i+1}$ where $M_{\leq i}$ is the torsion submodule of $M$ and $M_{\geq i+1}$ is a shifted $R$-point module.

We define a left ideal $J = J^{(i)}$ of $R$ by setting $J_{\leq i} = (R \cap \mathfrak{m}_{c_i})_{\leq i}$ and $J_{\geq i+1} =$

$(R \cap \mathfrak{m}_{c_i}^2)_{\geq i+1}$. To check that $J$ really is a left ideal of $R$, one calculates

$$R_1 J_i = \phi^i(\mathfrak{m}_{c_0})_1 \circ J_i = (\mathfrak{m}_{c_i})_1 \circ (R_i \cap \mathfrak{m}_{c_i}) \subseteq R_{i+1} \cap \mathfrak{m}_{c_i}^2 = J_{i+1}.$$

We will now classify the point ideals of $R$ which contain $J$. By Lemma 2.5.6, the Hilbert function of $R/J$ must be

$$\dim_k (R/J)_j = \begin{cases} 1 & j \leq i \\ t & j \geq i+1. \end{cases}$$

Then using Lemma 2.5.6 again, the natural injection

$$(R/J)_{\geq i+1} = \frac{(\mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1} \cap \cdots \cap \mathfrak{m}_{c_i})_{\geq i+1}}{(\mathfrak{m}_{c_0} \cap \mathfrak{m}_{c_1} \cap \cdots \cap \mathfrak{m}_{c_i}^2)_{\geq i+1}} \hookrightarrow (\mathfrak{m}_{c_i}/\mathfrak{m}_{c_i}^2)_{\geq i+1}$$

is an isomorphism of left $R$-modules, since the Hilbert functions on both sides are the same.

As a module over the polynomial ring $U$, we have an isomorphism

$$(\mathfrak{m}_{c_i}/\mathfrak{m}_{c_i}^2)_{\geq i+1} \cong \bigoplus_{j=1}^{t} (U/\mathfrak{m}_{c_i})_{\geq i+1}$$

which by the equivalence of categories $U$-Gr $\sim$ $S$-Gr translates to an $S$-isomorphism as follows:

$$_S(\mathfrak{m}_{c_i}/\mathfrak{m}_{c_i}^2)_{\geq i+1} \cong \bigoplus_{j=1}^{t} P(c_i)_{\geq i+1}.$$

By part (1), $P(c_i)_{\geq i+1} \cong P(c_{-1})[-i-1]$ is a shifted $R$-point module, so we conclude that $M = (R/J)_{\geq i+1}$ is a direct sum of $t$ isomorphic shifted $R$-point modules. Then every choice of a codimension-1 vector subspace $V = L/(J_{i+1})$ of $(R/J)_{i+1}$ generates a different $R$-submodule $N$ of $M$ with $M/N \cong {}_R P(c_i)_{\geq i+1}$, and then $J + RL$ is a point ideal for $R$. Clearly any point ideal containing $J$ must arise in this way, and the set of codimension 1 subspaces of $(R/J)_{i+1}$ is parameterized by $\mathbb{P}^{t-1}$. Thus the set of point ideals of $R$ which contain $J$ is naturally parameterized by a copy of $\mathbb{P}^{t-1}$.

For each $e \in \mathbb{P}^{t-1}$, we have a corresponding point ideal $I$ containing $J$ and we set $P(c_i, e) = R/I$. Then $P(c_i, e)_{\geq i+1} \cong {}_RP(c_i)_{\geq i+1}$ and $P(c_i, e)_{\leq i} \cong (R/J)_{\leq i} \cong {}_RP(c_i)_{\leq i}$.

Finally, note that all of the point ideals constructed above contain $(R \cap \mathfrak{m}_{c_i}^2)$. Conversely, if $I$ is any point ideal which contains $(R \cap \mathfrak{m}_{c_i}^2)$, then $J_{\leq i} + I_{\geq i+1}$ is also a point ideal, and so by Lemma 3.4.2 it follows that $I = J_{\leq i} + I_{\geq i+1} \supseteq J$ and $I$ is one of the point ideals we already constructed. Part (2) is now clear.

(3) Suppose that $M$ is an $R$-point module. Let $\mathcal{P}$ be the set of all $R$-modules isomorphic to a shift of one of the $R$-point modules constructed in parts (1) and (2) above. By Proposition 3.4.1, $M_{\geq n} \cong {}_RP(d)_{\geq n}$ for some $n \geq 0$ and $d \in \mathbb{P}^t$. For $m \gg 0$, $\varphi^m(d) \notin \{c_i\}_{i \geq 0}$ and so $M_{\geq m+n} \in \mathcal{P}$ by part (1) above. Thus to prove (3) it is enough by induction to show that given any $R$-point module $N$, if $N_{\geq 1} \in \mathcal{P}$ then $N \in \mathcal{P}$.

Let $N$ be an $R$-point module such that $N_{\geq 1} \in \mathcal{P}$. Let $I = \operatorname{ann}_R N_0$ be the point ideal of $N$. There are a number of cases. Suppose first that $N_{\geq 1} \cong {}_RP(d)[-1]$ for some $d \notin \{c_i\}_{i \geq -1}$. Since $(R \cap \mathfrak{m}_d)R_1 \subseteq I$, Lemma 3.4.3(1) implies that we have $(R \cap \mathfrak{m}_{\varphi^{-1}(d)})_{\geq m} \subseteq I$ for some $m \geq 0$. Note that $\varphi^{-1}(d) \notin \{c_i\}_{i \geq 0}$, so that $(R \cap \mathfrak{m}_{\varphi^{-1}(d)})$ is one of the point ideals appearing in part (1) above. Since $I$ is also a point ideal, Lemma 3.4.2 implies that $I = (R \cap \mathfrak{m}_{\varphi^{-1}(d)})$ and thus $N \cong {}_RP(\varphi^{-1}(d)) \in \mathcal{P}$.

If $N_{\geq 1} \cong {}_RP(c_{-1})$, then $(R \cap \mathfrak{m}_{c_{-1}})R_1 \subseteq I$ and by Lemma 3.4.3(2) we conclude that $(R \cap \mathfrak{m}_{c_0}^2)_{\geq m} \subseteq I$ for $m \gg 0$. But since $I$ is a point ideal, this implies that $(R \cap \mathfrak{m}_{c_0}^2) \subseteq I$. By part (2) above that this forces $N \cong P(c_0, e)$ for some $e \in \mathbb{P}^{t-1}$, and so $N \in \mathcal{P}$.

The last case is if $N_{\geq 1} \cong P(c_i, e)[-1]$ for some $i \geq 0$ and $e \in \mathbb{P}^{t-1}$. Then since $(R \cap \mathfrak{m}_{c_i}^2)R_1 \subseteq I$, by Lemma 3.4.3(3) we conclude that $(R \cap \mathfrak{m}_{c_{i+1}}^2)_{\geq m} \subseteq I$ for $m \gg 0$.

Then just as in the previous case this will force $N \cong P(c_{i+1}, e') \in \mathcal{P}$ for some $e' \in \mathbb{P}^{t-1}$.

Finally, for fixed $i$ the $P(c_i, e)$ are non-isomorphic for distinct $e$ by construction; then it follows from Lemma 3.4.3(4) that all of the point modules we have constructed in parts (1) and (2) are non-isomorphic. $\square$

Recall from §1.4 that a $k$-algebra $A$ is called *strongly (left) noetherian* if $A \otimes_k B$ is a left noetherian ring for all commutative noetherian $k$-algebras $B$. We also say that a left $A$-module $P$ is strongly noetherian if $P \otimes_k B$ is noetherian over $A \otimes_k B$ for all commutative noetherian $k$-algebras $B$. The strong noetherian property holds for all finitely generated commutative $k$-algebras. It also holds for many standard examples of noncommutative rings, including all twisted homogeneous coordinate rings of projective $k$-schemes, and the AS-regular algebras of dimension three [2, Section 4]. Artin and Zhang have shown the importance of the strong noetherian property for the study of the geometry of the point modules over an algebra, as we noted in §1.4. Let us now give a more detailed discussion of Artin and Zhang's theorem.

We want to make formal the notion of a scheme parameterizing the set of point modules. This is done by considering commutative extension rings of the base field. Let $A$ be a finitely $\mathbb{N}$-graded algebra over a field $k$. Given $h : \mathbb{Z} \to \mathbb{N}$ any function and $P \in A\text{-gr}$, we may consider the set

$$Q_P^h = \{\text{all graded factor modules of } P \text{ with Hilbert function } h\}.$$

For every commutative $k$-algebra $B$, we write $A_B = A \otimes_k B$ and $P_B = P \otimes_k B$. We may extend the definition of $Q_P^h$ to the rings $A_B$ as follows. If $B$ is finitely generated as a $k$-algebra, we set

$$Q_P^h(B) = \{\text{all graded flat quotients } V \text{of } P_B \text{ with rank } V_i = h(i) \text{ for all } i \geq 0\}.$$

For an arbitrary commutative $k$-algebra $B$, we then define

$$Q_P^h(B) = \varinjlim Q_P^h(B')$$

where the direct limit is taken over all finitely generated subalgebras $B'$ of $B$. The map $B \mapsto Q_P^h(B)$ is a contravariant functor from $\{rings\}$ to $\{sets\}$. Then to say that $Q_P^h(k)$ is *parameterized* by the $k$-scheme $Y$ is to say that the functor $Q_P^h$ is represented by $Y$; that is, $Q_P^h$ is naturally equivalent to the functor $C \mapsto \mathrm{Hom}_{k\text{-schemes}}(\mathrm{Spec}\, C, Y)$.

Now we may state Artin and Zhang's theorem.

**Theorem 3.4.7.** *[9] Let $A$ be a connected, finitely generated $\mathbb{N}$-graded algebra over an algebraically closed field $k$. Then for every strongly noetherian module $P \in A\text{-gr}$ and Hilbert function $h$, the set $Q_P^h(k)$ is parameterized by a commutative projective scheme over $k$.* $\square$

Restricting to the special case of the theorem where $P = A$ and $h$ is the constant function $h(i) = 1$ for all $i$, the authors prove the following corollary concerning point modules.

**Corollary 3.4.8.** *[9, Corollary E4.11, Corollary E4.12]. Let $A$ be a connected $\mathbb{N}$-graded strongly noetherian algebra over an algebraically closed field $k$.*

*(1) The point modules over $A$ are naturally parameterized by a commutative projective scheme over $k$.*

*(2) There is some $d \geq 0$ such that every point module $M$ for $A$ is uniquely determined by its truncation $M_{\leq d}$.* $\square$

The failure of the strong noetherian property for $R = R(\varphi, c)$ now follows immediately from the above corollary and the classification of point modules for $R$. This proves Theorem 1.4.5, and answers Question 1.4.3 from the introduction.

**Theorem 3.4.9.** *Assume the critical density condition. Then $R = R(\varphi, c)$ is a connected graded noetherian algebra, finitely generated in degree* 1, *which is noetherian but not strongly noetherian.* □

*Proof.* We only need to prove that $R$ is not strongly noetherian. For each $i \geq 0$, Theorem 3.4.5(2) provides a whole $\mathbb{P}^{t-1}$ of point modules $P(c_i, e)$ which have isomorphic truncations $P(c_i, e)_{\leq i}$. By Corollary 3.4.8(2), $R$ cannot be a strongly noetherian $k$-algebra. □

We remark that the point modules over $R$ still appear to have an interesting geometric structure. By Theorem 3.4.5, there is a single point module corresponding to each point $d \in \mathbb{P}^t \setminus \{c_i\}_{i \geq 0}$ and a $\mathbb{P}^{t-1}$-parameterized family of exceptional point modules corresponding to each point $c_i$ with $i \geq 0$. Since blowing up $\mathbb{P}^t$ at a point in some sense replaces that point by a copy of $\mathbb{P}^{t-1}$, the intuitive picture of the geometry of the point modules for $R$ is an infinite blowup of projective space at a countable point set.

## 3.5 Extending the base ring

Let $S = S(\varphi)$ and $R = R(\varphi, c)$ for $(\varphi, c)$ satisfying the critical density condition, and let $c_i = \varphi^{-i}(c) \in \mathbb{P}^t_k$ as usual. We now know by Theorem 3.4.9 that $R$ is not strongly noetherian, but this proof is quite indirect and it is not obvious which choice of extension ring $B$ makes $R \otimes_k B$ non-noetherian. In this section we construct such a noetherian commutative $k$-algebra $B$ which is even a UFD.

Let $B$ be an arbitrary commutative $k$-algebra which is a domain. We will use subscripts to indicate extension of the base ring, so we write $U_B = U \otimes_k B$, $S_B = S \otimes_k B$ and $R_B = R \otimes_k B$. The automorphism $\phi$ of $U$ extends uniquely to an automorphism of $U_B$ fixing $B$, which we also call $\phi$. We continue to identify the underlying $B$-module of $S_B$ with that of $U_B$, and we use juxtaposition for multiplication in $S_B$ and the symbol $\circ$ for multiplication in $U_B$, as in our current convention (see §3.1). The multiplication of $S_B$ is still given by $fg = \phi^n(f) \circ g$ for $f \in (S_B)_m$, $g \in (S_B)_n$; in other words, $S_B$ is the left Zhang twist of $U_B$ by the twisting system $\{\phi^i\}_{i \in \mathbb{N}}$, just as before.

Let $d$ be a point in $\mathbb{P}^t_k$. Since the homogeneous coordinates for $d$ are defined only up to a scalar multiple in $k^\times$, given $f \in U_B$ the expression $f(d)$ is defined up to a nonzero element of $k$; we will use this notation only in contexts where the ambiguity does not matter. For example, the condition $f(d) = 0$ makes sense and is equivalent to the condition $f \in \mathfrak{m}_d \circ U_B$, where $\mathfrak{m}_d \subseteq U$ and $\mathfrak{m}_d \circ U_B$ is a graded prime ideal of $U_B$.

The natural analog of Theorem 3.2.3 still holds in this setting:

**Proposition 3.5.1.** *For all $n \geq 0$, we have*

$$(R_B)_n = \{f \in (U_B)_n \textit{ such that } f(c_i) = 0 \textit{ for } 0 \leq i \leq n-1\}.$$

*Proof.* As subsets of $U_B$, using Theorem 3.2.3 we have

$$(R_B)_n = R_n \otimes B = (\cap_{i=0}^{n-1} \mathfrak{m}_{c_i})_n \otimes B = \cap_{i=0}^{n-1} (\mathfrak{m}_{c_i} \circ U_B)_n$$

and the proposition follows. ∎

We now give sufficient conditions on $B$ for the ring $R \otimes_k B$ to fail to have the left noetherian property.

**Proposition 3.5.2.** *Assume that $B$ is a UFD. Suppose that there exist nonzero homogeneous elements $f, g \in (U_B)_1$ satisfying the following conditions:*

*(1) $f(c_i)|g(c_i)$ for all $i \geq 0$.*

*(2) For all $i \gg 0$, $f(c_i)$ is not a unit of $B$.*

*(3) $\gcd(f, g) = 1$ in $U_B$.*

*Then $R \otimes_k B$ is not a left noetherian ring.*

*Proof.* Note that $U_B \cong B[x_0, x_1, \ldots, x_t]$ is a UFD, since $B$ is, so condition (3) makes sense.

For convenience, fix some homogeneous coordinates for the $c_i$. For each $n \geq 0$, we may choose a polynomial $\theta_n \in S_n$ with coefficients in $k$ such that $\theta_n(c_i) = 0$ for $-1 \leq i \leq n-2$ and $\theta_n(c_{n-1}) \neq 0$. This is possible, for example, by Lemma 2.5.7. By hypothesis (1), for each $n \geq 0$ we may write $\Omega_n = g(c_n)/f(c_n) \in B$. Now let $t_n = \theta_n(\Omega_n f - g) \in (S_B)_{n+1}$ for each $n \geq 0$.

Since $\phi(\theta_n)$ vanishes at $c_i$ for $0 \leq i \leq n-1$ and $[\Omega_n f - g](c_n) = 0$, the element $t_n = \phi(\theta_n) \circ (\Omega_n f - g)$ is in $(R_B)_{n+1}$, by Proposition 3.5.1. We will show that for $n \gg 0$ we have $t_{n+1} \notin \sum_{i=0}^{n} (R_B) t_i$, which will imply that $R_B$ is not left noetherian.

Suppose that $t_{n+1} = \sum_{i=0}^{n} r_i t_i$ for some $r_i \in (R_B)_{n+1-i}$. Writing out the explicit expressions for the $t_i$, this is

$$\theta_{n+1}(\Omega_{n+1} f - g) = \sum_{i=0}^{n} r_i \theta_i (\Omega_i f - g).$$

Considering these expressions in $U_B$, after some rearrangement we obtain (since $f, g$ have degree 1)

$$\phi[\theta_{n+1}\Omega_{n+1} - \sum_{i=0}^{n} r_i \theta_i \Omega_i] \circ f + \phi[-\theta_{n+1} + \sum_{i=0}^{n} r_i \theta_i] \circ g = 0.$$

Now by hypothesis (3), $g$ must divide the polynomial

$$h = \phi[\theta_{n+1}\Omega_{n+1} - \sum_{i=0}^{n} r_i\theta_i\Omega_i] = \Omega_{n+1}\phi(\theta_{n+1}) - \sum_{i=0}^{n} \Omega_i\phi^{i+1}(r_i) \circ \phi(\theta_i).$$

We note that $[\phi(\theta_{n+1})](c_n) = 0$ and $[\phi(\theta_n)](c_n) \in k^\times$ by the definition of the $\theta_i$, and $[\phi^{i+1}(r_i)](c_n) = 0$ for $0 \leq i \leq n-1$, since $r_i \in R_{n+1-i}$. Thus evaluating at $c_n$ we conclude that $g(c_n) \mid \Omega_n$. But since $\Omega_n = g(c_n)/f(c_n)$, this implies that $f(c_n)$ is a unit in $B$. For all $n \gg 0$, this contradicts hypothesis (2), and so $t_{n+1} \notin \sum_{i=0}^{n}(R_B)t_i$ for $n \gg 0$, as we wished to show. $\square$

Next, we construct a commutative noetherian ring $B$ which satisfies the hypotheses of Proposition 3.5.2. We shall obtain such a ring as an infinite blowup of affine space, to be defined presently. See [2, Section 1] for more details about this construction.

Let $A$ be a commutative domain, and let $X$ be the affine scheme Spec $A$. Suppose that $d$ is a closed nonsingular point of $X$ with corresponding maximal ideal $\mathfrak{p} \subseteq A$, and let $z_0, z_1, \ldots, z_r$ be some choice of generators of the ideal $\mathfrak{p}$ such that $z_0 \notin \mathfrak{p}^2$. The *affine blowup* of $X$ at $d$ (with denominator $z_0$) is $X' = \text{Spec } A'$ where $A' = A[z_1z_0^{-1}, z_2z_0^{-1} \ldots, z_rz_0^{-1}]$.

Consider the special case where $A = k[y_1, y_2, \ldots, y_t]$ is a polynomial ring, $X = \mathbb{A}^t$, and $d = (a_1, a_2, \ldots, a_t)$. The affine blowup of $\mathbb{A}^t$ at $d$ with the denominator $(y_1 - a_1)$ is $X' = \text{Spec } A'$ for the ring

$$A' = A[(y_2 - a_2)(y_1 - a_1)^{-1}, \ldots, (y_t - a_t)(y_1 - a_1)^{-1}].$$

Note that also $A' = k[y_1, (y_2 - a_2)(y_1 - a_1)^{-1}, \ldots, (y_t - a_t)(y_1 - a_1)^{-1}]$, so $A'$ is itself isomorphic to a polynomial ring in $t$ variables over $k$ and $X' = \mathbb{A}^t$ as well. The blowup map $X' \to X$ is an isomorphism outside of the closed set $\{y_1 = a_1\}$ of $X$.

Given a sequence of points $\{d_i = (a_{i1}, a_{i2}, \ldots, a_{it})\}_{i \geq 0}$ such that $a_{i1} \neq a_{j1}$ for $i \neq j$, we may iterate the blowup construction, producing a union of commutative domains

$$A \subseteq A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$$

where each $A_i$ is isomorphic to a polynomial ring in $t$ variables over $k$. In this case we set $B = \bigcup A_i$ and $Y = \text{Spec } B$, and call $Y$ (or $B$) the *infinite blowup* of $\mathbb{A}^t$ at the sequence of points $\{d_i\}$. Explicitly, $B = A[\{(y_j - a_{ij})(y_1 - a_{i1})^{-1} | 2 \leq j \leq t,\ i \geq 0\}]$.

That there should be some connection between such infinite blowups and the algebras $R(\varphi, c)$ is strongly suggested by the following result (compare Theorem 3.3.12).

**Theorem 3.5.3.** *[2, Theorem 1.5] The infinite blowup $B$ is a noetherian ring if and only if the set of points $\{d_i\}_{\geq 0}$ is a critically dense subset of $\mathbb{A}^t$.* □

Now we show the failure of the strong noetherian property for the noetherian rings $R(\varphi, c)$ explicitly.

**Theorem 3.5.4.** *Let $(\varphi, c) \in (\text{Aut}\, \mathbb{P}^t_k) \times \mathbb{P}^t_k$ satisfy the critical density condition. There is an affine patch $\mathbb{A}^t \subseteq \mathbb{P}^t$ such that $\{c_i\}_{i \in \mathbb{Z}} \subseteq \mathbb{A}^t$. Let $B$ be the infinite blowup of $\mathbb{A}^t$ at the points $\{c_i\}_{i \geq 0}$. Then $R = R(\varphi, c)$ is noetherian, and $B$ is a commutative noetherian $k$-algebra which is a UFD such that $R \otimes_k B$ is not a left noetherian ring.*

*Proof.* By changing coordinates, we may replace $\varphi$ by a conjugate without loss of generality, so we may assume that when represented as a matrix $\varphi$ is lower triangular. Also, we may multiply this matrix by a nonzero scalar without changing the automorphism of $\mathbb{P}^t$ it represents, and so we also assume that the top left entry of the matrix is 1.

By assumption the set of points $\{c_i\}_{i \in \mathbb{Z}}$ is critically dense in $\mathbb{P}^t$, and $R(\varphi, c)$ is noetherian. Let $X_0$ be the hyperplane $\{x_0 = 0\}$ of $\mathbb{P}^t$. Since $\varphi$ is upper triangular,

$\varphi(X_0) = X_0$, so if some $c_i \in X_0$ then $\{c_i\}_{i\in\mathbb{Z}} \subseteq X_0$ which contradicts the critical density condition. So certainly $\{c_i\}_{i\in\mathbb{Z}} \subseteq \mathbb{A}^t = \mathbb{P}^t \setminus X_0$. Since the top left entry of $\varphi$ is 1, we may fix homogeneous coordinates for the $c_i$ of the form $c_i = (1 : a_{i1} : a_{i2} : \cdots : a_{it})$. Let $y_i = x_i/x_0$, so that $k[y_1, y_2, \ldots y_t]$ is the coordinate ring of $\mathbb{A}^t$. In affine coordinates, $c_i = (a_{i1}, a_{i2}, \ldots, a_{it})$.

If $a_{i1} = a_{j1}$ for some $i < j$, then since $\varphi$ is lower triangular it follows that $a_{i1} = a_{k1}$ for all $k \in (j-i)\mathbb{Z}$. Then the hyperplane $\{a_{i1}x_0 - x_1 = 0\}$ of $\mathbb{P}^t$ contains infinitely many of the $c_i$, again contradicting the critical density of $\{c_i\}_{i\in\mathbb{Z}}$. So the scalars $\{a_{i1}\}_{i\in\mathbb{Z}}$ are all distinct, and the infinite blowup $B$ of $\mathbb{A}^t$ at the points $\{c_i\}_{i\geq 0}$ is well defined. The ring $B$ is generated over $k[y_1, y_2, \ldots, y_t]$ by the elements $\{(y_j - a_{ij})(y_1 - a_{i1})^{-1} \mid 2 \leq j \leq t,\ i \geq 0\}$. Clearly the points $\{c_i\}_{i\geq 0}$ must be critically dense subset of $\mathbb{A}^t$, since they are a critically dense subset of $\mathbb{P}^t$. Thus $B$ is noetherian by Theorem 3.5.3.

The ring $B$ is obtained as a directed union of $k$-algebras $A_i$ which are each isomorphic to a polynomial ring. In each ring $A_i$ the group of units is just $k^\times$, and so this is also the group of units of $B$. It follows that if $z \in A_i$ is an irreducible element of $B$, then $z$ is irreducible in $A_i$. Since $B$ is noetherian, every element of $B$ is a finite product of irreducibles, and the uniqueness of such a decomposition follows by the uniqueness in each UFD $A_i$. Thus $B$ is a UFD.

Fix the two elements $f = y_1x_0 - x_1$ and $g = y_2x_0 - x_2$ of $U_B \cong B[x_0, x_1, \ldots, x_t]$. Since $f$ and $g$ are homogeneous of degree 1 in the $x_i$ and are not divisible by any non-unit of $B$, it is clear that $f$ and $g$ are distinct irreducible elements of $U_B$, and so in particular $\gcd(f, g) = 1$. Now $f(c_i) = y_1 - a_{i1}$ and $g(c_i) = y_2 - a_{i2}$, so $\Omega_i = g(c_i)/f(c_i) = (y_2 - a_{i2})(y_1 - a_{i1})^{-1} \in B$ and thus $f(c_i)|g(c_i)$ for all $i \geq 0$.

Finally, $f(c_i) = (y_1 - a_{i1})$ is not in the group of units $k^\times$ of $B$. We see that all of the hypotheses of Proposition 3.5.2 are satisfied, and so $R \otimes_k B$ is not left noetherian. $\square$

## CHAPTER IV

# Homological Properties

We turn in this chapter to results of a more homological flavor. In particular, we will prove Theorems 1.5.6—1.5.10 from the introduction. In the preparatory first section, we begin by discussing some useful notation for some special categories of modules over the rings $S = S(\varphi)$ and $R = R(\varphi, c)$, and we prove some elementary results concerning these categories. The purpose of the remainder of the section is to collect together various needed homological definitions and lemmas, especially about the properties of $\underline{\text{Ext}}$ and $\underline{\text{Tor}}$ over the rings $R$ and $S$.

In §4.2 we examine the properties of 2-sided ideals of $R$, and show that they are very closely related to the 2-sided ideals of $S$. Because it is an easy consequence, we give a complete description of the graded prime spectrum of $R$ in terms of that of $S$, although we do not use this result elsewhere in the thesis. A second consequence of our results on ideals of $R$ is the proof in §4.3 that $R$ is a maximal order in its Goldie ring of fractions.

In §4.4 we use spectral sequence techniques to show that $R$ satisfies the $\chi_1$ condition but that $R$ fails $\chi_i$ for $i \geq 2$. It then follows quickly from the noncommutative Serre's finiteness theorem proved by Artin and Zhang that the category $R$-qgr is not equivalent to $A$-qgr for any graded algebra $A$ which satisfies $\chi_2$. Next, we discuss the notion of cohomological dimension, and show that $R$-proj has cohomological dimension at most $t$. The proof uses another spectral sequence to reduce the calculation of cohomology in $R$-proj to a more tractable calculation over the ring $S$. Last, in §4.6 we give a brief discussion of the notion of Krull dimension, and show that it coincides with GK-dimension for modules over the ring $R$. This final section is included for reference purposes only, since it may be useful for further work; it is not essential to any other results in the thesis.

## 4.1 Special subcategories and homological lemmas

Let $S = S(\varphi)$ and $R = R(\varphi, c)$, and assume the critical density condition (Hypothesis 3.3.14), in particular that $R$ is noetherian. First, we introduce some notation for the subcategories of $S$-Gr and $R$-Gr which are generated by the "distinguished" $S$-point modules $P(c_i)$, which will be useful throughout this chapter.

**Definition 4.1.1.** (1) Let $S$-dist be the full subcategory of $S$-gr consisting of all $S$-modules $M$ with a finite $S$-module filtration whose factors are either torsion or a shift of $P(c_i)$ for some $i \in \mathbb{Z}$.
(2) Let $R$-dist be the full subcategory of $R$-gr consisting of all $R$-modules $M$ having a finite $R$-module filtration whose factors are either torsion or a shift of the module ${}_RP(c_i)$ for some $i \in \mathbb{Z}$.

Note that by Theorem 3.4.5(1),(2), ${}_RP(c_i)$ is finitely generated for any $i \in \mathbb{Z}$ and so part (2) of the definition makes sense. We also define $S$-Dist to be the full subcategory of $S$-Gr consisting of those modules $M$ such that $N \in S$-dist for every finitely generated submodule $N$ of $M$. The subcategory $R$-Dist of $R$-Gr is defined similarly.

The following basic facts are left to the reader; we shall use them without comment below.

**Lemma 4.1.2.** *(1)* $S$-dist *and* $S$-Dist *are Serre subcategories of* $S$-Gr*.*

*(2)* $R$-dist *and* $R$-Dist *are Serre subcategories of* $R$-Gr*.*

*(3)* ${}_R(S/R)$ *is in* $R$-Dist *(use Corollary 3.3.6).* $\square$

The following result helps to clarify the relationship between the special categories over the two rings.

**Lemma 4.1.3.** *(1) Let* ${}_SM \in S$-gr*. Then* ${}_RM \in R$-dist *if and only if* ${}_SM \in S$-dist*.*

*(2) Let* ${}_SM \in S$-Gr*. Then* ${}_RM \in R$-Dist *if and only if* ${}_SM \in S$-Dist*.*

*Proof.* (1) Suppose that ${}_SM \in S$-dist. Then it follows directly from Definition 4.1.1 that ${}_RM \in R$-dist. Conversely, suppose that ${}_RM \in R$-dist. Clearly $\mathrm{GK}_R(M) \leq 1$, so we have $\mathrm{GK}_S(M) \leq 1$ since we can measure GK-dimension using the Hilbert function. By Lemma 3.1.3, $M$ has a finite filtration over $S$ with cyclic critical factors, which must in this case be shifts of ${}_Sk$ and $S$-point modules. Suppose that a shift of $P(d)$ is one of the factors occurring. Then $N = {}_RP(d) \in R$-dist. By the definition of $R$-dist, some tail of $N$ is isomorphic to a shift of some ${}_RP(c_i)$ for some $i \in \mathbb{Z}$. Using Lemma 3.1.2, this forces ${}_RP(d) \cong {}_RP(c_j)$ for some $j \in \mathbb{Z}$ and so by Lemma 3.4.3(4) we have $d = c_j$. Thus the only point modules which may occur as factors in the $S$-filtration of $M$ are shifts of the $P(c_j)$ for $j \in \mathbb{Z}$ and so $M \in S$-dist.

(2) Suppose that ${}_SM \in S$-Dist. Let $N$ be any finitely generated $R$-submodule of $M$. Then $SN \in S$-dist and so $SN \in R$-dist by part (1), and then $N \in R$-dist. Thus ${}_RM \in R$-Dist. On the other hand, suppose that ${}_RM \in R$-Dist. Let $L$ be a finitely generated $S$-submodule of $M$. Certainly ${}_RS \notin R$-Dist, so necessarily $\mathrm{GK}_S(L) < \mathrm{GK}(S) = t + 1$. Then it is clear from a finite filtration of $L$ by cyclic $S$-modules and Lemma 3.3.16(2) that ${}_RL$ is finitely generated. So ${}_RL \in R$-dist and

then ${}_SL \in S$-dist by part (1). We conclude that ${}_SM \in S$-Dist. $\square$

An important property of the distinguished point modules $P(c_i)$ is that they have zero annihilator.

**Lemma 4.1.4.** *(1) If* $M \in R$-Dist*, then either* ${}_RM$ *is torsion or else* $\operatorname{ann}_R M = 0$.
*(2) If* $N \in S$-Dist*, then either* ${}_SN$ *is torsion or else* $\operatorname{ann}_S N = 0$.

*Proof.* Consider the $S$-point module $P(c_i)$ for some $i \in \mathbb{Z}$. By Lemma 3.1.2(1), $P(c_i)$ has point sequence $c_i, c_{i-1}, c_{i-2}, \ldots$. Then $\operatorname{ann}_S P(c_i) = \cap_{j=0}^{\infty} \mathfrak{m}_{c_{i-j}}$, and by the critical density of the points $\{c_i\}$ we conclude that $\operatorname{ann}_S P(c_i) = 0 = \operatorname{ann}_R P(c_i)$.

Now both parts follow quickly from the definitions of the categories $R$-Dist and $S$-Dist. $\square$

In the rest of this section, we gather some definitions and lemmas concerning homological algebra over the rings $R$ and $S$.

Let $A$ be a connected $\mathbb{N}$-graded $k$-algebra, finitely generated in degree 1, and let $k = (A/A_{\geq 1})$. Recall the $\chi$ conditions from Definition 1.5.1: $A$ satisfies $\chi_i$ if $\dim_k \underline{\operatorname{Ext}}^j(k, M) < \infty$ for all $M \in A$-gr and all $0 \leq j \leq i$, and $A$ satisfies $\chi$ if $A$ satisfies $\chi_i$ for all $i \geq 0$. In addition, we will say that $\chi_i(M)$ *holds* for a particular module $M \in A$-Gr if $\underline{\operatorname{Ext}}_A^j(k, M) < \infty$ for $0 \leq j \leq i$. If $M \in A$-gr, the *grade* of $M$ is the number $j(M) = \min\{i | \underline{\operatorname{Ext}}_A^i(M, A) \neq 0\}$. We say that $A$ is *Cohen-Macaulay* if $j(M) + \operatorname{GK}(M) = \operatorname{GK}(A)$ for all $M \in A$-gr.

The ring $S$ obtains many nice homological properties simply because it is a Zhang twist of a commutative polynomial ring.

**Lemma 4.1.5.** *(1)* $S$ *has global dimension* $t+1$.
*(2)* $S$ *is Cohen-Macaulay.*

*(3) $S$ is Artin-Schelter regular (in the sense of Definition 1.3.2).*

*(4) $S$ satisfies $\chi$.*

*Proof.* All of these properties are standard for the polynomial ring $U$. Properties (1)-(3) follow for the Zhang twist $S$ of $U$ by [43, Propositions 5.7, 5.11]. Then since $S$ is Artin-Schelter regular it satisfies $\chi$ [8, Theorem 8.1]. $\square$

For an $\mathbb{N}$-graded algebra $A$ and $M, N \in A$-gr, recall the definition of $\underline{\mathrm{Ext}}^i(M, N)$ from §2.1. If $L$ is a $\mathbb{Z}$-graded right $A$-module, then the $k$-space $\mathrm{Tor}_i^A(L, N)$ has a natural $\mathbb{Z}$-grading as well. We emphasize this fact by writing $\underline{\mathrm{Tor}}_i^A(L, N)$ for this vector space. Let us gather here some elementary facts about $\underline{\mathrm{Ext}}$ and $\underline{\mathrm{Tor}}$.

**Lemma 4.1.6.** *[8, Propositions 2.4, 3.1] Let $A$ be a finitely generated connected $\mathbb{N}$-graded $k$-algebra. Let $M$ be a graded left $A$-module, let $L$ be a graded right $A$-module, and let $\{N_\alpha\}$ be an arbitrary indexed set of graded left $A$-modules.*

*(1) $\underline{\mathrm{Ext}}_A^i(M, \bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha \underline{\mathrm{Ext}}_A^i(M, N_\alpha)$.*

*(2) $\underline{\mathrm{Tor}}_i^A(L, \bigoplus_\alpha N_\alpha) \cong \bigoplus_\alpha \underline{\mathrm{Tor}}_i^A(L, N_\alpha)$, and similarly in the other coordinate.*

*(3) If $A$ is noetherian, $L$ is finitely generated, and $M$ is bounded, then $\underline{\mathrm{Tor}}_i^A(L, M)$ is also bounded.*

*(4) If $L$ is a left bounded module, then the left bound of $\underline{\mathrm{Tor}}_i^A(L, A/A_{\geq n})$ tends to $\infty$ with $n$.*

*(5) If $N$ is left or right bounded, the same is true of $\underline{\mathrm{Ext}}^i(M, N)$ for all $i \geq 0$.* $\square$

The next proposition shows that the critical density of the set $\{c_i\}_{i \in \mathbb{Z}}$, besides characterizing the noetherian property for $R$, also has implications for the homological properties of the $S$-point modules $P(c_i)$. As usual, we identify the graded left ideals of $S$ and the graded ideals of $U$.

**Proposition 4.1.7.** *Assume the critical density condition, and let $N \in S$-gr.*

*(1)* $\dim_k \underline{\mathrm{Ext}}^p_S(P(c_i), N) < \infty$ *for* $0 \le p \le t-1$ *and any* $i \in \mathbb{Z}$.

*(2) Let* $M \in S$-dist. *Then* $\dim_k \underline{\mathrm{Ext}}^p_S(M, N) < \infty$ *for* $0 \le p \le t-1$.

*Proof.* (1) Since $N$ is finitely generated, it is easy to see that $E = \underline{\mathrm{Ext}}^p_S(P(c_i), N)$ is finitely graded, in other words each graded piece is finite dimensional over $k$. So it is enough to show that $E$ is bounded. Note that $E$ is automatically left bounded since $N$ is, by Lemma 4.1.6(5). It remains to show that $E$ is right bounded. Using a finite filtration of $N$ by cyclic modules, one reduces quickly to the case where $N$ is cyclic, say $N = S/I$.

In case $I = 0$, $E = \underline{\mathrm{Ext}}^p_S(P(c_i), S) = 0$ for $0 \le p \le t-1$ by the Cohen-Macaulay property of $S$ (Lemma 4.1.5(2)).

Now assume that $I \neq 0$. By Lemma 2.3.3(3), we have for each $n \ge 0$ the $k$-space isomorphism

$$\underline{\mathrm{Ext}}^p_S(S/\mathfrak{m}_{c_i}, S/I)_n \cong \underline{\mathrm{Ext}}^p_U(U/\mathfrak{m}_{c_i}, U/\phi^{-n}(I))_n.$$

Now $\phi^{-n}(I) \subseteq \mathfrak{m}_{c_i}$, or equivalently $I \subseteq \mathfrak{m}_{c_{i+n}}$, can hold for at most finitely many $n$, since the points $\{c_i\}_{i \in \mathbb{Z}}$ are critically dense. Thus for $n \gg 0$ we have $\phi^{-n}(I) \not\subseteq \mathfrak{m}_{c_i}$, and the module $U/(\phi^{-n}(I) + \mathfrak{m}_{c_i})$ is bounded. By Corollary 2.4.8, there is some fixed $d \ge 0$ such that $\underline{\mathrm{Ext}}^p_U(U/\mathfrak{m}_{c_i}, U/\phi^{-n}(I))_n = 0$ as long as $n \ge d$. We conclude that $\underline{\mathrm{Ext}}^p_S(S/\mathfrak{m}_{c_i}, S/I)_n = 0$ for $n \gg 0$, as we wish.

(2) Since $M \in S$-dist, we may choose a finite filtration of $M$ with factors which are shifts of the point modules $P(c_i)$ or ${}_Sk$. Since $S$ satisfies $\chi$ by Lemma 4.1.5(4), $\dim_k \underline{\mathrm{Ext}}^p_S({}_Sk, N) < \infty$ for all $p \ge 0$, and now the statement follows by part (1). □

To study homological algebra over $R$, we will generally try to reduce to calculations over the ring $S$. In particular, we will often use the following convergent spectral sequence, which is valid for any graded modules ${}_RM$ and ${}_SN$ [31, Equation (2.2)]:

$$\underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S,M),N) \underset{p}{\Rightarrow} \underline{\mathrm{Ext}}^{p+q}_R(M,N). \tag{4.1.8}$$

We also note for reference the 5-term exact sequence arising from this spectral sequence [27, 11.2]:

$$\begin{aligned} 0 \to \underline{\mathrm{Ext}}^1_S(S\otimes_R M,N) \to \underline{\mathrm{Ext}}^1_R(M,N) \to \underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}^R_1(S,M),N) \\ \to \underline{\mathrm{Ext}}^2_S(S\otimes_R M,N) \to \underline{\mathrm{Ext}}^2_R(M,N). \end{aligned} \tag{4.1.9}$$

In order to make effective use of the spectral sequence, we need some information about $\underline{\mathrm{Tor}}$.

**Lemma 4.1.10.** *Fix some $n \geq 0$ and let $M = R/R_{\geq n}$.*

*(1) As left $R$-modules, $\underline{\mathrm{Tor}}^R_q(S,M) \cong \underline{\mathrm{Tor}}^R_q(S/R,M)$ for any $q \geq 1$.*

*(2) $\underline{\mathrm{Tor}}^R_q(S,M) \in S$-dist for $q \geq 0$.*

*Proof.* (1) For $q \geq 2$, the desired isomorphism follows immediately from the long exact sequence in $\underline{\mathrm{Tor}}^R_i(-,M)$ associated to the short exact sequence of $R$-bimodules $0 \to R \to S \to S/R \to 0$. For the case $q = 1$, consider the end of this long exact sequence of left $R$-modules:

$$\ldots \to 0 \to \underline{\mathrm{Tor}}_1(S,M) \to \underline{\mathrm{Tor}}_1(S/R,M) \xrightarrow{\theta} M \xrightarrow{\psi} S\otimes_R M \to S/R\otimes M \to 0.$$

Now $S\otimes_R M \cong S/SR_{\geq n}$, and the map $\psi$ is just the natural map $R/R_{\geq n} \to S/SR_{\geq n}$. It is clear that $\psi$ is an injection, so $\theta$ is the zero map and $\underline{\mathrm{Tor}}_1(S,M) \cong \underline{\mathrm{Tor}}_1(S/R,M)$.

(2) When $q = 0$, we have that $\underline{\mathrm{Tor}}^R_0(S,M) = S\otimes_R M \cong S/SR_{\geq n}$. Note that $S/R_{\geq n} \in R$-Dist, since both $S/R$ and $R/R_{\geq n}$ are in $R$-Dist (see Lemma 4.1.2). Then the image $S/SR_{\geq n}$ of $S/R_{\geq n}$ is in $R$-Dist also. If instead $q \geq 1$, then $\underline{\mathrm{Tor}}^R_q(S,M) \cong \underline{\mathrm{Tor}}^R_q(S/R,M)$ by part (1). Computing $N = \underline{\mathrm{Tor}}^R_q(S/R,M)$ using a free resolution of $M$, we see that $N$ is a subfactor of some direct sum of copies of $(S/R)$, so $N \in R$-Dist.

For all $q \geq 0$ we have $\underline{\mathrm{Tor}}_q^R(S, M) \in R\text{-Dist}$, so $\underline{\mathrm{Tor}}_q^R(S, M) \in S\text{-Dist}$ by Lemma 4.1.3. Then computing $\underline{\mathrm{Tor}}_q^R(S, M)$ using a resolution of $M$ by free $R$-modules of finite rank, it follows that actually $\underline{\mathrm{Tor}}_q^R(S, M) \in S\text{-dist}$. $\square$

One easy consequence of the spectral sequence is the following useful fact.

**Lemma 4.1.11.** $\underline{\mathrm{Ext}}_R^1({}_Rk, R) = 0$.

*Proof.* Consider the long exact sequence in $\underline{\mathrm{Ext}}_R(k, -)$ associated to the short exact sequence $0 \to R \to S \to S/R \to 0$:

$$\ldots \to \underline{\mathrm{Hom}}_R(k, S/R) \to \underline{\mathrm{Ext}}_R^1(k, R) \to \underline{\mathrm{Ext}}_R^1(k, S) \to \ldots \tag{4.1.12}$$

Now ${}_R(S/R)$ is torsionfree, since it is isomorphic to a direct sum of point modules by Corollary 3.3.6(2). Thus $\underline{\mathrm{Hom}}_R(k, S/R) = 0$.

To analyze the group $\underline{\mathrm{Ext}}_R^1(k, S)$, we use the beginning of the 5-term exact sequence (4.1.9) for $M = {}_Rk$ and $N = S$:

$$0 \longrightarrow \underline{\mathrm{Ext}}_S^1(S \otimes_R k, S) \longrightarrow \underline{\mathrm{Ext}}_R^1(k, S) \longrightarrow \underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}_1^R(S, k), S) \longrightarrow \ldots \tag{4.1.13}$$

Now by Lemma 4.1.10(2), $\underline{\mathrm{Tor}}_i^R(S, k)$ is in $S$-dist for all $i \geq 0$; in particular, $\mathrm{GK}_S(S \otimes_R k) \leq 1$ and $\mathrm{GK}_S(\underline{\mathrm{Tor}}_1^R(S, k)) \leq 1$. Then $\underline{\mathrm{Ext}}_S^1(S \otimes_R k, S) = 0$ by the Cohen-Macaulay property of $S$ (Lemma 4.1.5(2)) and $\underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}_1^R(S, k), S) = 0$ since $S$ is a domain with $\mathrm{GK}(S) = t + 1 > 1$. Thus by (4.1.13) $\underline{\mathrm{Ext}}_R^1(k, S) = 0$, and by (4.1.12) $\underline{\mathrm{Ext}}_R^1(k, R) = 0$ as well. $\square$

## 4.2 Two-sided ideals and prime ideals

In the present section, we will examine the 2-sided ideals of $R = R(\varphi, c)$, assuming as always the critical density condition on $(\varphi, c)$. We see first that the graded ideal structure of $R$ is strongly controlled by that of $S = S(\varphi)$.

**Proposition 4.2.1.** *Let $I$ be a graded ideal of $R$. Then* $\dim_k(SIS \cap R)/I < \infty$.

*Proof.* If $I = 0$ the result is obvious, so we may assume that $I \neq 0$. As a left $R$-module, $SI/I$ is a surjective image of some direct sum of copies of ${}_R(S/R)$. By Lemma 4.1.2(3) we conclude that $(SI \cap R)/I \in R$-Dist. Since $(SI \cap R)/I$ is killed on the left by $I$, Lemma 4.1.4 implies that $(SI \cap R)/I$ is torsion, so $\dim_k(SI \cap R)/I < \infty$. A similar argument shows that $\dim_k(SIS \cap R)/(IS \cap R) < \infty$. Then using Lemma 3.2.2(1), the same property must hold on the right side, and so $\dim_k(SIS \cap R)/(SI \cap R) < \infty$. Now altogether we have $\dim_k(SIS \cap R)/I < \infty$. □

We also need a related lemma.

**Lemma 4.2.2.** *Let $I$ be a graded ideal of $R$. Then $SIS/SI \in S$-dist.*

*Proof.* By Lemma 4.1.3, it will be enough to show that $SIS/SI \in R$-Dist. By Proposition 4.2.1, $(SIS \cap R)/I$ is a torsion $R$-module, so we need only show that $M = SIS/(SIS \cap R) \in R$-Dist. But given any $f \in (SIS)_m$, the left ideal $J = (\cap_{i=1}^m \mathfrak{m}_{c_{-i}}) \cap R$ of $R$ satisfies $Jf \in SIS \cap R$, by Theorem 3.2.3. The module $R/J$ embeds in $S/(\cap_{i=1}^m \mathfrak{m}_{c_{-i}})$, which is clearly in $S$-dist, so $R/J \in R$-dist by Lemma 4.1.3. Thus every cyclic submodule of $M$ is in $R$-dist, so we conclude that $M \in R$-Dist. □

We may now classify the graded prime ideals of $R = R(\varphi, c)$ in terms of the graded primes of $S = S(\varphi)$.

**Theorem 4.2.3.** *Assume the critical density condition. The graded prime ideals of $R$ are exactly the ideals of the form $Q \cap R$ for a graded prime $Q$ of $S$.*

*Proof.* Let $P$ be a graded prime ideal of $R$, and set $\overline{R} = R/P$. If $\mathrm{GK}(\overline{R}) = 0$, then clearly $P = R_{\geq 1}$ and $P = Q \cap R$ for $Q = S_{\geq 1}$. Assume then that $\mathrm{GK}(\overline{R}) \geq 1$. By Proposition 4.2.1, there is a proper graded ideal $Q = SPS$ of $S$ such that $(Q \cap R)/P$

is torsion over $R$. The noetherian prime ring $\overline{R}$ can not have a nonzero torsion ideal, so $Q \cap R = P$. Now enlarge $Q$ if necessary so that $Q$ is maximal among graded ideals of $S$ with the property that $Q \cap R = P$. If $I$, $J$ are ideals of $S$ satisfying $I \supseteq Q$, $J \supseteq Q$ and $IJ \subseteq Q$, then $(I \cap R)(J \cap R) \subseteq (Q \cap R) = P$, so either $I \cap R = P$ or $J \cap R = P$. Thus either $I = Q$ or $J = Q$ and $Q$ is a prime ideal of $S$.

Conversely, let $Q$ be a graded prime of $S$ and let $\overline{S} = S/Q$ and $P = Q \cap R$. If $Q = 0$ or $Q = S_{\geq 1}$, then obviously $P$ is a prime ideal of $R$. Thus we may assume that $Q \neq 0$ and that $\mathrm{GK}(\overline{S}) \geq 1$. Also, since ${}_RR \subseteq {}_RS$ is an essential extension by Lemma 3.2.5, we can assume that $P \neq 0$. Suppose that $I, J$ are ideals of $R$ with $I \supseteq P$, $J \supseteq P$, and $IJ \subseteq P$. Let $M = SISJS/SIJS$. Then ${}_SM$ is a homomorphic image of a direct sum of copies of $SIS/SI$, so $M \in S$-dist by Lemma 4.2.2. Since $0 \neq SIJS \subseteq \operatorname{ann}_S M$, ${}_SM$ is torsion by Lemma 4.1.4(2). Then $SISJS/(SISJS \cap Q) \cong (SISJS + Q)/Q$ is also torsion over $S$. But $\overline{S}$ may not have any nonzero torsion ideals, so $SISJS \subseteq Q$. Finally, either $SIS \subseteq Q$ or $SJS \subseteq Q$, and so either $I \subseteq P$ or $J \subseteq P$, and $P$ is prime. □

Finally, we note that it is easy to see exactly what the graded prime ideals of $S(\varphi)$ are.

**Lemma 4.2.4.** *Let $S = S(\varphi)$, and identify the homogeneous left ideals of $S$ with the homogeneous ideals of $U$. Then a homogeneous left ideal $I$ of $S$ is a prime ideal if and only if $I = \bigcap_{i=0}^{n-1} \phi^i(P)$ for some homogeneous prime $P$ of $U$ of finite order $n$ under $\phi$.*

*Proof.* Recall the definition of saturation for graded ideals of $U$, from §2.4. First we claim that if $I$ is saturated, then $I$ is a 2-sided ideal of $S$ if and only if $\phi(I) = I$. This follows from [3, Lemma 4.4] since $S$ may be described as a twisted homogeneous coordinate ring, but it is also easy to see directly: if $IS = I$, then in $U$ we have in particular that $\phi(I) \circ U_1 \subseteq I$. Then $(\phi(I) + I)/I$ is a torsion $U$-module, so since $I$ is saturated, $\phi(I) \subseteq I$. Since $U$ is finite dimensional in each degree this forces $\phi(I) = I$. Conversely, if $\phi(I) = I$ then clearly $IS = IU = I$.

Now suppose that $I$ is a graded prime ideal of $S$. Obviously $S_{\geq 1} = U_{\geq 1}$ is the unique maximal graded prime of $S$ and it has the correct form, so assume that $I \subsetneq S_{\geq 1}$. Then $S/I$ has no nonzero torsion ideals, so $I$ is saturated, and thus $\phi(I) = I$. We claim that $I$ is a radical ideal of $U$. Let $J = \sqrt{I}$. Then $J$ is homogeneous and $\phi(J) = J$. Thus $J/I$ is a nilpotent ideal of the prime ring $S/I$, and so $J = I$, proving the claim. Now if $P_1, \dots P_m$ are the distinct associated homogeneous primes of $I$ in $U$, then the action of $\phi$ must permute these primes. If $O_1, \dots O_n$ are the orbits of this action, then setting $J_i = \cap\{P \mid P \in O_i\}$, the $J_i$ are homogeneous $\phi$-invariant ideals of $U$ with $J_i \supseteq I$ and $J_1 \circ J_2 \circ \cdots \circ J_n = J_1 J_2 \dots J_n \subseteq I$. Since $I$ is prime in $S$, $n = 1$ and so $I$ is the intersection of a single orbit of primes. This forces $I = \bigcap_{i=0}^{n-1} \phi^i(P)$ for some graded prime $P$ of $U$ of order $n$ under $\phi$.

Conversely, let $I = \bigcap_{i=0}^{n-1} \phi^i(P)$ for some homogeneous prime $P$ of $U$ of finite order $n$ under $\phi$. If $P = U_{\geq 1}$ then $I = S_{\geq 1}$ is certainly prime, so we may assume that $P \subsetneq U_{\geq 1}$. Then $I$ is saturated. Suppose that $JK \subseteq I$ for ideals $J \supseteq I$ and $K \supseteq I$ of $S$. We may replace $J, K$ by their saturations, so by the initial claim of the proof, $J$ and $K$ are invariant under $\phi$; since $I$ is saturated, we still have $JK \subseteq I$. This means that $J \circ K \subseteq I \subseteq P$ in $U$. Since $P$ is prime in $U$, either $J \subseteq P$ or $K \subseteq P$. If $J \subseteq P$, then since $J$ is $\phi$-invariant, $J \subseteq \bigcap_{i=0}^{n-1} \phi^i(P) = I$. Similarly, if $K \subseteq P$ then $K \subseteq I$. Thus $I$ is a prime ideal of $S$. $\square$

## 4.3 The maximal order property

Let $A$ be a noetherian domain with Goldie quotient ring $Q$. We say $A$ is a *maximal order* in $Q$ if given any ring $T$ with $A \subseteq T \subseteq Q$ and nonzero elements $a, b$ of $A$ with $aTb \subseteq A$, we have $T = A$. If $A$ is commutative, then $A$ is a maximal order if and only if $A$ is integrally closed in its fraction field [23, Proposition 5.1.3].

We are interested in an equivalent formulation of the maximal order property. For any left ideal $I$ of $A$, we define $\mathcal{O}_r(I) = \{q \in Q | Iq \subseteq I\}$ and $\mathcal{O}_l(I) = \{q \in Q | qI \subseteq I\}$. Then $A$ is a maximal order if and only if $\mathcal{O}_r(I) = A = \mathcal{O}_l(I)$ for all nonzero ideals $I$ of $A$ [23, Proposition 5.1.4]. If $A$ is an $\mathbb{N}$-graded algebra with a graded ring of fractions $D$, then for any homogeneous ideal $I$ of $A$ we may also define $\mathcal{O}^g_r(I) = \{q \in D | Iq \subseteq I\}$ and $\mathcal{O}^g_l(I) = \{q \in D | qI \subseteq I\}$. In the graded case we have the following criterion for the maximal order property.

**Lemma 4.3.1.** *Let $A$ be an $\mathbb{N}$-graded noetherian domain which has a graded quotient ring $D$ and Goldie quotient ring $Q$. Then $A$ is a maximal order if and only if $\mathcal{O}^g_r(I) = A = \mathcal{O}^g_l(I)$ holds for all homogeneous nonzero ideals $I$ of $A$.*

This result is stated in [35, Lemma 2], but since we had trouble tracking down the reference given there we will supply a brief proof here.

*Proof.* We may write $D \cong T[z, z^{-1}; \sigma]$ for some division ring $T$ and automorphism $\sigma$ of $T$ (see §2.1). Then since $T$ is a maximal order, it follows by [22, Propositions IV.2.1, V.2.3] that $D$ is a maximal order in $Q$.

Assume that $\mathcal{O}^g_r(J) = A = \mathcal{O}^g_l(J)$ for all homogeneous ideals $J$ of $A$. Let $I$ be any ideal of $A$, and let $q \in \mathcal{O}_r(I)$. Then $DI$ is a 2-sided ideal of $D$ [17, Theorem 9.20], and also $q \in \mathcal{O}_r(DI)$. Since $D$ is a maximal order in $Q$, this forces $q \in D$.

Given any $d = \sum d_i \in D$ where $d_i \in D_i$, let $n$ be maximal such that $d_n \neq 0$ and set $\widetilde{d} = d_n$. Let $\widetilde{I}$ be the 2-sided homogeneous ideal generated by $\widetilde{a}$ for all $a \in I$. Write $q = \sum_{i=m}^{n} d_i$; then since $Iq \subseteq I$, we have $\widetilde{I} d_n \subseteq \widetilde{I}$ and so $d_n \in \mathcal{O}_r^g(\widetilde{I}) = A$. Then $q - d_n \in \mathcal{O}_r(I)$. By induction on $n - m$ we get that $q - d_n \in A$ and so $q \in A$. Thus $\mathcal{O}_r(I) = A$, and an analogous argument gives $\mathcal{O}_l(I) = A$, so $A$ is a maximal order. The opposite implication is trivial. □

Now let $S = S(\varphi)$ and $R = R(\varphi, c)$ and assume the critical density condition. Our next goal is to show that $R = R(\varphi, c)$ is a maximal order. First, we note that the ring $S$ has this property.

**Lemma 4.3.2.** *$S = S(\varphi)$ is a maximal order.*

*Proof.* By [43, Theorem 5.11], $S$ is ungraded Cohen-Macaulay and Auslander-regular, since $U$ has both properties; also, since $S$ is graded it is trivially stably free. By [28, Theorem 2.10], any ring satisfying these three properties is a maximal order. □

Recall that $R$ and $S$ have the same graded quotient ring $D$ (Lemma 3.2.5). For any graded left $R$-submodules $M, N$ of $D$, we identify $\underline{\mathrm{Hom}}_R(M, N)$ with $\{d \in D \mid Md \subseteq N\}$. Similarly, if $M, N$ are graded left $S$-submodules of $D$ we identify $\underline{\mathrm{Hom}}_S(M, N)$ and $\{d \in D \mid Md \subseteq N\}$.

**Proposition 4.3.3.** *Let $I$ be a nonzero homogeneous ideal of $R$. Then $\mathcal{O}_l^g(I) \subseteq S$ and $\mathcal{O}_r^g(I) \subseteq S$.*

*Proof.* Consider $\mathcal{O}_r^g(I)$ for some nonzero homogeneous ideal $I$ of $R$. We have that

$$\mathcal{O}_r^g(I) = \{q \in D | Iq \subseteq I\} \subseteq \{q \in D | SIq \subseteq SI\} = \underline{\mathrm{Hom}}_S({}_S SI, {}_S SI).$$

We will show that $\underline{\mathrm{Hom}}_S(SI, SI) \subseteq S$. Since $S$ is a maximal order by Lemma 4.3.2, we know that $\mathcal{O}_r^g(SIS) = \underline{\mathrm{Hom}}_S(SIS, SIS) = S$. Set $M = SIS/SI$, and note that by Lemma 4.2.2, $M \in S$-dist.

We have the following long exact sequence in $\underline{\mathrm{Ext}}$:

$$0 \to \underline{\mathrm{Hom}}_S(M, SIS) \to \underline{\mathrm{Hom}}_S(SIS, SIS) \to \underline{\mathrm{Hom}}_S(SI, SIS)$$
$$\to \underline{\mathrm{Ext}}^1_S(M, SIS) \to \dots$$

Now $\underline{\mathrm{Hom}}_S(M, SIS) = 0$, since $S$ is a domain with $GK(S) = t + 1 \geq 3$ and $\mathrm{GK}(M) \leq 1$. Also, $\dim_k \underline{\mathrm{Ext}}^1_S(M, SIS) < \infty$, by Proposition 4.1.7(2). We see that $\underline{\mathrm{Hom}}(SI, SIS)$ is a right $S$-submodule of $D$ which is an essential finite-dimensional extension of $\underline{\mathrm{Hom}}_S(SIS, SIS) = S$. Since $\underline{\mathrm{Ext}}^1_S(k_S, S_S) = 0$ (for example, by a right-sided version of Lemma 4.1.5(2)), $S$ has no nontrivial finite dimensional extensions and so it must be that $\underline{\mathrm{Hom}}_S(SI, SIS) = S$. Finally, $\underline{\mathrm{Hom}}_S(SI, SI) \subseteq \underline{\mathrm{Hom}}_S(SI, SIS) = S$ and so $\mathcal{O}^g_r(I) \subseteq S$.

The proof that $\mathcal{O}^g_l(I) \subseteq S$ follows by applying the same argument in the extension of rings $R^{op} \subseteq S^{op}$, which is valid by Lemma 3.2.2(1). □

Now we may complete the proof that $R$ is a maximal order.

**Theorem 4.3.4.** *Assume the critical density condition, so that $R$ is noetherian. Then $R$ is a maximal order.*

*Proof.* Let $I$ be any nonzero homogeneous ideal of $R$. Then $\underline{\mathrm{Hom}}_R({}_RI, {}_RI) = \mathcal{O}^g_r(I) \subseteq S$, by Proposition 4.3.3. Set $M = (\underline{\mathrm{Hom}}_R(I, I))/R$; then ${}_RM$ is a submodule of ${}_R(S/R)$, so $M \in R$-Dist by Lemma 4.1.2. Since $IM = 0$, Proposition 4.1.4(1) implies that $M$ is a torsion module. But $\underline{\mathrm{Ext}}^1_R(k, R) = 0$ by Lemma 4.1.11, and so $R$ may not have any nontrivial torsion extensions. Since $R \subseteq \underline{\mathrm{Hom}}_R(I, I)$ is an essential extension, this forces $\mathcal{O}^g_r(I) = \underline{\mathrm{Hom}}(I, I) = R$. Applying the same argument in $R^{op}$, we get $\mathcal{O}^g_l(I) = R$ as well. Thus $R$ is a maximal order by Lemma 4.3.1. □

## 4.4 The $\chi$ condition and $R$-proj

We begin this section by discussing some definitions from the theory of noncommutative projective schemes which we have not needed until now. See also [8] for more details.

Let $A$ be a noetherian $\mathbb{N}$-graded ring which is finitely generated in degree 1. Recall from §1.2 that the *noncommutative projective scheme* $A$-proj is defined to be the ordered pair $(A\text{-qgr}, \mathcal{A})$, where $A$-qgr is the quotient category $A\text{-gr}/A\text{-tors}$ as defined in §2.1, and $\mathcal{A}$, the *distinguished object*, is the image of ${}_A A$ in $A$-qgr. We write $A\text{-proj} \cong B\text{-proj}$ if there is an equivalence of categories $A\text{-qgr} \sim B\text{-qgr}$ under which the distinguished objects correspond. The shift functor $M \mapsto M[1]$, which is an autoequivalence of the category $A$-Gr, descends naturally to an autoequivalence of $A$-Qgr, for which we use the same notation. For $\mathcal{M}, \mathcal{N} \in A\text{-Qgr}$ we define

$$\underline{\mathrm{Ext}}^i(\mathcal{M}, \mathcal{N}) = \bigoplus_{i=-\infty}^{\infty} \mathrm{Ext}^i(\mathcal{M}, \mathcal{N}[i]).$$

Cohomology groups are defined for $A$-Proj by setting $\mathrm{H}^i_A(\mathcal{N}) = \mathrm{Ext}^i(\mathcal{A}, \mathcal{N})$ for any $\mathcal{N} \in A\text{-Qgr}$, where Ext is calculated in the category $A$-Qgr. We also define the graded cohomology groups $\underline{\mathrm{H}}^i_A(\mathcal{N}) = \underline{\mathrm{Ext}}^i(\mathcal{A}, \mathcal{N})$. The section functor $\omega$ (see §2.1) may also be described using cohomology as $\omega(\mathcal{M}) = \underline{\mathrm{H}}^0(\mathcal{M})$ for all $\mathcal{M} \in A\text{-Qgr}$.

In this section, we will analyze the $\chi$ conditions (see §4.1) for $R = R(\varphi, c)$, assuming the critical density condition throughout. The reader may easily prove the following simple facts.

**Lemma 4.4.1.** *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $R$-gr, and let $N \in R$-gr.*

*(1) If $\chi_1(M')$ and $\chi_1(M'')$ hold then $\chi_1(M)$ holds.*

*(2) If $\chi_1(M)$ holds then $\chi_1(M')$ holds.*

*(3) If* $\dim_k N < \infty$ *then* $\chi_1(N)$ *holds.* $\square$

To prove $\chi_1$ for $R$ we will reduce to the case of $S$-modules.

**Proposition 4.4.2.** *Suppose that* $N \in S$-gr. *Then* $\chi_1({}_R N)$ *holds.*

*Proof.* Consider the first 3 terms of the 5-term exact sequence (4.1.9) for $M = {}_R k$:

$$0 \longrightarrow \underline{\mathrm{Ext}}^1_S(S \otimes_R k, N) \longrightarrow \underline{\mathrm{Ext}}^1_R(k, N) \longrightarrow \underline{\mathrm{Hom}}_S(\underline{\mathrm{Tor}}^R_1(S, k), N) \longrightarrow \dots$$

Now $\underline{\mathrm{Tor}}^R_i(S, k)$ is in $S$-dist for any $i \geq 0$, by Lemma 4.1.10(2). Then by Proposition 4.1.7(2), we conclude that $\dim_k \underline{\mathrm{Ext}}^j_S(\underline{\mathrm{Tor}}^R_i(S, k)), N) < \infty$ for $j = 0, 1$ and $i \geq 0$. Thus $\dim_k \underline{\mathrm{Ext}}^1_R(k, N) < \infty$. $\square$

We now prove Theorem 1.5.6, which together with Theorem 4.3.4 answers Questions 1.5.5 and 1.5.8 from the introduction.

**Theorem 4.4.3.** *Assume the critical density condition and let* $R = R(\varphi, c)$.

*(1)* $R$ *satisfies* $\chi_1$.

*(2)* $\underline{\mathrm{Ext}}^2_R(k, R)$ *is not bounded, and* $\chi_i$ *fails for all* $i \geq 2$.

*Proof.* (1) By Lemma 4.4.1(1) and induction it is enough to show that $\chi_1(M)$ holds for all graded cyclic $R$-modules $M$.

Let $R/I$ be an arbitrary graded cyclic left $R$-module. If $I = 0$, then $\chi_1({}_R R)$ holds by Lemma 4.1.11. Assume then that $I \neq 0$. Consider the exact sequence (3.3.15). Now $\chi_1({}_R(S/SI))$ holds by Proposition 4.4.2. By Lemma 3.3.16(1), both $(SI \cap R)/I$ and $S/(R+SI)$ have finite filtrations with factors which are either torsion or shifted $R$-point modules with a compatible $S$-module structure. Then $\chi_1((SI \cap R)/I)$ and $\chi_1(S/(R+SI))$ hold, by Proposition 4.4.2 and Lemma 4.4.1(1),(3). Finally, applying Lemma 4.4.1(1),(2) to (3.3.15) we get that $\chi_1(R/I)$ holds.

(2) Consider the long exact sequence in $\underline{\mathrm{Ext}}_R(k,-)$ that arises from the short exact sequence of $R$-modules $0 \to R \to S \to S/R \to 0$:

$$\dots \to \underline{\mathrm{Ext}}^1_R(k,S) \to \underline{\mathrm{Ext}}^1_R(k,S/R) \to \underline{\mathrm{Ext}}^2_R(k,R) \to \dots \tag{4.4.4}$$

Now $\underline{\mathrm{Ext}}^1_R(k,S) = 0$, as in the proof of Lemma 4.1.11. On the other hand,

$$\underline{\mathrm{Ext}}^1_R(k,S/R) \cong \bigoplus_{i=1}^{\infty} \underline{\mathrm{Ext}}^1_R(k,P(c_{-1}))[-i]$$

by Corollary 3.3.6(2), since $\underline{\mathrm{Ext}}$ commutes with direct sums in the second coordinate by Lemma 4.1.6(1). By Theorem 3.4.5(2), it is clear that the point module $P(c_{-1})$ has a nontrivial extension by $k[1]$, since any point module $P(c_0,e)$ defined there satisfies $(P(c_0,e)[1])_{\geq 0} \cong {}_R P(c_{-1})$. Thus $\underline{\mathrm{Ext}}^1_R(k,P(c_{-1})) \neq 0$, and so $\bigoplus_{i=1}^{\infty} \underline{\mathrm{Ext}}^1_R(k,P(c_{-1}))[-i] \cong \underline{\mathrm{Ext}}^1_R(k,S/R)$ is not right bounded. Then by the exact sequence (4.4.4), $\underline{\mathrm{Ext}}^2_R(k,R)$ is also not right bounded. In particular, we have $\dim_k \underline{\mathrm{Ext}}^2_R(k,R) = \infty$ and $\chi_i$ fails for $R$ for all $i \geq 2$ by definition. □

We see next that the failure of $\chi_i$ for $R$ for $i \geq 2$ is reflected in the cohomology of $R$-proj. Recall the noncommutative version of Serre's finiteness theorem which was proved by Artin and Zhang (Theorem 1.5.4). We give a slightly stronger restatement here which better suits our purposes.

**Theorem 4.4.5.** *Let $A$ be a left noetherian finitely $\mathbb{N}$-graded algebra which satisfies $\chi_1$. Then $A$ satisfies $\chi_i$ for some $i \geq 2$ if and only if the following two conditions hold:*

*(1)* $\dim_k \mathrm{H}^j(\mathcal{N}) < \infty$ *for all* $0 \leq j < i$ *and all* $\mathcal{N} \in A$-qgr.

*(2)* $\underline{\mathrm{H}}^j(\mathcal{N})$ *is right bounded for all* $1 \leq j < i$ *and all* $\mathcal{N} \in A$-qgr.

*Proof.* This follows immediately from the proof of [8, Theorem 7.4]. □

**Lemma 4.4.6.** *Let $A$ be a left noetherian finitely $\mathbb{N}$-graded algebra satisfying $\chi_i$. Then $\dim_k \operatorname{Ext}^j(\mathcal{M}, \mathcal{N}) < \infty$ for $0 \le j < i$ and for all $\mathcal{M}, \mathcal{N} \in A$-qgr.*

*Proof.* Let $\mathcal{A} = \pi(A)$. Since any $M \in A$-gr is an image of some finite sum of shifts of $A$, in $A$-qgr there is an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{F} \to \mathcal{M} \to 0$$

where we have $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{A}[d_i]$ for some integers $d_i \in \mathbb{Z}$. Then $\operatorname{Ext}^j(\mathcal{F}, \mathcal{N}) = \bigoplus_{i=1}^n \operatorname{Ext}^j(\mathcal{A}, \mathcal{N}[-d_i]) = \bigoplus_{i=1}^n \mathrm{H}^j(\mathcal{N}[-d_i])$ and so $\dim_k \operatorname{Ext}^j(\mathcal{F}, \mathcal{N}) < \infty$ for all $0 \le j < i$ by Theorem 4.4.5.

We induct on $j$. If $j = 0$ then there is an exact sequence $0 \to \operatorname{Hom}(\mathcal{M}, \mathcal{N}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{N})$ from which it follows that $\dim_k \operatorname{Hom}(\mathcal{M}, \mathcal{N}) < \infty$. For $0 < j < i$, we consider the long exact sequence

$$\ldots \to \operatorname{Ext}^{j-1}(\mathcal{M}', \mathcal{N}) \to \operatorname{Ext}^j(\mathcal{M}, \mathcal{N}) \to \operatorname{Ext}^j(\mathcal{F}, \mathcal{N}) \ldots$$

Since $\dim_k \operatorname{Ext}^{j-1}(\mathcal{M}', \mathcal{N}) < \infty$ by the induction hypothesis, $\dim_k \operatorname{Ext}^j(\mathcal{M}, \mathcal{N}) < \infty$ as well. This completes the induction step and the proof. $\square$

We can make the failure of the Serre's finiteness theorem for $R$-proj explicit.

**Lemma 4.4.7.** *Let $\mathcal{R} = \pi(R) \in R$-qgr be the distinguished object of $R$-proj. Then $\dim_k \mathrm{H}^1(\mathcal{R}) = \infty$.*

*Proof.* Set $\mathcal{S} = \pi(S) \in R$-Qgr. The exact sequence $0 \to R \to S \to S/R \to 0$ descends to an exact sequence $0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{S}/\mathcal{R} \to 0$ in $R$-qgr. For $\mathcal{M} \in R$-Qgr, the cohomology $\mathrm{H}^0(\mathcal{M})$ may be identified with the zeroeth graded piece of the module $\omega(\mathcal{M})$, where $\omega$ is the section functor. Recall also that for torsionfree $M \in A$-Gr, $\omega\pi(M)$ is the largest essential extension of $M$ by a torsion module.

Since $\underline{\mathrm{Ext}}^1_R(k, S) = 0$ by the proof of Lemma 4.1.11, ${}_RS$ has no nontrivial torsion extensions and so $\omega(\mathcal{S}) = S$. In particular, $\dim_k \mathrm{H}^0(\mathcal{S}) = \dim_k S_0 = 1$. On the other hand, $S/R = \bigoplus_{i=1}^{\infty} P(c_{-1})[-i]$ is an infinite direct sum of shifted $R$-point modules by Corollary 3.3.6(2). For each $i \geq 0$, we may choose some $R$-point module $P(c_i, e_i)$ by Theorem 3.4.5(2) which satisfies $P(c_i, e_i)_{\geq i+1} \cong P(c_{-1})[-i-1]$. Then $M = \bigoplus_{i=0}^{\infty} P(c_i, e_i)$ is an essential extension of $S/R$ by a torsion module, so $M \subseteq \omega(\mathcal{S}/\mathcal{R})$ and it follows that $\dim_k \mathrm{H}^0(\mathcal{S}/\mathcal{R}) \geq \dim_k M_0 = \infty$. Now the long exact sequence in cohomology forces $\dim_k \mathrm{H}^1(\mathcal{R}) = \infty$ as well. $\square$

The following result, which proves Theorem 1.5.7 from the introduction, shows that the category $R$-qgr is necessarily something quite different from any of the standard examples of noncommutative schemes.

**Theorem 4.4.8.** *Let $R = R(\varphi, c)$ for $(\varphi, c)$ satisfying the critical density condition.*
*(1) Suppose that $A$ is a left noetherian finitely $\mathbb{N}$-graded $k$-algebra which satisfies $\chi_2$. Then the categories $A$-qgr and $R$-qgr are not equivalent.*
*(2) $R$-qgr is not equivalent to $\mathrm{coh}\, X$ for any commutative projective scheme $X$.*

*Proof.* (1) The proof is immediate from Lemmas 4.4.6 and 4.4.7.
(2) This follows from part (1) and the usual commutative Serre's theorem (Theorem 1.2.1). $\square$

## 4.5 Cohomological dimension

In this section, we study the behavior of cohomology in $R$-proj further. In particular, we will show that $R$ has finite cohomological dimension. Let us recall the definition:

**Definition 4.5.1.** Let $A$ be a connected finitely generated $\mathbb{N}$-graded algebra. The *cohomological dimension* of $A$-proj is defined to be

$$\mathrm{cd}(A\text{-proj}) = \max\{i \mid \mathrm{H}^i(\mathcal{N}) \neq 0 \text{ for some } \mathcal{N} \in A\text{-qgr}\}$$

if this number is finite; otherwise we set $\mathrm{cd}(A\text{-proj}) = \infty$.

We remark that it is not known if there exists any graded algebra $A$ such that $\mathrm{cd}(A\text{-proj}) = \infty$.

It is also useful to define the cohomological dimension of an single object $\mathcal{N} \in A$-Qgr by $\mathrm{cd}_A(\mathcal{N}) = \max\{i \mid \mathrm{H}^i(\mathcal{N}) \neq 0\}$. Let us note how the cohomological dimension of objects in $A$-Qgr behaves with respect to exact sequences.

**Lemma 4.5.2.** *Let* $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \longrightarrow 0$ *be a short exact sequence in* $A$-Qgr*, with* $\mathrm{cd}(\mathcal{N}') < \infty$*,* $\mathrm{cd}(\mathcal{N}) < \infty$*, and* $\mathrm{cd}(\mathcal{N}'') < \infty$*. Then*

*(1)* $\mathrm{cd}(\mathcal{N}) \leq \max(\mathrm{cd}(\mathcal{N}'), \mathrm{cd}(\mathcal{N}''))$*.*

*(2)* $\mathrm{cd}(\mathcal{N}'') \leq \max(\mathrm{cd}(\mathcal{N}'), \mathrm{cd}(\mathcal{N}))$*.*

*(3)* $\mathrm{cd}(\mathcal{N}') \leq \max(\mathrm{cd}(\mathcal{N}), \mathrm{cd}(\mathcal{N}'')) + 1$*.*

*Proof.* All three assertions quickly follow from the long exact sequence in cohomology associated to the given short exact sequence. □

For $i \geq 1$, the graded cohomology groups in $A$-Proj may be expressed in terms of a kind of local cohomology over the ring $A$, as in the commutative case.

**Lemma 4.5.3.** *[8, Proposition 7.2(2)] Let* $\mathcal{N} = \pi(N) \in A$-Qgr *for some* $N \in A$-Gr*. Then as* $k$*-spaces,*

$$\underline{\mathrm{H}}^i_A(\mathcal{N}) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^{i+1}_A(A/A_{\geq n}, N)$$

*for* $i \geq 1$*.* □

Now let $S = S(\varphi)$ and $R = R(\varphi, c)$, and assume the critical density condition as usual. It is easy to compute the cohomological dimension of $S$-proj:

**Lemma 4.5.4.** $\mathrm{cd}(S\text{-proj}) = \mathrm{GK}(S) - 1 = t$.

*Proof.* Since $S$ is AS-regular by Lemma 4.1.5(3), this follows immediately from [8, Theorem 8.1]. $\square$

We also have the following version of Grothendieck's vanishing theorem.

**Lemma 4.5.5.** *Let $N \in S$-gr. Then $\mathrm{H}^i_S(\pi(N)) = 0$ for $i \geq \mathrm{GK}_S(N)$.*

*Proof.* The equivalence of categories $S$-gr $\sim U$-gr (see §2.3) obviously sends torsion objects to torsion objects, and so it descends to an isomorphism $S$-proj $\sim U$-proj. Thus it is enough to prove that $\mathrm{H}^i_U(\pi(M)) = 0$ for $i \geq \mathrm{GK}_U(M)$ and all $M \in U$-gr. If $\mathrm{GK}(M) = 0$ the statement is trivial since $\pi(M) = 0$, so assume that $\mathrm{GK}(M) \geq 1$. Now for $i \geq 1$, we can express $\underline{\mathrm{H}}^i_U(\pi(M))$ in terms of local cohomology, by Lemma 4.5.3: $\underline{\mathrm{H}}^i_U(\pi(M)) = \lim_{n\to\infty} \underline{\mathrm{Ext}}^{i+1}_U(U/U_{\geq n}, M)$. By Grothendieck's vanishing theorem [13, 6.1.2], $\lim_{n\to\infty} \underline{\mathrm{Ext}}^j_U(U/U_{\geq n}, M) = 0$ for $j > \mathrm{GK}(M)$, so the result follows. $\square$

The main machinery we will use to study the cohomological dimension of $R$ is the next proposition, which reduces the calculation of the cohomology $H^i_R(\pi(N))$ for $N \in S$-Gr to a homological calculation over the ring $S$ only. First we need some technical lemmas.

**Lemma 4.5.6.** *For any $N \in S$-Gr there is a convergent spectral sequence of the form*

$$E_2^{pq} = \lim_{n\to\infty} \underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S, R/R_{\geq n}), N) \underset{p}{\Rightarrow} \lim_{n\to\infty} \underline{\mathrm{Ext}}^{p+q}_R(R/R_{\geq n}, N).$$

*Proof.* Consider the spectral sequence (4.1.8) for arbitrary ${}_RM \in R$-Gr:

$$\underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S, M), N) \underset{p}{\Rightarrow} \underline{\mathrm{Ext}}^{p+q}_R(M, N).$$

Let $\mathcal{C}$ be the category of all $\mathbb{N}$-indexed directed systems of modules in $R$-Gr of the form

$$\ldots \to M_n \to \ldots \to M_1 \to M_0.$$

Let $\mathcal{D}$ be the analogous category of directed systems of modules in $S$-Gr. Both of these categories have enough projectives and injectives. For example, if $P$ is a projective object of $R$-Gr, then any object in $\mathcal{C}$ of the form

$$\ldots \to 0 \to 0 \to 0 \to P \xrightarrow{\cong} P \xrightarrow{\cong} \ldots \xrightarrow{\cong} P \tag{4.5.7}$$

is projective, and clearly every object in $\mathcal{C}$ is an image of a direct sum of objects of this form. See [41, Exercises 2.3.7, 2.3.8] for more details. The functor $S \otimes_R - :$ $R$-Gr $\to S$-Gr extends to a functor $G : \mathcal{C} \to \mathcal{D}$. We also have a functor $F : \mathcal{D} \to \mathrm{Ab}$ defined by $\{L_n\}_{n \in \mathbb{N}} \mapsto \lim_{n \to \infty} \underline{\mathrm{Hom}}_S(L_n, N)$, where Ab is the category of abelian groups. It is easy to see that $G$ is right exact and $F$ is left exact. Finally, $G$ sends any direct sum of objects in $\mathcal{C}$ of the form in (4.5.7) to a projective object in $\mathcal{D}$. Then corresponding to the composition of functors $F \circ G$ is a Grothendieck spectral sequence (see [27, Theorem 11.40])

$$E_2^{p,q} = R^p F(L_q G(M_.)) \underset{p}{\Rightarrow} R^{p+q} FG(M_.)$$

which unravels to the spectral sequence required by the lemma when we take $M_n = R/R_{\geq n}$ for all $n \geq 0$. $\square$

**Lemma 4.5.8.** *For any $n \geq 0$ and $N \in S$-Gr, $\underline{\mathrm{Ext}}^i_S(S_{\geq n}/SR_{\geq n}, N) = 0$ for $i > t$.*

*Proof.* Fix $n \geq 0$. If $J = (\cap_{i=0}^{n-1} \mathfrak{m}_{c_i}) \subseteq U$, then by Lemma 2.5.8, $J$ is generated in degrees $\leq n$ and so using Theorem 3.2.3 we may identify $SR_{\geq n}$ and $J_{\geq n}$. There is a natural injection of $U$-modules $U_{\geq n}/J_{\geq n} \to \bigoplus_{i=0}^{n-1} (U/m_{c_i})_{\geq n}$, which must be an isomorphism since both sides have the same Hilbert function by Lemma 2.5.6. Then by the equivalence of categories between $U$-gr and $S$-gr it follows that $S_{\geq n}/SR_{\geq n}$ is a direct sum of shifted $S$-point modules. Thus we reduce to showing that $\underline{\mathrm{Ext}}^i_S(M,N)=0$ for $i>t$ when $M$ is a point module over $S$. But since point modules over $U$ have projective dimension $t$ by the Auslander-Buchsbaum formula ([16, Theorem 19.9]), the same is true of point modules over $S$ by the equivalence of categories. The result follows. $\square$

**Proposition 4.5.9.** *Let* $N \in S$-Gr.

*(1) As graded vector spaces, for all* $m \geq 1$ *we have*

$$\underline{\mathrm{H}}^m_R(\pi(N)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^{m+1}_S(S/SR_{\geq n}, N).$$

*(2)* $\underline{\mathrm{H}}^i_R(\pi(N)) \cong \underline{\mathrm{H}}^i_S(\pi(N))$ *for all* $i \geq t+1$*. In particular,* $\mathrm{cd}_R(\pi(N)) \leq t$*.*

*(3) If* $\underline{\mathrm{H}}^t_S(\pi(N))=0$ *then* $\underline{\mathrm{H}}^t_R(\pi(N))=0$*.*

*Proof.* (1) We use the spectral sequence of Lemma 4.5.6:

$$E_2^{pq} = \lim_{n\to\infty} \underline{\mathrm{Ext}}^p_S(\underline{\mathrm{Tor}}^R_q(S, R/R_{\geq n}), N) \underset{p}{\Rightarrow} \lim_{n\to\infty} \underline{\mathrm{Ext}}^{p+q}_R(R/R_{\geq n}, N).$$

Our goal is to show that $E_2^{pq}=0$ for any pair of indices $p,q$ with $q \geq 1$.

Fix $q \geq 1$. For fixed $n \geq 0$, we claim first that there is some $n' \geq n$ such that the natural map $\psi_1 : \underline{\mathrm{Tor}}^R_q(S, R/R_{\geq n'}) \to \underline{\mathrm{Tor}}^R_q(S, R/R_{\geq n})$ is 0. Using Lemma 3.2.2, Corollary 3.3.6(2) holds just as well on the right side and so there is a right point module $P$ of $R$ such that $S/R \cong \bigoplus_{i=1}^{\infty} P[-i]$ as right $R$-modules. Now by part (1) of Lemma 4.1.10 and the fact that $\underline{\mathrm{Tor}}$ commutes with direct sums (Lemma 4.1.6(2)) we get a commutative diagram

$$\begin{array}{ccc}
\underline{\mathrm{Tor}}_q^R(S, R/R_{\geq n'}) & \xrightarrow{\psi_1} & \underline{\mathrm{Tor}}_q^R(S, R/R_{\geq n}) \\
\downarrow \cong & & \downarrow \cong \\
\underline{\mathrm{Tor}}_q^R(S/R, R/R_{\geq n'}) & \xrightarrow{\psi_2} & \underline{\mathrm{Tor}}_q^R(S/R, R/R_{\geq n}) \\
\downarrow \cong & & \downarrow \cong \\
\bigoplus_{i=1}^{\infty} \underline{\mathrm{Tor}}_q^R(P, R/R_{\geq n'})[-i] & \xrightarrow{\psi_3} & \bigoplus_{i=1}^{\infty} \underline{\mathrm{Tor}}_q^R(P, R/R_{\geq n})[-i]
\end{array}$$

where the vertical maps are vector space isomorphisms and the $\psi_i$ are the natural maps. Now $T^n = \underline{\mathrm{Tor}}_q^R(P, R/R_{\geq n})$ is bounded, since $P_R$ is finitely generated and $R/R_{\geq n}$ is bounded, by Lemma 4.1.6(3). Also, the left bound $l(n)$ of $T^n$ satisfies $\lim_{n\to\infty} l(n) = \infty$, by Lemma 4.1.6(4). It follows that for $n' \gg n$ the natural map $\theta : T^{n'} \to T^n$ is 0. The restriction of the map $\psi_3$ to any summand is just a shift of the map $\theta$, so $\psi_3 = 0$ for $n' \gg n$. Finally, the commutative diagram gives $\psi_1 = 0$ for $n' \gg n$. This proves the claim.

Write ${}_nE_2^{pq} = \underline{\mathrm{Ext}}_S^p(\underline{\mathrm{Tor}}_q^R(S, R/R_{\geq n}), N)$. Since $\psi_1 = 0$ for $n' \gg n$, the natural map ${}_nE_2^{pq} \to {}_{n'}E_2^{pq}$ is also zero for $n' \gg n$. Since $n$ was arbitrary, we have $E_2^{pq} = \lim_{n\to\infty} {}_nE_2^{pq} = 0$.

Therefore only the $E_2^{pq}$ with $q = 0$ are possibly nonzero, and the spectral sequence collapses, giving an isomorphism of vector spaces for all $m \geq 1$ (using also Lemma 4.5.3) as follows:

$$\underline{\mathrm{H}}_R^m(\pi(N)) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_R^{m+1}(R/R_{\geq n}, N) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^{m+1}(\underline{\mathrm{Tor}}_0^R(S, R/R_{\geq n}), N)$$
$$= \lim_{n\to\infty} \underline{\mathrm{Ext}}_S^{m+1}(S/SR_{\geq n}, N).$$

(2) For each $n \geq 0$, we have the short exact sequence

$$0 \to S_{\geq n}/SR_{\geq n} \to S/SR_{\geq n} \to S/S_{\geq n} \to 0$$

and for any $N \in S$-Gr we have the associated long exact sequence in $\underline{\mathrm{Ext}}^i_S(-, N)$. The direct limit of a directed system of exact sequences is exact [41, Theorem 2.6.15], and so taking the direct limit of all of the long exact sequences for various $n$ we get a long exact sequence

$$\begin{aligned}(4.5.10)\qquad \ldots \to \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_S(S/S_{\geq n}, N) \to \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_S(S/SR_{\geq n}, N) \\ \to \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_S(S_{\geq n}/SR_{\geq n}, N) \to \ldots\end{aligned}$$

Note that for $i \geq 1$, $\lim_{n\to\infty} \underline{\mathrm{Ext}}^{i+1}_S(S/S_{\geq n}, N) = \underline{\mathrm{H}}^i_S(\pi(N))$, and by part (1), $\lim_{n\to\infty} \underline{\mathrm{Ext}}^{i+1}_S(S/SR_{\geq n}, N) = \underline{\mathrm{H}}^i_R(\pi(N))$. Now by Lemma 4.5.8, we have that $\underline{\mathrm{Ext}}^i_S(S_{\geq n}/SR_{\geq n}, N) = 0$ for $i > t$ and any $n \geq 0$, and so it follows from (4.5.10) that $\underline{\mathrm{H}}^i_S(\pi(N)) \cong \underline{\mathrm{H}}^i_R(\pi(N))$ as vector spaces for all $i > t$. Now since $S$-proj has cohomological dimension $t$ by Lemma 4.5.4, we have $\underline{\mathrm{H}}^i_S(\pi(N)) = 0$ for $i > t$ and so also $\underline{\mathrm{H}}^i_R(\pi(N)) = 0$ for $i > t$. Thus by definition $\mathrm{cd}_R(\pi(N)) \leq t$.

(3) This result also follows immediately from the long exact sequence (4.5.10). □

We now have our main result concerning the cohomological dimension of $R$, which proves Theorem 1.5.10 from the introduction. The theorem suggests that cohomological dimension may be finite even for rings which fail the $\chi$ condition, thus supporting the conjecture that cohomological dimension should be a finite number for all graded rings.

**Theorem 4.5.11.** *Assume the critical density condition. The graded algebra $R = R(\varphi, c)$ has finite cohomological dimension, in particular* $\mathrm{cd}(R\text{-proj}) \leq t = \mathrm{GK}(R) - 1$.

*Proof.* We show first that the cohomological dimension of $R$-proj is indeed a finite number. As usual, we reduce to the case of $S$-modules.

Let $M = R/I$ be an arbitrary cyclic left $R$-module. Suppose first that $I \neq 0$. We have the exact sequence (3.3.15), which descends to an exact sequence in $R$-Qgr:

$$0 \to \pi((SI \cap R)/I) \to \pi(R/I) \to \pi(S/SI) \to \pi(S/(R + SI)) \to 0.$$

Now $S/SI$ is an $S$-module already, and $(SI \cap R)/I$ and $(S/R + SI)$ both have finite left $R$-module filtrations where each factor has a compatible $S$-module structure, by Lemma 3.3.16(1). By Lemma 4.5.2(1) and Proposition 4.5.9(2), we conclude that $\mathrm{cd}_R(\pi(S/SI)) \leq t$, $\mathrm{cd}_R(\pi((SI \cap R)/I)) \leq t$, and $\mathrm{cd}_R(\pi(S/(R + SI))) \leq t$. A further application of Lemma 4.5.2(1),(3) gives $\mathrm{cd}_R(\pi(R/I)) \leq t + 1$.

Suppose instead that $I = 0$ and so $M = R$. We have the following exact sequence in $R$-Qgr:

$$0 \to \pi(R) \to \pi(S) \to \pi(S/R) \to 0.$$

By Proposition 4.5.9(2) we have $\mathrm{cd}_R(\pi(S)) \leq t$. Now $S/R \cong \bigoplus_{i=1}^{\infty} {}_R P(c_{-1})[-i]$ by Corollary 3.3.6(2); since cohomology commutes with direct sums [8, Proposition 7.2(4)], $\mathrm{cd}_R(\pi(S/R)) \leq t$ by Proposition 4.5.9(2) also. We conclude that $\mathrm{cd}_R(\pi(R)) \leq t + 1$ by Lemma 4.5.2(3).

Thus for any cyclic graded left $R$-module $R/I$, $\mathrm{cd}_R(\pi(R/I)) \leq t + 1$. Any $N \in R$-gr has a finite filtration by cyclic modules, so by Lemma 4.5.2(1) $\mathrm{cd}_R(\pi(N)) \leq t + 1$. Thus $\mathrm{cd}(R\text{-proj}) \leq t + 1$, and in particular $R$-proj has finite cohomological dimension.

To complete the proof of the theorem, we need to improve the bound on the cohomological dimension of $R$-proj we just calculated. By [8, Proposition 7.10(1)], if $A$ is a graded ring such that $\mathrm{cd}(A\text{-proj}) < \infty$, then $\mathrm{cd}(A\text{-proj}) = \mathrm{cd}(\pi(A))$; in other words, it is enough to calculate the cohomological dimension of the distinguished object. Thus we need only show that $\underline{\mathrm{H}}_R^{t+1}(\pi(R)) = 0$.

Consider the long exact sequence in cohomology associated to the short exact sequence $0 \to \pi(R) \to \pi(S) \to \pi(S/R) \to 0$:

$$(4.5.12) \qquad \dots \to \underline{\mathrm{H}}^t_R(\pi(S/R)) \to \underline{\mathrm{H}}^{t+1}_R(\pi(R)) \to \underline{\mathrm{H}}^{t+1}_R(\pi(S)) \dots$$

Let $P = P(c_{-1})$. Now $\underline{\mathrm{H}}^t_S(\pi(P)) = 0$ by Lemma 4.5.5 since $GK_S(P) = 1$ and $t \geq 2$. Then $\underline{\mathrm{H}}^t_R(\pi(P)) = 0$ by Proposition 4.5.9(3). Then since $S/R \cong \bigoplus_{j=1}^{\infty} P(c_{-1})[-j]$ by Corollary 3.3.6(2), it follows that $\underline{\mathrm{H}}^t_R(\pi(S/R)) = 0$. Also, $\underline{\mathrm{H}}^{t+1}_R(\pi(S)) = 0$ is immediate from Proposition 4.5.9(2). We conclude from (4.5.12) that $\underline{\mathrm{H}}^{t+1}_R(\pi(R)) = 0$. □

We remark that we expect that the exact value of the cohomological dimension of $R$-proj is $t$, but we have not yet succeeded in showing this.

Before leaving the subject of cohomological dimension, we wish to mention another approach to cohomology for noncommutative graded algebras which is provided by the work of Van Oystaeyen and Willaert on schematic algebras [38, 39, 40]. An algebra graded $A$ is called *schematic* if it has enough Ore sets to give an open cover of $A$-proj; we shall not concern ourselves here with the formal definition. For such algebras one can define a noncommutative version of Čech cohomology which gives the same cohomology groups as the cohomology theory we studied above.

It turns out that the theory of schematic algebras is of no help in computing the cohomology of $R$-proj. Indeed, if $A$ is a connected $\mathbb{N}$-graded noetherian schematic algebra then $\underline{\mathrm{Ext}}^n_A({}_Ak, A)$ is torsion as a right $A$-module for all $n \in \mathbb{N}$ [40, Proposition 3], hence finite dimensional over $k$. But we saw in Theorem 4.4.3 that $\dim_k \underline{\mathrm{Ext}}^2_R({}_Rk, R) = \infty$. Thus we have incidentally proven the following proposition.

**Proposition 4.5.13.** *Assume the critical density condition. Then $R = R(\varphi, c)$ is a connected $\mathbb{N}$-graded noetherian domain, generated in degree 1, which is not schematic.*

□

The previously known non-schematic algebras have not been generated in degree 1 [40, page 12].

## 4.6 GK and Krull dimension for $R(\varphi, c)$

The GK-dimension is the most important dimension function in noncommutative geometry, and it is certainly sufficient for our purposes above. However, for some applications the *Krull dimension* is a better way of measuring the size of modules. This dimension depends only on the lattice of submodules of a module, and so is even defined for objects in an abelian category. We prove below that Krull and GK-dimension correspond for modules over the ring $R(\varphi, c)$, assuming the critical density condition. This section is not used elsewhere in this thesis; we include it only because it may be useful for later reference.

Let us define the Krull dimension.

**Definition 4.6.1.** Let $A$ be any ring. We define the Krull dimension of the zero module to be $\mathrm{Kdim}(0) = -1$. For convenience let the ordinal numbers start with $-1$. Suppose that the class of $A$-modules with Krull dimension $\alpha$ has been defined for all ordinals $\alpha < \beta$. Then we set $\mathrm{Kdim}(M) = \beta$ for every module $M$ such that

(1) $\mathrm{Kdim}(M)$ has not already been defined to be an ordinal less than $\beta$;

(2) Given any descending chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ of submodules $M$, $\mathrm{Kdim}(M_j/M_{j-1}) < \beta$ (in particular, is already defined) for all $j \gg 0$.

If $N$ never gets assigned a dimension in this inductive process then we leave $\mathrm{Kdim}(N)$ undefined. We set $\mathrm{Kdim}(A) = \mathrm{Kdim}({}_A A)$ if this exists.

If $M$ is a noetherian module then $\mathrm{Kdim}(M)$ is defined. The Krull dimension is always an exact dimension function, and it agrees with the usual Krull dimension, defined using chains of prime ideals, for noetherian modules over a commutative ring. See [17, Chapter 13] for proofs of these facts.

Let $U = k[x_0, x_1, \ldots, x_t]$ be the polynomial ring for some $t \geq 2$, and let $S = S(\varphi)$ and $R = R(\varphi, c)$ for some $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$. It is standard that every $M \in U\text{-gr}$ has a Hilbert polynomial, that is some $f \in \mathbb{Q}[z]$ such that $f(n) = \dim_k M_n$ for $n \gg 0$. Since $S$ is a twist of the commutative polynomial ring $U$, and the equivalence of categories $S\text{-Gr} \sim U\text{-Gr}$ preserves Hilbert functions, it is clear that every $M \in S\text{-gr}$ also has a Hilbert polynomial. We may easily show the same for $R$-modules.

**Lemma 4.6.2.** *Let $M \in R\text{-gr}$. Then $M$ has a Hilbert polynomial.*

*Proof.* Since $M$ has a finite filtration with cyclic factors, we reduce quickly to the case where $M$ is cyclic. If $M = R$ then $M$ has a Hilbert polynomial by Lemma 3.2.4. If $M = R/I$ with $I \neq 0$, then we have the exact sequence (3.3.15). Now $S/SI \in S\text{-gr}$ and so $S/SI$ has a Hilbert polynomial. By 3.3.16, the left $R$-modules $(SI \cap R)/I$ and $S/(R + SI)$ each have a finite filtration where the are factors either finite dimensional over $k$ or $R$-point modules, and so each module also has a Hilbert polynomial. Thus $R/I$ also has a Hilbert polynomial by the exact sequence. □

The existence of Hilbert polynomials for all $M \in R\text{-gr}$ immediately implies the following.

**Lemma 4.6.3.** *If $M \in R\text{-gr}$, then $\operatorname{GK}(M) \geq \operatorname{Kdim}(M)$.* □

*Proof.* Call an $\mathbb{N}$-graded $k$-algebra $A$ *left finitely partitive* if given any $M \in A\text{-gr}$, there is some $n \geq 0$ such that every chain of graded modules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_m$$

with $\operatorname{GK}(M_i/M_{i+1}) = \operatorname{GK}(M)$ for all $0 \leq i \leq m-1$ satisfies $m \leq n$. Since every $M \in R\text{-gr}$ has a Hilbert polynomial by Lemma 4.6.2, $R$ is left finitely partitive (see the proof of [23, Corollary 8.4.9(ii)]). Further, because $R$ is left finitely partitive it follows that $\mathrm{GK}(M) \geq \mathrm{Kdim}(M)$ for all $M \in R$-gr [23, Proposition 8.3.18]. □

If $N \in U$-gr, then $\mathrm{GK}(N) = \mathrm{Kdim}(N)$, since both dimension functions are equal to the usual commutative Krull dimension. The equivalence of categories $S$-gr $\sim$ $U$-gr clearly preserves both GK-dimension and Krull dimension and so $\mathrm{GK}(N) = \mathrm{Kdim}(N)$ also holds for all $N \in S$-gr. Now we may prove the equality of the two dimensions for $R$-modules.

**Proposition 4.6.4.** *Let $R = R(\varphi, c)$. For all $M \in R$-gr, $\mathrm{GK}(M) = \mathrm{Kdim}(M)$.*

*Proof.* Both dimensions are exact for modules in $R$-gr, so since $M$ has a finite filtration with cyclic factors we may reduce to the case where $M$ is cyclic.

We consider several cases depending on the GK-dimension of $M$. Note first that $\mathrm{GK}(M) = 0$ if and only if $M$ has finite length, if and only if $\mathrm{Kdim}(M) = 0$. If $\mathrm{GK}(M) = 1$, then $0 < \mathrm{Kdim}(M) \leq 1$ by Lemma 4.6.3, and so $\mathrm{Kdim}(M) = 1$.

Next, suppose that $\mathrm{GK}(M) = d$ with $1 < d < (t+1)$. Since $\mathrm{GK}({}_R R) = t + 1$, we have $M = R/I$ for some $I \neq 0$. Consider the exact sequence (3.3.15); By Lemma 3.3.16, all terms of the sequence are noetherian $R$-modules, and it is clear that $(SI \cap R)/I$ and $S/(R + SI)$ both have GK-dimension at most 1 over $R$, and thus also Krull dimension at most 1. By the exactness of Krull and GK-dimension we conclude that $\mathrm{GK}_R(R/I) = \mathrm{GK}_R(S/SI)$ and $\mathrm{Kdim}_R(R/I) = \mathrm{Kdim}_R(S/SI)$. Now since GK-dimension depends only on the Hilbert polynomial for noetherian graded modules, $\mathrm{GK}_R(S/SI) = \mathrm{GK}_S(S/SI) = \mathrm{Kdim}_S(S/SI)$, and since Krull dimension depends only on the lattice of submodules, $\mathrm{Kdim}_S(S/SI) \leq \mathrm{Kdim}_R(S/SI)$. We conclude that $\mathrm{GK}_R(R/I) \leq \mathrm{Kdim}_R(R/I)$; the opposite inequality follows from Lemma 4.6.3, so $\mathrm{GK}_R(M) = \mathrm{Kdim}_R(M)$.

Finally, suppose that $\mathrm{GK}(M) = t + 1$, so that $M = R$. If $J$ is any nonzero principal left ideal of $R$, then by a Hilbert function calculation clearly $\mathrm{GK}(R/J) = t$ and so $t = \mathrm{GK}(R/J) = \mathrm{Kdim}(R/J)$ by the previous case. Since $R$ is a domain, $t = \mathrm{Kdim}(R/J) < \mathrm{Kdim}(R)$ [17, Proposition 13.7]. But also $\mathrm{Kdim}(R) \leq GK(R) = t+1$ by Lemma 4.6.3 and so $\mathrm{Kdim}(R) = t + 1 = \mathrm{GK}(R)$ in this last case. $\square$

# CHAPTER V

## Examples

In the previous chapters, we studied the rings $R = R(\varphi, c)$, assuming the case that $R$ is noetherian, or equivalently (by Theorem 3.3.12) that $\mathcal{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$ is a critically dense set of points (Hypothesis 3.3.14). In this chapter, we study this critical density condition in detail. First, we show that for generic choices of $\varphi$ and $c$ the set $\mathcal{C}$ is indeed critically dense, justifying our assumption of the noetherian case above. On the other hand, we show that for automorphisms $\varphi$ represented by matrices which are not nearly diagonalizable, $\mathcal{C}$ is not even Zariski dense.

If one is willing to restrict one's attention to base fields $k$ with $\operatorname{char} k = 0$, the analysis of the critical density of the set $\mathcal{C}$ is much simpler. In §5.2 we show using a theorem of Cutkosky and Srinivas that in this case the critical density of $\mathcal{C}$ is equivalent to the density of the set $\mathcal{C}$, which is easy to analyze in particular examples. We give an explicit example to show, however, that if $k$ has positive characteristic then $\mathcal{C}$ may very well be dense but not critically dense.

For completeness, we also briefly discuss the rings $R(\varphi, c)$ in the case where $c$ is of finite order under $\varphi$, so that $\mathcal{C}$ is a finite point set. We have excluded this possibility ever since §3.2, and Theorem 3.3.12 does not correctly characterize the noetherian property for $R(\varphi, c)$ in this case. In fact, it is easy to show in case $c$ has finite order under $\varphi$ that $R(\varphi, c)$ is strongly noetherian and satisfies $\chi$, so that these rings have none of the interesting behavior of the infinite order case.

Finally, in §5.3 we introduce rings generated by Eulerian derivatives, which was the context in which rings of the form $R(\varphi, c)$ first appeared in the literature [19]. We translate our earlier results into this language, and show that they solve several open questions in [19].

## 5.1 The critical density property

In this section, we will show that $\mathcal{C} = \{\varphi^i(c)\}_{i\in\mathbb{Z}}$ is critically dense for generic choices of $\varphi$ and $c$. Throughout this section we write $c_i = \varphi^{-i}(c)$. Also, we will identify automorphisms of $\mathbb{P}^t$ with elements of $\mathrm{PGL}_{t+1}(k) = \mathrm{GL}_{t+1}(k)/k^\times$ [18, page 151], where we let matrices in $\mathrm{GL}_{t+1}(k)$ act on the left on the homogeneous coordinates $(a_0 : a_1 : \cdots : a_t)$ of $\mathbb{P}^t$, considered as column vectors.

Let us define precisely our intended meaning of the word "generic".

**Definition 5.1.1.** A subset $U$ of a variety $X$ is *generic* if its complement is contained in a countable union of proper closed subvarieties of $X$. A property associated to points of $X$ holds *generically* if it holds for all points of some generic subset $U$ of $X$.

If the base field $k$ is uncountable, a generic subset is intuitively very large. For example, if $k = \mathbb{C}$ then a property which holds generically holds "almost everywhere" in the sense of Lebesgue measure. For any results below which involve genericity we will assume that $k$ is uncountable.

We begin our analysis of the critical density property by showing that for fixed $\varphi$, the choice of $c$ is not too important. Let us fix some notation for the next lemma. Let $\varphi \in PGL_{t+1}(k)$ be represented by a matrix $L \in \mathrm{GL}_{t+1}(k)$ (L is unique up to scalar multiple). Then let $V = \sum_{i\in\mathbb{Z}} kL^i \subseteq M_{t+1}(k) = \mathbb{A}^{(t+1)^2}$. Note that since $L$

satisfies its characteristic polynomial, the dimension of $V$ as a $k$-vector space is at most $t+1$. Then $\mathbb{P}V$ is a projective space of dimension at most $t$ which contains $\{\varphi^i\}_{i\in\mathbb{Z}}$.

**Lemma 5.1.2.** *Fix $\varphi \in \operatorname{Aut}\mathbb{P}^t$. Let $c, d \in \mathbb{P}^t$, and set $c_i = \varphi^{-i}(c)$, $d_i = \varphi^{-i}(d)$, $\mathcal{C} = \{c_i\}_{i\in\mathbb{Z}}$ and $\mathcal{D} = \{d_i\}_{i\in\mathbb{Z}}$. Let $\mathbb{P}V$ be the projective space constructed above.*
*(1) If $\mathcal{C}$ is contained in a hyperplane of $\mathbb{P}^t$, then no infinite subset of $\mathcal{C}$ is critically dense in $\mathbb{P}^t$.*
*(2) If $\mathcal{C}$ is not contained in a hyperplane of $\mathbb{P}^t$, then there is an isomorphism $\psi : \mathbb{P}V \cong \mathbb{P}^t$ such that $\psi(\varphi^i) = \varphi^i(c)$ for all $i \in \mathbb{Z}$. In particular, if $\mathcal{D}$ is also not contained in a hyperplane of $\mathbb{P}^t$ then there is an automorphism $\theta$ of $\mathbb{P}^t$ with $\theta(c_i) = d_i$ for all $i \in \mathbb{Z}$.*

*Proof.* (1) In this case, any infinite subset of $\mathcal{C}$ fails to be critically dense by definition.
(2) Consider $\mathbb{P}^t$ as the set of lines in $\mathbb{A}^{t+1}$, and let $\widetilde{c} \in \mathbb{A}^{t+1}$ be a non-zero point lying on $c \in \mathbb{P}^t$. There is a natural linear evaluation map $\widetilde{\psi} : V \to \mathbb{A}^{t+1}$ defined by $N \mapsto N(\widetilde{c})$. Since $\mathbb{P}\operatorname{Im}(\widetilde{\psi})$ is a linear space containing the set $\mathcal{C}$, by hypothesis $\mathbb{P}\operatorname{Im}(\widetilde{\psi}) = \mathbb{P}^t$ and so $\widetilde{\psi}$ must be surjective. Then since $\dim V \leq t+1$, $\widetilde{\psi}$ is an isomorphism. This descends to an isomorphism $\psi : \mathbb{P}V \to \mathbb{P}^t$ which sends $\varphi^i$ to $\varphi^i(c)$ for all $i \in \mathbb{Z}$. The last statement is now obvious. $\square$

The following technical lemma handles the combinatorics involved in the next theorem.

**Lemma 5.1.3.** *Fix $d \geq 1$, and set $N = \binom{t+d}{d}$. Let $U = k[x_0, x_1, \ldots, x_t]$ be the polynomial ring, and give monomials in $U$ the lexicographic order with respect to some fixed ordering of the variables. Let $f_1, f_2, \ldots, f_N$ be the monomials of degree $d$ in $U = k[x_0, x_1, \ldots, x_t]$, enumerated so that $f_1 < f_2 < \ldots f_N$ in the lex order.*

*Fix some sequence of distinct nonnegative integers* $a_1 < a_2 < \cdots < a_N$. *Then the polynomial* $\det(f_i^{a_j}) \in U$ *is nonzero.*

*Proof.* (1) Set $F = \det(f_i^{a_j}) \in U$. Let $S_N$ be the symmetric group on $N$ elements, with identity element 1; then $F$ is a sum of terms of the form $h_\sigma = \pm \prod_{i=1}^N f_i^{a_{\sigma(i)}}$ for $\sigma \in S_N$. It is easy to see that the monomial $f_1^{a_1} f_2^{a_2} \dots f_N^{a_N}$ is the unique largest in the lex order occurring among the $h_\sigma$, and that it occurs only in $h_1$ and thus may not be cancelled by any other term. $\square$

Now we prove that $\mathcal{C}$ is critically dense for $\varphi$ a suitably general diagonal matrix, and $c$ chosen from an open set of $\mathbb{P}^t$. The next theorem proves Proposition 3.3.13. We recall the following standard definition:

**Definition 5.1.4.** Let $\mathcal{S}$ be a set of points in $\mathbb{P}^t$. We say that $\mathcal{S}$ is in *general position* if given any $0 \neq f \in U_d$, the degree $d$ hypersurface of $\mathbb{P}^t$ defined by the vanishing of $f$ contains at most $\binom{t+d}{d}$ points of the set $\mathcal{S}$.

**Theorem 5.1.5.** *Let* $\varphi = \operatorname{diag}(1, p_1, p_2, \dots, p_t)$ *with* $\{p_1, p_2, \dots p_t\}$ *algebraically independent over the prime subfield of* $k$. *Let* $c = (b_0 : b_1 : \cdots : b_t) \in \mathbb{P}^t$ *with* $b_i \neq 0$ *for all* $0 \leq i \leq t$. *Then* $\mathcal{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$ *is in general position and* $R(\varphi, c)$ *is noetherian.*

*Proof.* We set $c_n = \varphi^{-n}(c)$, so we have the explicit formula $c_{-n} = (b_0 : b_1 p_1^n : b_2 p_2^n : \cdots : b_t p_t^n)$. The matrix $(b_i p_i^j)_{0 \leq i,j \leq t}$ has nonzero determinant, since the $b_i \neq 0$ and $(p_i^j)$ is a Vandermonde matrix, with the $p_i$ distinct. Thus the points $c_0, c_{-1}, \dots c_{-t}$ have linear span equal to $\mathbb{P}^t$. Now by Lemma 5.1.2(2), we may replace $c$ by any other point such that the $c_i$ do not all lie on a hyperplane; we choose $c = (1 : 1 : \dots 1)$ for convenience, so that $c_{-n} = (1 : p_1^n : p_2^n : \cdots : p_t^n)$ for $n \in \mathbb{Z}$.

Now we will prove that the set of points $\mathcal{C}$ is in general position. Suppose that this fails, so there is some $d \geq 1$ and a sequence of $N = \binom{t+d}{d}$ integers $a_1 < a_2 < \cdots < a_N$

such that the points $c_{a_1}, c_{a_2}, \ldots, c_{a_N}$ lie on a degree $d$ hypersurface in $\mathbb{P}^t$. We may assume that the $a_i$ are nonnegative, since if the $\{c_{a_i}\}$ lie on a degree $d$ hypersurface then the same is true of the points $\{\varphi^{-m}(c_{a_i})\} = \{c_{a_i+m}\}$ for any $m \in \mathbb{Z}$. Let $f_1, f_2, \ldots f_N$ be the distinct degree $d$ monomials in the variables $x_i$ of $U$. It follows that $\det(f_i(c_{a_j})) = 0$.

Given the explicit formula for $c_n$, we have

$$\det(f_i(c_{a_j})) = [\det(f_i^{a_j})](1 : p_1^{-1} : p_2^{-1} : \cdots : p_t^{-1}) = 0.$$

Now by Lemma 5.1.3 the polynomial $\det(f_i^{a_j})$ is a nonzero homogeneous element of $U$, which clearly has coefficients in the prime subfield of $k$. Thus $p_1^{-1}, p_2^{-1}, \ldots p_t^{-1}$ satisfy some nonzero non-homogeneous relation with coefficients in the prime subfield of $k$, contradicting the hypothesis on the $\{p_i\}$.

Thus the set $\mathcal{C}$ is in general position, and it follows immediately that $\mathcal{C}$ is critically dense in $\mathbb{P}^t$. Certainly then $c$ has infinite order under $\varphi$, so that Hypothesis 3.2.1 holds. Then $R(\varphi, c)$ is noetherian by Theorem 3.3.12. $\square$

Next, let us show that for generic choices of $\varphi$ and $c$ (in the sense of Definition 5.1.1), the ring $R(\varphi, c)$ is noetherian. Because of Lemma 3.2.2(2), for every fixed $c$ we get the same class of rings $\{R(\varphi, c) \mid \varphi \in \operatorname{Aut} \mathbb{P}^t\}$. Thus we might as well fix some arbitrary $c$ and vary $\varphi$ only.

**Theorem 5.1.6.** *Assume that the base field $k$ is uncountable. Fix $c \in \mathbb{P}^t$. There is a generic subset $Y$ of $X = \operatorname{Aut} \mathbb{P}^t$ such that $R(\varphi, c)$ is noetherian for all $\varphi \in Y$.*

*Proof.* By Lemma 3.2.2(2) there is no harm in assuming that $c = (1 : 1 : \cdots : 1)$. Choose some homogeneous coordinates $(z_{ij})_{0 \le i,j \le t}$ for $X \subseteq \mathbb{P}(M_{t+1}(k))$. Just as in the proof of Theorem 5.1.5, we see that $\mathcal{C} = \{c_i\}_{i \in \mathbb{Z}}$ fails to be in general position if and only if there exists some $d \ge 1$ and some choice of $N = \binom{t+d}{d}$ nonnegative integers $a_1 < a_2 < \cdots < a_N$ such that $\det f_i(c_{a_j}) = 0$, where the $f_i$ are the degree $d$ monomials in $U$.

Each condition $\det f_i(c_{a_j}) = 0$ is a closed condition in the coordinates of $X$; moreover it does not hold identically, otherwise for no choice of $\varphi$ would $\mathcal{C}$ be in general position, in contradiction to Theorem 5.1.5. There are countably many such conditions, and so the complement $Y$ of the union of all of these closed subsets is generic by definition. Thus for $\varphi \in Y$ we have that $\mathcal{C}$ is in general position and so $R(\varphi, c)$ is noetherian, by Theorem 3.3.12. $\square$

Let us also note some examples where the critical density property fails. If the matrix $L$ representing the automorphism $\varphi$ is not almost diagonalizable, then the set $\mathcal{C} = \{\varphi^i(c)\}_{i\in\mathbb{Z}}$ will not even be dense in $\mathbb{P}^t$.

**Example 5.1.7.** (1) Suppose that $t = 2$ and

$$\varphi = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

with $c \in \mathbb{P}^2$ any point. Then $\mathcal{C}$ is not dense in $\mathbb{P}^2$.

(2) Similarly, suppose that $t = 3$ and

$$\varphi = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $c \in \mathbb{P}^3$ any point. Again, $\mathcal{C}$ is not dense in $\mathbb{P}^3$.

(3) Represent $\varphi$ as a matrix $L \in M_{t+1}(k)$. If the Jordan canonical form of $L$ has a Jordan block of size $\geq 3$ or more than one Jordan block of size 2, then given any $c \in \mathbb{P}^t$, the set $\mathcal{C}$ is not dense in $\mathbb{P}^t$.

*Proof.* (1) If $\operatorname{char} k = p > 0$, then $\varphi \in \mathrm{PGL}_3(\mathbb{F}_p)$ and so $\varphi$ has finite order. Then $\mathcal{C}$ is finite and so certainly not dense (k is infinite).

Assume then that $\operatorname{char} k = 0$. Suppose that $c = (0 : 0 : 1)$, so that we have the formula $c_{-n} = (n(n-1)/2 : n : 1)$ for $n \in \mathbb{Z}$. The polynomial $f = x_0x_2 + \frac{1}{2}x_2x_1 - \frac{1}{2}x_1^2$ vanishes at $(1 : n : n(n-1)/2)$ for every $n \in \mathbb{Z}$, and so $\{c_i\}_{i\in\mathbb{Z}}$ is not dense. It is easy to see that the linear span of $c_0, c_{-1}, c_{-2}$ is all of $\mathbb{P}^t$, and so by Lemma 5.1.2, $\mathcal{C}$ will not be dense for any choice of $c$.

(2) Suppose first that $\lambda = 1$ and $c = (a_0 : a_1 : a_2 : a_3)$. If one of $a_1, a_3$ is nonzero, then all of the $c_i$ lie on the hyperplane $a_3x_1 - a_1x_3 = 0$; if instead $a_1 = a_3 = 0$ then all of the $c_i$ lie on the hyperplane $a_2x_0 - a_0x_2 = 0$.

Assume then that $\lambda \neq 1$. If $\operatorname{char} k = p > 0$, then it is easy to see that $\mathcal{C}$ is contained in a finite union of hyperplanes, so it is not dense. Now let $\operatorname{char} k = 0$. The rest of the proof is similar to part (1); we take $c = (0 : 1 : 0 : 1)$, so that $c_{-n} = (n\lambda^{n-1} : \lambda^n : n : 1)$ for all $n \in \mathbb{Z}$, and one may check that the linear span of the $\{c_i\}_{\in\mathbb{Z}}$ is all of $\mathbb{P}^3$. Note that all of the $c_i$ lie on the hypersurface $x_1x_2 - \lambda x_0x_3 = 0$, so $\mathcal{C}$ is not dense. The result for an arbitrary point $c$ follows from Lemma 5.1.2.

(3) Suppose that $X$ is a linear subvariety of $\mathbb{P}^t$ with $\varphi(X) = X$, so that $\varphi$ restricts to an automorphism of $X$, and $\theta$ is any linear projection map $\mathbb{P}^t \to X$. Then given a subset $\mathcal{D}$ of points of $\mathbb{P}^t$, if $\mathcal{D}$ is dense in $\mathbb{P}^t$ then $\theta(\mathcal{D})$ is dense in $X$.

Now if the canonical form of $L$ has a Jordan block of size at least 3, then multiply $L$ by a scalar (which does not change the automorphism of $\mathbb{P}^t$) so that this block has eigenvalue 1; after changing coordinates, one sees that there is some linear subvariety $\mathbb{P}^2 \cong X$ of $\mathbb{P}^t$ with $\varphi(X) = X$ such that $\varphi$ acts on $X$ as the matrix of part (1). Then if $\theta$ is a projection map $\mathbb{P}^t \to X$, the set $\theta(\mathcal{C})$ will not be dense in $X$, and so $\mathcal{C}$ is not dense in $\mathbb{P}^t$, regardless of the choice of $c$. Similarly, if $L$ has at least two Jordan blocks of size 2 then $\varphi$ acts on some $\mathbb{P}^3 \cong X \subseteq \mathbb{P}^t$ as a matrix of the form in part (2), and so again $\mathcal{C}$ can not be dense. $\square$

### 5.2 Improvements in characteristic 0 and the finite order case

The result of the preceding section that $R(\varphi, c)$ is noetherian for generic choices of $\varphi$ and $c$ holds for a field of arbitrary characteristic. In the case where $\operatorname{char} k = 0$, one can get a better result with less work, using the following theorem of Cutkosky and Srinivas.

**Theorem 5.2.1.** *[15, Theorem 7] Let $G$ be a connected commutative algebraic group defined over an algebraically closed field $k$ of characteristic $0$. Suppose that $g \in G$ is such that the cyclic subgroup $H = \langle g \rangle$ is dense in $G$. Then any infinite subset of $H$ is dense in $G$.*

In our situation, we can derive the following consequence.

**Lemma 5.2.2.** *Let $\operatorname{char} k = 0$. Let $(\varphi, c) \in (\operatorname{Aut} \mathbb{P}^t) \times \mathbb{P}^t$, and set $\mathcal{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$. Then $\mathcal{C}$ is critically dense in $\mathbb{P}^t$ if and only if $\mathcal{C}$ is Zariski dense in $\mathbb{P}^t$.*

*Proof.* If $\mathcal{C}$ is critically dense in $\mathbb{P}^t$, then $\mathcal{C}$ is of course dense in $\mathbb{P}^t$ by definition.

Now assume that $\mathcal{C}$ is dense. Choosing a matrix $L \in \mathrm{GL}_{t+1}(k)$ to represent $\varphi$, set $V = \sum_{i \in \mathbb{Z}} kL^i \subseteq M_{t+1}(k)$. Since $\mathcal{C}$ is certainly not contained in a hyperplane of $\mathbb{P}^t$, by Lemma 5.1.2(2) there is an isomorphism of varieties $\psi : \mathbb{P}V \cong \mathbb{P}^t$ where $\mathbb{P}V$ is a projective subvariety of $\mathbb{P}M_{t+1}(k)$ such that $\{\varphi^i\}_{i \in \mathbb{Z}} \subseteq \mathbb{P}V$ is a dense set, and $\psi(\varphi^i) = \varphi^i(c)$ for all $i \in \mathbb{Z}$. Now let $G = \mathbb{P}V \cap \mathrm{PGL}_{t+1}(k)$. Then $G$ is an algebraic group, since it is the closure of the subgroup $H = \{\varphi^i\}_{i \in \mathbb{Z}}$ of $\mathrm{PGL}_{t+1}(k)$ [12, Proposition 1.3]. Since any two elements of $V$ commute, $G$ is commutative. Note also that $G$ is an open subset of a projective space, so $G$ is irreducible and in particular connected. Finally, we always assume that the field $k$ is algebraically closed, so the hypotheses of Theorem 5.2.1 are all satisfied.

Now via the automorphism $\psi$, the set $H = \{\varphi^i\}_{i \in \mathbb{Z}}$ is dense in $\mathbb{P}V$, so that $H$ is certainly a dense subgroup of $G$. But then $H$ is critically dense in $G$ by Theorem 5.2.1, and thus $H$ is critically dense in $\mathbb{P}V$. Finally, applying $\psi$ again we get that $\mathcal{C}$ is critically dense in $\mathbb{P}^t$. □

Thus in case $\operatorname{char} k = 0$, the question of the noetherian property for $R(\varphi, c)$ reduces to the question of the density of $\mathcal{C} = \{\varphi^i(c)\}_{i \in \mathbb{Z}}$, which is easy to analyze for particular choices of $\varphi$ and $c$. in particular, $\mathcal{C}$ will be dense if and only if $c$ is not contained in a proper closed set $X \subsetneq \mathbb{P}^t$ with $\varphi(X) = X$. Let us note some specific examples. Note that part (1) of the following example is a significant improvement over Theorem 5.1.5 if the field has zero characteristic.

**Example 5.2.3.** Let $\operatorname{char} k = 0$. (1) Suppose that $\varphi = \operatorname{diag}(1, p_1, \dots, p_t)$, and that the multiplicative subgroup of $k^\times$ generated by $p_1, p_2, \dots p_t$ is $\cong \mathbb{Z}^t$. Let $c$ be the point $(a_0 : a_1 : \cdots : a_t)$. Then $R(\varphi, c)$ is noetherian if and only if $\prod_{i=0}^t a_i \neq 0$.

(2) Let

$$\varphi = \begin{bmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & p_2 & & \\ & & & \cdots & \\ & & & & p_t \end{bmatrix}$$

such that the multiplicative subgroup of $k^\times$ generated by the $p_2, \dots p_t$ is $\cong \mathbb{Z}^{t-1}$. Let $c = (a_0 : a_1 : \cdots : a_t) \in \mathbb{P}^t$. Then $R(\varphi, c)$ is noetherian if and only if $\prod_{i=1}^t a_i \neq 0$.

*Proof.* (1) Let $\phi$ be the automorphism of $U$ corresponding to $\varphi$; explicitly (up to scalar multiple), $\phi(x_i) = p_i x_i$, if we set $p_0 = 1$. Suppose that $I$ is a graded ideal of $U$ with $\phi(I) = I$. Then if we choose $m \gg 0$ such that $I_m \neq 0$, then there is some $0 \neq f \in I_m$ with $\phi(f) \in kf$, since the action of $\phi$ on the finite dimensional vector space $I_m$ has an eigenvector. If $f = \sum b_I x_I$ (where $I$ is a multi-index), then $\phi(f) = \sum b_I p_I x_I$. The hypothesis on the $p_i$ forces $p_I$ to be distinct for distinct multi-indices $I$ of degree $m$, so $f$ must be a scalar multiple of a monomial in the $x_i$. Thus any closed set $X \subsetneq \mathbb{P}^t$ with $\varphi(X) = X$ is contained in the union of hyperplanes $\cup_{i=0}^{t}\{x_i = 0\}$. It follows that if all $a_i \neq 0$ then $\mathcal{C}$ is dense. Conversely, if some $a_i = 0$ then $\mathcal{C}$ is contained in the hyperplane $x_i = 0$. Now apply Lemma 5.2.2 and Theorem 3.3.12.

(2) The automorphism $\phi$ of $U$ corresponding to $\varphi$ is given by $\phi(x_0) = x_0 + x_1$, $\phi(x_1) = x_1$, and $\phi(x_i) = p_i x_i$ for $2 \leq i \leq t$. If $I$ is a graded ideal of $U$ with $\phi(I) = I$, then as above there is some $0 \neq f \in U_m$ with $\phi(f) \in kf$. We leave it to the reader to show that this forces $f$ to be scalar multiple of a monomial in $x_1, x_2, \ldots x_t$ only; then the result easily follows as in part (1). □

Lemma 5.2.2 and Example 5.2.3(1) fail in positive characteristic. The next example, which we thank Mel Hochster for suggesting, shows this explicitly.

**Example 5.2.4.** Let $k$ have characteristic $p > 0$ and let $y \in k$ be transcendental over the prime subfield $\mathbb{F}_p$. Suppose that $t = 2$, and let $\varphi = \mathrm{diag}(1, y, y+1)$ and $c = (1 : 1 : 1)$. Then there is a line $X \subseteq \mathbb{P}^2$ such that $c_n \in X$ if and only if $n = \pm p^j$ for some $j \in \mathbb{N}$. The set $\mathcal{C} = \{c_i\}_{i \in \mathbb{Z}}$ is not critically dense, but it is a Zariski dense set, and the multiplicative subgroup of $k^\times$ generated by $y$ and $y+1$ is isomorphic to $\mathbb{Z}^2$. The ring $R(\varphi, c)$ is neither left nor right noetherian.

*Proof.* We have $c_{-n} = \varphi^n(c) = (1, y^n, (y+1)^n)$, so $c$ has infinite order under $\varphi$. If $n = p^j$ for some $j \geq 0$, then $(y+1)^n = y^n + 1$. Therefore $c_{-n}$ is on the line

$X = \{x_0 + x_1 - x_2 = 0\}$ for all $n = p^j$. On the other hand, suppose that $n \geq 0$ is not a power of $p$. Then some binomial coefficient $\binom{n}{i}$ with $0 < i < n$ is not divisible by $p$, and the binomial expansion of $(y+1)^n$ contains the nonzero term $\binom{n}{i}y^i$. Since $y$ is transcendental over $\mathbb{F}_p$, this implies $(y+1)^n \neq y^n + 1$ and so $c_{-n}$ is not on the curve $X$. Thus for $n \geq 0$, $c_{-n}$ is on $X$ if and only if $n$ is a power of $p$. A similar argument shows the same result for negative $n$.

It is now immediate that the set $\mathcal{C}$ is not a critically dense subset of $\mathbb{P}^2$, and it follows from Theorem 3.3.12 that $R(\varphi, c)$ is neither left nor right noetherian. It is easy to see since $y$ is transcendental over $\mathbb{F}_p$ that the multiplicative subgroup of $k^\times$ generated by $y$ and $y+1$ is isomorphic to $\mathbb{Z}^2$.

Now consider the Zariski closure $\overline{\mathcal{C}}$ of $\mathcal{C}$. Since the line $X$ contains infinitely many points of $\mathcal{C}$, $X \subseteq \overline{\mathcal{C}}$. For all $n \in \mathbb{Z}$, $\varphi^n(X)$ also contains infinitely many points of $\mathcal{C}$, and so $\bigcup_{n\in\mathbb{Z}} \varphi^n(X) \subseteq \overline{\mathcal{C}}$. Finally, one checks that the curves $\varphi^n(X)$ are distinct irreducible curves for all $n \in \mathbb{Z}$. It follows that $\overline{\mathcal{C}} = \mathbb{P}^2$, and $\mathcal{C}$ is a Zariski dense set. $\square$

Throughout this thesis, we have been assuming Hypothesis 3.2.1, that is that the points $c_i = \phi^{-i}(c)$ are all distinct. We would now like to switch gears and touch briefly on the case where $c$ has finite order under $\varphi$. It turns out that in this case it is very easy to prove that $R(\varphi, c)$ has very nice properties. In particular, these rings have none of the unusual features with respect to the noetherian property or the $\chi$ condition that one finds in the infinite order case, which is why we excluded them.

**Lemma 5.2.5.** *Assume that $c$ has finite order under $\varphi$, and let $R = R(\varphi, c)$. Then*

*(1) $R$ is strongly noetherian.*

*(2) $R$ satisfies $\chi$.*

*Proof.* (1) Let $n = \min\{i > 0 \mid \varphi^i(c) = c\}$, and let $R' = k \oplus R_n \oplus R_{2n} \oplus \dots$ be the Veronese ring of degree $n$ of $R$. Setting $V = (\mathfrak{m}_c)_1 \subseteq U_1$, then by the definition of $R(\varphi, c)$,

$$R_{nb} = V^{nb} = \phi^{nb-1}(\mathfrak{m}_c)_1 \circ \dots \circ \phi^1(\mathfrak{m}_c)_1 \circ (\mathfrak{m}_c)_1 = \Big[\prod_{i=0}^{n-1} (\mathfrak{m}_{c_i})^b\Big]_{nb}$$

for all $b \geq 1$. Let $I$ be the ideal $(\prod_{i=0}^{n-1} \mathfrak{m}_{c_i})$ of $U$. Then $I$ is generated in degree $n$, and if we set

$$T = k \oplus I_n \oplus (I^2)_{2n} \oplus \dots,$$

then $T$ is a graded subalgebra of $U$ which is finitely generated in degree $n$. Now since $I = \phi(I)$, the automorphism $\phi$ of $U$ restricts to a graded automorphism of $T$, and it is immediate that the ring $R' \subseteq S$ is the left Zhang twist of $T \subseteq U$ by the twisting system $\{\phi^i\}_{i \in \mathbb{N}}$ (see §2.3). If $B$ is any noetherian commutative $k$-algebra, then $\phi$ extends to an automorphism of $U \otimes B$ which fixes $B$, and $R' \otimes B$ is again a Zhang twist of $T \otimes B$. Since it is a commutative finitely generated $B$-algebra, $T \otimes B$ is noetherian, and since the noetherian property passes to Zhang twists [43, Theorem 5.1], $R' \otimes B$ is noetherian. Finally, since $R$ is finitely generated as a left and right $R'$-module, $R \otimes B$ is noetherian. In conclusion, $R$ is strongly noetherian.

(2) Keep the notation from part (1). The commutative finitely generated connected graded $k$-algebra $T$ automatically satisfies $\chi$ [8, Proposition 3.11]. Let $\theta : T\text{-Gr} \sim R'\text{-Gr}$ be the equivalence of categories which follows from the Zhang twist construction. By Lemma 2.3.3(1), it follows that $\theta({}_T k[n]) \cong {}_{R'} k[n]$ for all $n \in \mathbb{Z}$. Then by Lemma 2.3.3(2) we have $\underline{\mathrm{Ext}}^i_{R'}(k, \theta(N)) \cong \underline{\mathrm{Ext}}^i_T(k, N)$ as $k$-spaces, for all $N \in T\text{-gr}$ and all $i$. Since $T$ satisfies $\chi$, it follows that $R'$ also satisfies $\chi$. Finally, the $\chi$ property passes upwards in finite ring extensions [8, Theorem 8.3], so $R$ satisfies $\chi$ on the left. Then applying Lemma 3.2.2(1), $R$ has $\chi$ on the right also. □

## 5.3 Algebras generated by Eulerian derivatives

The original motivation for our study of the algebras $R(\varphi, c)$ came from the results of Jordan [19] on algebras generated by two Eulerian derivatives. In this final section we show that Jordan's examples are special cases of the algebras $R(\varphi, c)$, and so we may use our previous results to answer the main open question of [19], namely whether algebras generated by two Eulerian derivatives are ever noetherian. In fact we will prove that an algebra generated by a generic finite set of Eulerian derivatives is noetherian.

Fix a Laurent polynomial algebra $k[y^{\pm 1}] = k[y, y^{-1}]$ over the base field $k$.

**Definition 5.3.1.** For $p \in k \setminus \{0, 1\}$, we define the operator $D_p \in \operatorname{End}_k k[y^{\pm 1}]$ by the formula $f(y) \mapsto \dfrac{f(py) - f(y)}{py - y}$. For $p = 1$, we define $D_1 \in \operatorname{End}_k k[y^{\pm 1}]$ by the formula $f \mapsto df/dy$. For any $p \neq 0$, we call $D_p$ an *Eulerian Derivative.*

It is also useful to let $y^{-1}$ be notation for the operator $y^{-1} : y^i \mapsto y^{i-1}$ for $i \in \mathbb{Z}$.

We now consider algebras generated by a finite set of Eulerian derivatives. There are naturally two cases, depending on whether $D_1$ is one of the generators.

**Theorem 5.3.2.** *Suppose that $\{p_1, \dots p_t\} \in k \setminus \{0, 1\}$ are distinct, and assume that the multiplicative subgroup of $k^\times$ these scalars generate has rank $t$. Let $R = k\langle D_{p_1}, D_{p_2}, \dots, D_{p_t}\rangle$. Then $R \cong R(\varphi, c)$ for*

$$\varphi = \operatorname{diag}(1, p_1^{-1}, p_2^{-1}, \dots, p_t^{-1}) \text{ and } c = (1 : 1 : \cdots : 1).$$

*$R$ is noetherian if either* $\operatorname{char} k = 0$ *or if the $\{p_i\}$ are algebraically independent over the prime subfield of $k$.*

*Proof.* Set $p_0 = 1$ and let $w_i = y^{-1} + (p_i - 1)D_{p_i}$ for $0 \leq i \leq t$. The automorphism $\phi$ of $U$ corresponding to $\varphi$ is given (up to scalar multiple) by $\phi : x_i \mapsto p_i^{-1} x_i$ for

$0 \le i \le t$. An easy calculation shows that $S(\varphi)$ has relations $\{x_j x_i - p_j^{-1} p_i x_i x_j\}$ for $0 \le i < j \le t$; clearly these relations generate the ideal of relations for $S(\varphi)$, since $S(\varphi)$ has the Hilbert function of a polynomial ring in $t+1$ variables.

As in [19, Section 2], it is straightforward to prove the identities $w_j w_i - p_j^{-1} p_i w_i w_j$ for all $0 \le i, j \le t$, so there is a surjection of algebras given by

$$\begin{aligned} \psi : S(\varphi) \quad &\longrightarrow \quad k\langle y^{-1}, D_{p_1}, \ldots, D_{p_t}\rangle \subseteq \operatorname{End} k[y^{\pm 1}] \\ x_i \quad &\mapsto \quad w_i, \qquad 0 \le i \le t. \end{aligned}$$

Now the hypothesis that the $\{p_i\}$ generate a rank $t$ subgroup of $k^\times$ ensures that $\psi$ is injective: this is proved for the case $t = 2$ in [19, Proposition 1]; the proof in general is analogous. Thus $\psi$ is an isomorphism of algebras. Then one checks that the image under $\psi$ of the subalgebra $R(\varphi, c)$ of $S(\varphi)$ is $k\langle D_{p_1}, D_{p_2}, \ldots, D_{p_t}\rangle = R$.

The noetherian property for $R$ follows from Example 5.2.3(1) in case $\operatorname{char} k = 0$, or from Theorem 5.1.5 if the the $\{p_i\}$ are algebraically independent over the prime subfield of $k$. $\square$

The case where $D_1$ is one of the generators is very similar, and we only sketch the proof.

**Theorem 5.3.3.** *Assume that* $\operatorname{char} k = 0$. *Let* $\{p_2, p_3, \ldots p_t\} \in k \setminus \{0, 1\}$ *be distinct, and assume that the multiplicative subgroup of* $k^\times$ *that the* $\{p_i\}$ *generate is of rank* $t-1$. *Let* $R = k\langle D_1, D_{p_2}, D_{p_3}, \ldots, D_{p_t}\rangle$. *Then* $R \cong R(\varphi, c)$ *for*

$$\varphi = \begin{bmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & {p_2}^{-1} & & \\ & & & \cdots & \\ & & & & p_t^{-1} \end{bmatrix} \text{ and } c = (0 : 1 : 1 : \cdots : 1).$$

*The ring $R$ is noetherian.*

*Proof.* Let $p_1 = 1$, and let $w_i = y^{-1} + (p_i - 1)D_{p_i}$ for $1 \leq i \leq t$. As in the preceding proposition, one calculates the relations for the algebra $S(\varphi)$, and using these and the identities in [19, section 2], one gets an algebra surjection

$$\begin{aligned} \psi : S(\varphi) &\longrightarrow k\langle y^{-1}, D_1, D_{p_2}, \ldots, D_{p_t}\rangle \\ x_0 &\mapsto -D_1 \\ x_i &\mapsto w_i, \quad 1 \leq i \leq t. \end{aligned}$$

The hypothesis on the $\{p_i\}$ implies that $\psi$ is an isomorphism, by an analogous proof to that of [19, Proposition 1]. Then $R(\varphi, c)$ is mapped isomorphically onto $R$. The noetherian property for $R$ follows from Example 5.2.3(2). □

The results above easily imply that a ring generated by a generic set of Eulerian derivatives is noetherian.

**Theorem 5.3.4.** *Assume that $k$ is uncountable. Let $V_i$ be the closed set $\{y_i = 0\}$ in $\mathbb{A}^t$. There is a generic subset $Y \subseteq \mathbb{A}^t \setminus \cup_{i=1}^t V_i$ such that if $(p_1, p_2, \ldots, p_t) \in Y$ then $R = k\langle D_{p_1}, D_{p_2}, \ldots D_{p_t}\rangle$ is noetherian.*

*Proof.* Let $k[y_1, y_2, \ldots y_t]$ be the coordinate ring of $\mathbb{A}^t$, and write $V(f)$ for the vanishing set in $\mathbb{A}^t$ of $f \in k[y_1, y_2, \ldots y_t]$. Let $\mathbb{F}$ be the prime subfield of $k$, and set $A = \mathbb{F}[y_1, y_2, \ldots y_t]$. The set $Y$ of points $(p_1, p_2, \ldots, p_t) \subseteq \mathbb{A}^t$ where the $\{p_i\}$ are algebraically independent over $\mathbb{F}$ is the complement in $\mathbb{A}^t$ of $\bigcup_{f \in A} V(f)$. But since $\mathbb{F}$ is countable, $A$ is also countable and so $Y$ is a generic subset of $\mathbb{A}^t$ (Definition 5.1.1). Now apply Theorem 5.3.2. □

We can also produce an example of a ring generated by Eulerian derivatives that is *not* noetherian, which was also lacking in [19].

**Proposition 5.3.5.** *Assume that* $\operatorname{char} k = p > 0$ *and that* $k$ *has transcendence degree at least* $1$ *over its prime subfield* $\mathbb{F}_p$*. Then there exist scalars* $p_1, p_2 \in k$ *such that the ring* $k\langle D_{p_1}, D_{p_2}\rangle$ *is not noetherian.*

*Proof.* let $y \in k$ be transcendental over $\mathbb{F}_p$. Consider the ring $R(\varphi, c)$ of Example 5.2.4, where $\varphi = \operatorname{diag}(1, y, y+1)$ and $c = (1 : 1 : 1)$. As in Example 5.2.4, the scalars $y, y+1$ generate a rank 2 multiplicative subgroup of the field $k$, so setting $p_1 = y$ and $p_2 = y + 1$ we have $R(\varphi, c) \cong k\langle D_{p_1}, D_{p_2}\rangle$ by Theorem 5.3.2. But as we saw in Example 5.2.4, $R(\varphi, c)$ is not noetherian. $\square$

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# ABSTRACT

Examples of Generic Noncommutative Surfaces

by

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We study a class of noncommutative surfaces, and their higher dimensional analogues, which come from generic subalgebras of twisted homogeneous coordinate rings of projective space. Such rings provide answers to several open questions in noncommutative projective geometry. Specifically, these rings $R$ are the first known graded algebras over a field $k$ which are noetherian but not strongly noetherian: in other words, $R \otimes_k B$ is not noetherian for some choice of commutative noetherian extension ring $B$. This answers a question of Artin, Small, and Zhang. The rings $R$ are also maximal orders, but they do not satisfy all of the $\chi$ conditions of Artin and Zhang. In particular, they satisfy the $\chi_1$ condition but not $\chi_i$ for $i \geq 2$, answering a question of Stafford and Zhang and a question of Stafford and Van den Bergh. Finally, we show that these algebras have finite cohomological dimension.