Unit CL

Basic Counting and Listing

Section 1: Lists with Repetitions

We begin with some matters of terminology and notation. Two words that we shall often use are *set* and *list*. (Lists are also called *strings*.) Both words refer to collections of objects. There is no standard notation for lists. Some of those in use are

$$\begin{aligned} &\text{apple banana pear peach} \\ &\text{apple, banana, pear, peach} \\ \text{and}\quad &(\text{apple}, \text{banana}, \text{pear}, \text{peach}). \end{aligned}$$

The notation for sets is standard: the items are separated by commas and surrounded by curly brackets as in

$$\{\text{apple}, \text{banana}, \text{pear}, \text{peach}\}.$$

The curly bracket notation for sets is so well established that you can normally assume it means a set — but beware, some mathematical software systems use { } (curly brackets) for lists.

What is the difference between a set and a list? "Set" means a collection of distinct objects in which the order doesn't matter. Thus

$$\{\text{apple}, \text{peach}, \text{pear}\} \quad \text{and} \quad \{\text{peach}, \text{apple}, \text{pear}\}$$

are the same sets, and the set $\{\text{apple}, \text{peach}, \text{apple}\}$ is the same as the set $\{\text{apple}, \text{peach}\}$. In other words, repeated elements are treated as if they occurred only once. Thus two sets are the same if and only if each element that is in one set is in both. In a list, order is important and repeated objects are usually allowed. Thus

$$(\text{apple}, \text{peach}) \quad (\text{peach}, \text{apple}) \quad \text{and} \quad (\text{apple}, \text{peach}, \text{apple})$$

are three different lists. Two lists are the same if and only if they have exactly the same items in exactly the same positions. Thus, "sets" and "lists" represent different concepts: A list is *always ordered* and a set has *no repeated elements.*

Example 1 (Using the terminology) People, in their everyday lives, deal with the issues of "order is important" and "order is not important." Imagine that Tim, Jane, and Linda are going to go shopping for groceries. Tim makes a note to remind himself to get apples and bananas. Tim's note might be written out in an orderly manner, or might just be words randomly placed on a sheet of paper. In any case, the purpose of the note is to remind him to buy some apples and bananas and, we assume, the order in which these items are noted is not important. The number of apples and bananas is not specified in the note. That will be determined at the store after inspecting the quality of the apples and bananas. The best model for this note is a set. Tim might have written {apples, bananas}. We have added the braces to emphasize that we are talking about sets. Suppose Jane wrote {bananas, apples} and Linda wrote {apples, bananas, apples}. Linda was a bit forgetful and wrote apples twice. It doesn't matter. All three sets are the same and all call for the purchase of some apples and some bananas. If Linda's friend Mary had made the note {peaches, bananas, oranges} and Linda and Mary had decided to combine their notes and go shopping together, they would have gone to the store to get {apples, peaches, bananas, oranges}.

There are times when order is important for notes regarding shopping trips or daily activities. For example, suppose Tim makes out the list (dentist, bookstore, groceries). It may be that he regards it as important to do these chores in the order specified. The dentist appointment may be at eight in the morning. The bookstore may not be open until nine in the morning. He may be planning to purchase milk at the grocery store and does not want the milk to be sitting in the car while he goes to the bookstore. In a list where order matters, the list (dentist, bookstore, groceries, dentist) would be different than (dentist, bookstore, groceries). The first list directs Tim to return to the dentist after the groceries, perhaps for a quick check that the cement on his dental work is curing properly.

In addition to the sets and lists described above, there is another concept that occurs in both everyday life and in mathematics. Suppose Tim, Jane, and Linda happen to go the grocery store and are all standing in line at the checkout counter with bags in hand containing their purchases. They compare purchases. Tim says "I purchased 3 bananas and 2 apples." Jane says, "I purchased 2 bananas and 3 apples." Linda says, "I purchased 3 apples and 2 bananas." Jane and Linda now say in unison "Our purchases are the same!" Notice that repetition (how many bananas and apples) now matters, but as with sets, order doesn't matter (Jane and Linda announced their purchases in different order but concluded their purchases were the same). We might use the following notation: Tim purchased {2 apples, 3 bananas}, Jane purchased {3 apples, 2 bananas}, Linda purchased {2 bananas, 3 apples}. Another alternative is to write {apple, apple, banana, banana, banana} for Tim's purchase. All that matters is the number of apples and bananas, so we could have written

{apple, banana, apple, banana, banana}

for Tim's purchase. Such collections, where order doesn't matter, but repetition does matter are called *multisets* in mathematics. Notice that if Tim and Jane dumped their purchases into the same bag they would have the combined purchase {5 apples, 5 bananas}. Combining multisets requires that we keep track of repetitions of objects. In this chapter, we deal with sets and lists. We will have some brief encounters with multisets later in our studies. □

To summarize the concepts in the previous example:

List: an ordered collection. Whenever we refer to a list, we will indicate whether the elements must be distinct.[1]

Set: a collection of distinct objects where order does not matter.

[1] A list is sometimes called a *string*, a *sequence* or a *word*. Lists are sometimes called vectors and the elements components.

Multiset: a collection of objects (repeats allowed) where order does not matter.[2]

The terminology "k-list" is frequently used in place of the more cumbersome "k-long list." Similarly, we use k-set and k-multiset. Vertical bars (also used for absolute value) are used to denote the number of elements in a set or in a list. We call $|A|$ "the number of elements in A" or, alternatively, "the cardinality of A." For example, if A is an n-set, then $|A| = n$.

We want to know how many ways we can do various things with a set. Here are some examples, which we illustrate by using the set $S = \{x, y, z\}$.

1. How many ways can we *list*, without repetition, all the elements of S? This means, how many ways can we arrange the elements of S in a list so that each element of S appears exactly once in each of the lists. For $S = \{x, y, z\}$, there are six ways: xyz, xzy, yxz, yzx, zxy and zyx. Notice that we have written the list (x, y, z) simply as xyz since there is no possibility of confusion. (These six lists are all called permutations of S. People often use Greek letters like π and σ to indicate a permutation of a set.)
2. How many ways can we construct a k-list of distinct elements from a set? When $k = |S|$, this is the previous question. If $k = 2$ and $S = \{x, y, z\}$, there are six ways: xy, xz, yx, yz, zx and zy.
3. If the list in the previous question is allowed to contain repetitions, what is the answer? There are nine ways for $S = \{x, y, z\}$: xx, xy, xz, yx, yy, yz, zx, zy and zz.
4. If, in Questions 2 and 3, the order in which the elements appear doesn't matter, what are the answers? For $S = \{x, y, z\}$ and $k = 2$, the answers are three and six, respectively. We are forming 2-sets and 2-multisets from the elements of S. The 2-sets are $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$. The 2-multisets are the three 2-sets plus $\{x, x\}$, $\{y, y\}$ and $\{z, z\}$.
5. How many ways can the set S be partitioned into a collection of k pairwise disjoint nonempty smaller sets?[3] With $k = 2$, the set $S = \{x, y, z\}$ has three such: $\{\{x\}, \{y, z\}\}$, $\{\{x, y\}, \{z\}\}$ and $\{\{x, z\}, \{y\}\}$.

We will learn how to answer these questions without going through the time-consuming process of listing all the items in question as we did for our illustration.

How many ways can we construct a k-list (repeats allowed) using an n-set? Look at our illustration in Question 3 above. The first entry in the list could be x, y or z. After any of these there were three choices (x, y or z) for the second entry. Thus there are $3 \times 3 = 9$ ways to construct such a list. The general pattern should be clear: There are n ways to choose each list entry. Thus

Theorem 1 (k-lists with repetitions) *There are n^k ways to construct a k-list from an n-set.*

This calculation illustrates an important principle:

Theorem 2 (Rule of Product) *Suppose structures are to be constructed by making a sequence of k choices such that, (1) the ith choice can be made in c_i ways, a number*

[2] *Sample* and *selection* are often used in probability and statistics, where it may mean a list or a multiset, depending on whether or not it is ordered.

[3] In other words, each element of S appears in *exactly one* of the smaller sets.

independent of what choices were made previously, and (2) each structure arises in exactly one way in this process. Then, the number of structures is $c_1 \times \cdots \times c_k$.

"Structures" as used above can be thought of simply as elements of a set. We prefer the term structures because it emphasizes that the elements are built up in some way; in this case, by making a sequence of choices. In the previous calculation, the structures are k-lists, which are built up by adding one element at a time. Each element is chosen from a given n-set and $c_1 = c_2 = \ldots = c_k = n$.

Definition 1 (Cartesian Product) *If $C_1, \ldots, C_k$ are sets, the Cartesian product of the sets is written $C_1 \times \cdots \times C_k$ and consists of all k-lists $(x_1, \ldots, x_k)$ with $x_i \in C_i$ for $1 \le i \le k$.*

For example, $\{1, 2\} \times \{x\} \times \{a, b, c\}$ is a set containing the six lists $1xa$, $1xb$, $1xc$, $2xa$, $2xb$ and $2xc$.

A special case of the Rule of Product is the fact that the number of elements in $C_1 \times \cdots \times C_k$ is the product $|C_1| \cdots |C_k|$. Here C_i is the collection of i^{th} choices and $c_i = |C_i|$. This is only a special case because the Rule of Product would allow the *collection* C_i to depend on the previous choices $x_1, \ldots, x_{i-1}$ as long as the *number* c_i of possible choices does not depend on $x_1, \ldots, x_{i-1}$.

Here is a property associated with Cartesian products that we will find useful in our later discussions.

Definition 2 (Lexicographic order) *If $C_1, \ldots, C_k$ are lists of distinct elements, we may think of them as sets and form the Cartesian product $P = C_1 \times \cdots \times C_k$. The lexicographic order on P is defined by saying that $(a_1, \ldots, a_k) <_L (b_1, \ldots, b_k)$ if and only if there is some $t \le k$ such that $a_i = b_i$ for $i < t$ and $a_t < b_t$. Usually we write $(a_1, \ldots, a_k) < (b_1, \ldots, b_k)$ instead of $(a_1, \ldots, a_k) <_L (b_1, \ldots, b_k)$, because it is clear from the context that we are talking about lexicographic order.*

Often we say *lex order* instead of lexicographic order. If all the C_i's equal

$$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$$

then lex order is simply numerical order of k digit integers with leading zeroes allowed. Suppose that all the C_i's equal (<space>, A, B, …, Z). If we throw out those elements of P that have a letter following a space, the result is dictionary order. For example, BAT, BATTERS and BATTLE are in lex order. Why? All agree in the first three positions. The fourth position of BAT is <space>, which precedes all letters in our order. Similarly, BATTERS comes before BATTLE because they first differ in the fifth position and E < L Unlike these two simple examples, the C_i's usually vary with i.

Example 2 (A simple count) The North-South streets in Rectangle City are named using the numbers 1 through 12 and the East-West streets are named using the letters A through H. The most southwesterly intersection occurs where 1 and A streets meet. How many blocks are within the city?

Each block can be labeled by the streets at its southwesterly corner. These labels have the form (x, y) where x is between 1 and 11 inclusive and y is between A and G. (If you don't see why 12 and H are missing, draw a picture and look at southwesterly corners.) By the Rule of Product there are $11 \times 7 = 77$ blocks. In this case the structures can be taken to be the descriptions of the blocks. Each description has two parts: the names of the north-south and East-West streets at the block's southwest corner. □

Example 3 (Counting galactic names) In a certain land on a planet in a galaxy far away the alphabet contains only 5 letters which we will transliterate as A, I, L, S and T in that order. All names are 6 letters long, begin and end with consonants and contain two vowels which are not adjacent to each other. Adjacent consonants must be different. The list begins with LALALS, LALALT, LALASL, LALAST, LALATL, LALATS, LALILS and ends with TSITAT, TSITIL, TSITIS, TSITIT. How many possible names are there?

The possible positions for the two vowels are $(2, 4)$, $(2, 5)$ and $(3, 5)$. Each of these results in two isolated consonants and two adjacent consonants. Thus the answer is the product of the following factors:

1. choose the vowel locations (3 ways);
2. choose the vowels ($2 \times 2 = 4$ ways);
3. choose the isolated consonants ($3 \times 3 = 9$ ways);
4. choose the adjacent consonants ($3 \times 2 = 6$ ways).

The answer is $3 \times 4 \times 9 \times 6 = 648$. This construction can be interpreted as a Cartesian product as follows. C_1 is the set of lists of possible positions for the vowels, C_2 is the set of lists of vowels in those positions, and C_3 and C_4 are sets of lists of consonants. Thus

$$C_1 = \{(2,4), (2,5), (3,5)\} \quad C_2 = \{\mathrm{AA}, \mathrm{AI}, \mathrm{IA}, \mathrm{II}\}$$

$$C_3 = \{\mathrm{LL}, \mathrm{LS}, \mathrm{LT}, \mathrm{SL}, \mathrm{SS}, \mathrm{ST}, \mathrm{TL}, \mathrm{TS}, \mathrm{TT}\} \quad C_4 = \{\mathrm{LS}, \mathrm{LT}, \mathrm{SL}, \mathrm{ST}, \mathrm{TL}, \mathrm{TS}\}.$$

For example, ((2,5), IA, SS, ST) in the Cartesian product corresponds to the word SISTAS. □

Here's another important principle, the proof of which is self evident:

Theorem 3 (Rule of Sum) *Suppose a set T of structures can be partitioned into sets $T_1, \ldots, T_j$ so that each structure in T appears in exactly one T_i, then*

$$|T| = |T_1| + \cdots + |T_j|.$$

Example 4 (Counting galactic names again) We redo the previous example using the Rule of Sums. The possible vowel (V) and consonant (C) patterns for names are CCVCVC, CVCCVC and CVCVCC. Since these patterns are disjoint and cover all cases, we may compute the number of names of each type and add the results together. For the first pattern we have a product of six factors, one for each choice of a letter: $3\times2\times2\times3\times2\times3 = 216$. The other two patterns also give 216, for a total of $216+216+216 = 648$ names.

This approach has a wider range of applicability than the method we used in the previous example. We were only able to avoid the Rule of Sum in the first method because each pattern contained the same number of vowels, isolated consonants and adjacent consonants. Here's an example that requires the Rule of Sum. Suppose a name consists of only four letters, namely two vowels and two consonants, constructed so that the vowels are not adjacent and, if the consonants are adjacent, then they are different. There are three patterns: CVCV, VCVC, VCCV. By the Rule of Product, the first two are each associated with 36 names, but VCCV is associated with only 24 names because of the adjacent consonants. Hence, we cannot choose a pattern and then proceed to choose vowels and consonants. On the other hand, we can apply the Rule of Sum to get a total of 96 names. $\square$

Example 5 (Smorgasbord College committees) Smorgasbord College has four departments which have 6, 35, 12 and 7 faculty members. The president wishes to form a faculty judicial committee to hear cases of student misbehavior. To avoid the possibility of ties, the committee will have three members. To avoid favoritism the committee members will be from different departments and the committee will change daily. If the committee only sits during the normal academic year (165 days), how many years can pass before a committee must be repeated?

If T is the set of all possible committees, the answer is $|T|/165$. Let T_i be the set of committees with no members from the ith department. By the Rule of Sum $|T| = |T_1|+|T_2|+|T_3|+|T_4|$. By the Rule of Product

$$|T_1| = 35\times12\times7 = 2940 \qquad |T_3| = 35\times6\times7 = 1470$$
$$|T_2| = 6\times12\times7 = 504 \qquad |T_4| = 35\times12\times6 = 2520.$$

Thus the number of years is $7434/165 = 45^+$. Due to faculty turnover, a committee need never repeat — if the president's policy lasts that long. $\square$

Whenever we encounter a new technique, there are two questions that arise:

• *When* is it used? • *How* is it used?

For the Rules of Sum and Product, the answers are intertwined:

> Suppose you wish to count the number of structures in a set and that you can describe how to construct the structures in terms of subconstructions that are connected by "ands" and "ors." *If this leads to the construction of each structure in a unique way*, then the Rules of Sum and Product apply. To use them, replace "ands" by products and "ors" by sums. Whenever you write something like "Do A AND do B," it should mean "Do A AND then do B" because the Rule of Product requires that the choices be made sequentially. Remember that the number of ways to do B *must not depend on* the choice for A.

Example 6 (Applying the sum–product rules) To see how this technique is applied, let's look back at Example 5. A committee consists of either

1. One person from Dept. 1 AND one person from Dept. 2 AND one person from Dept. 3, OR
2. One person from Dept. 1 AND one person from Dept. 2 AND one person from Dept. 4, OR
3. One person from Dept. 1 AND one person from Dept. 3 AND one person from Dept. 4, OR
4. One person from Dept. 2 AND one person from Dept. 3 AND one person from Dept. 4.

The number of ways to choose a person from a department equals the number of people in the department. □

Until you become comfortable using the Rules of Sum and Product, look for "and" and "or" in what you do. This is an example of the useful tactic:

Step 1: Break the problem into parts.
Step 2: Work on each piece separately.
Step 3: Put the pieces together.

Here Step 1 is getting a phrasing with "ands" and "ors;" Step 2 is calculating each of the individual pieces; and Step 3 is applying the Rules of Sum and Product.

Exercises for Section 1

The following exercises will give you additional practice on lists with repetition and the Rules of Sum and Product.

In each exercise, indicate how you are using the Rules of Sum and Product.

1.1. Suppose a bookshelf contains five discrete math texts, two data structures texts, six calculus texts, and three Java texts. (All the texts are different.)

(a) How many ways can you choose one of the texts?

(b) How many ways can you choose one of each type of text?

1.2. How many different three digit positive integers are there? (No leading zeroes are allowed.) How many positive integers with at most three digits? What are the answers when "three" is replaced by "n?"

1.3. Prove that the number of subsets of a set S, including the empty set and S itself, is $2^{|S|}$.

1.4. Suppose $n > 1$. An n-digit number is a list of n digits where the first digit in the list is not zero.

(a) How many n-digit numbers are there?

(b) How many n-digit numbers contain no zeroes?

(c) How many n-digit numbers contain at least one zero?
Hint: Use (a) and (b).

1.5. For this exercise, we work with the ordinary alphabet of 26-letters.

(a) Define a "4-letter word" to be any list of 4 letters that contains *at least* one of the vowels A, E, I, O and U. How many 4-letter words are there?

(b) Suppose, instead, we define a "4-letter word" to be any list of 4 letters that contains *exactly* one of the vowels A, E, I, O and U. How many 4-letter words are there?

1.6. In a certain land on a planet in a galaxy far away the alphabet contains only 5 letters which we will transliterate as A, I, L, S and T in that order. All names are 5 letters long, begin and end with consonants and contain two vowels which are not adjacent to each other. Adjacent consonants must be different. How many names are there?

1.7. A *composition* of a positive integer n is a list of positive integers (called *parts*) that sum to n. The four compositions of 3 are 3; 2,1; 1,2 and 1,1,1.

(a) By considering ways to insert plus signs and commas in a list of n ones, obtain the formula 2^{n-1} for the number of compositions of n. To avoid confusion with the Rule of Sum, we'll write this plus sign as $\oplus$. (The four compositions 3; 2,1; 1,2 and 1,1,1 correspond to $1 \oplus 1 \oplus 1$; $1 \oplus 1{,}1$; $1{,}1 \oplus 1$ and 1,1,1, respectively.)

(b) List all compositions of 4.

(c) List all compositions of 5 with 3 parts.

1.8. In Example 3 we found that there were 648 possible names. Suppose that these are listed in the usual dictionary order. The last word in the first third of the dictionary is LTITIT (the 216^{th} word). The first word in the middle third is SALALS. Explain.

1.9. There is another possible lexicographic order on the names in Example 3 (Counting galactic names) that gives rise to a "nonstandard" lex order on this list of names. Using the interpretation of the list of names as the Cartesian product of the lists $C_1 \times C_2 \times C_3 \times C_4$, we can lexicographically order the entire list of names based on the following linear orderings of the C_i, $i = 1, 2, 3, 4$:

$$C_1 = ((2,4),(2,5),(3,5)) \quad C_2 = (\mathrm{AA}, \mathrm{AI}, \mathrm{IA}, \mathrm{II})$$

$$C_3 = (\mathrm{LL}, \mathrm{LS}, \mathrm{LT}, \mathrm{SL}, \mathrm{SS}, \mathrm{ST}, \mathrm{TL}, \mathrm{TS}, \mathrm{TT}) \quad C_4 = (\mathrm{LS}, \mathrm{LT}, \mathrm{SL}, \mathrm{ST}, \mathrm{TL}, \mathrm{TS}).$$

What are the first seven and last seven entries in this lex ordering?
Hint: The lex ordering can be done entirely in terms of the sets C_i and then translated to the names as needed. Thus the first two entries in the list $C_1 \times$

$C_2 \times C_3 \times C_4$ in lex order are (2,4)(AA)(LL)(LS) and (2,4)(AA)(LL)(LT). The last two are (3,5)(II)(TT)(TL) and (3,5)(II)(TT)(TS). These translate to LALALS and LALALT for the first two and TLITIT and TSITIT for the last two.

1.10. Recall that the size of a multiset is the number of elements it contains. For example, the size of $\{a, a, b\}$ is three.

(a) How many 4-element multisets are there whose elements are taken from the set $\{a, b, c\}$? (An element may be taken more than once; for example, the multiset $\{c, c, c, c\}$.)

(b) How many multisets are there whose elements are taken from the set $\{a, b, c\}$?

Section 2: Lists Without Repetition

What happens if we do not allow repeats in our list? Suppose we have n elements to choose from and wish to form a k-list with no repeats. How many lists are there? We can choose the first entry in the list AND choose the second entry AND $\cdots$ AND choose the kth entry. There are $n - (i - 1) = n - i + 1$ ways to choose the ith entry since $i - 1$ elements have been removed from the set to make the first part of the list. By the Rule of Product, the number of lists is $n(n-1)\cdots(n-k+1)$. Using the notation $n!$ for the product of the first n integers and writing $0! = 1$, you should be able to see that this answer can be written as $n!/(n-k)!$, which is often designated by $(n)_k$ and called the *falling factorial.* Some authors write the falling factorial as $n^{\underline{k}}$. We have proven

Theorem 4 (k-lists without repetition) *When repeats are not allowed, there are $n!/(n-k)! = (n)_k$ k-lists that can be constructed from an n-set. (When $k > n$ the answer is zero.)*

When $k = n$, a list without repeats is simply a *linear ordering* of the set. We frequently say "ordering" instead of "linear ordering." An ordering is sometimes called a "permutation" of S. Thus, we have proven that a set S can be (linearly) ordered in $|S|!$ ways.

Example 7 (Lists without repeats) How many lists without repeats can be formed from a 5-set? There are $5! = 120$ 5-lists without repeats, $5!/1! = 120$ 4-lists without repeats, $5!/2! = 60$ 3-lists, $5!/3! = 20$ 2-lists and $5!/4! = 5$ 1-lists. By the Rule of Sum, this gives a total of 325 lists, or 326 if we count the empty list. □

Example 8 (Linear arrangements) How many different ways can 100 people be arranged in the seats in a classroom that has exactly 100 seats? Each seating is simply an ordering of the people. Thus the answer is 100!. Simply writing 100! probably gives you little idea of the size of the number of seatings. A useful approximation for factorials is given by *Stirling's formula.*

Theorem 5 (Stirling's formula) $\sqrt{2\pi n}\,(n/e)^n$ *approximates* $n!$ *with a relative error less than* $1/10n$.

We say that $f(x)$ approximates $g(x)$ with a *relative error* of $|f(x)/g(x) - 1|$. Thus, the theorem states that $\sqrt{2\pi n}\,(n/e)^n/n!$ differs from 1 by less than $1/10n$. When relative error is multiplied by 100, we obtain "percentage error." By Stirling's formula, we find that 100! is nearly 9.32×10^{157}, which is much larger than estimates of the number of atoms in the universe. □

We can extend the ideas of the previous example. Suppose we still have 100 seats but have only 95 people. We need to think a bit more carefully than before. One approach is to put the people in some order, select a list of 95 seats, and then pair up people and seats so that the first person gets the first seat, the second person the second seat, and so on. By the general formula for lists without repetition, the answer is $100!/(100-95)! = 100!/120$. We can also solve this problem by thinking of the people as positions in a list and the seats as entries! Thus we want to form a 95-list using the 100 seats. According to Theorem 4, this can be done in $100!/(100-95)!$ ways.

Lists can appear in many guises. As seen in the previous paragraph, the people could be thought of as the positions in a list and the seats the things in the list. Sometimes it helps to find a reinterpretation like this for a problem. At other times it is easier to tackle the problem starting over again from scratch. These methods can lead to several approaches to a problem. That can make the difference between a solution and no solution or between a simple solution and a complicated one. You should practice using both methods, even on the same problem.

Example 9 (Circular arrangements) How many ways can n people be seated on a Ferris wheel with exactly one person in each seat? Equivalently, we can think of this as seating the people at a circular table with n chairs. Two seatings are defined to be "the same" if one can be obtained from the other by rotating the Ferris wheel (or rotating the seats around the table).

If the people were seated in a straight line instead of in a circle, the answer would be $n!$. Can we convert the circular seating into a linear seating (i.e., a list)? In other words, *can we convert the unsolved problem to a solved one*? Certainly — simply cut the circular arrangement between two people and unroll it. Thus, to arrange n people in a linear ordering,

first arrange them in a circle AND then cut the circle.

According to our AND/OR technique, we must prove that each linear arrangement arises in *exactly one way* with this process.

- Since a linear seating can be rolled up into a circular seating, it can also be obtained by unrolling that circular seating. Hence each linear seating arises *at least once.*
- Since the people at the circular table are all different, the place we cut the circle determines who the first person in the linear seating is, so each cutting of a circular seating gives a different linear seating. Obviously two different circular seatings cannot give the same linear seating. Hence each linear seating arises *at most once.*

Putting these two observations together, we see that each linear seating arises *exactly once.* By the Rule of Product,

$$n! = (\text{number of circular arrangements}) \times n.$$

Hence the number of circular arrangements is $n!/n = (n-1)!$.

Our argument was somewhat indirect. We can derive the result by a more direct argument. For convenience, let the people be called 1 through n. We can read off the people in the circular list starting with person 1. This gives a linear ordering of $\{1, \ldots, n\}$ that starts with 1. Conversely, each such linear ordering gives rise to a circular ordering. Thus the number of circular orderings equals the number of such linear orderings. Having listed person 1, there are $(n-1)!$ ways to list the remaining $n-1$ people.

If we are making circular necklaces using n distinct beads, then the arguments we have just given prove that there are $(n-1)!$ possible necklaces provided we are not allowed to flip necklaces over.

What happens if the beads are not distinct? For example, suppose there are three blue beads and three yellow beads. There are just two linear arrangements associated with the circular arrangement BYBYBY, namely (B,Y,B,Y,B,Y) and (Y,B,Y,B,Y,B). But there are six linear arrangements associated with the circular arrangement BBBYYY. Thus, the approach we used for distinct beads fails, because the number of lists associated with a necklace depends on the necklace. For now, you only need to be aware of this complication. □

We need not insist on "no repetitions at all" in lists. There are natural situations in which some repetitions are allowed and others are not allowed. The following example illustrates one such way that this can happen.

Example 10 (Words from a collection of letters — first try) How many "words" of length k can be formed from the letters in ERROR when no letter may be used more often than it appears in ERROR? (A "word" is any list of letters, pronounceable or not.) You can imagine that you have 5 tiles, namely one E, one O, and three R's. The answer is not 3^k even though we are using 3 different letters. Why is this? Unlimited repetition is not allowed so, for example, we cannot have EEE. On the other hand, the answer is not $(3)_k$ since R can be repeated some. Also, the answer is not $(5)_k$ even though we have 5 tiles. Why is this? The formula $(5)_k$ arises if we have 5 *distinct* objects; however, our 3 tiles with R are identical. At present, all we can do is carefully list the possibilities. Here they are in alphabetical order.

$k = 1:$ E, O, R

$k = 2:$ EO, ER, OE, OR, RE, RO, RR

$k = 3:$ EOR, ERO, ERR, OER, ORE, ORR, REO, RER, ROE, ROR, RRE, RRO, RRR

$k = 4:$ EORR, EROR, ERRO, ERRR, OERR, ORER, ORRE, ORRR, REOR, RERO, RERR, ROER, RORE, RORR, RREO, RRER, RROE, RROR, RRRE, RRRO

$k = 5:$ EORRR, ERORR, ERROR, ERRRO, OERRR, ORERR, ORRER, ORRRE, REORR, REROR, RERRO, ROERR, RORER, RORRE, RREOR, RRERO, RROER, RRORE, RRREO, RRROE

This is obviously a tedious process. We shall return to this type of problem in the next section. □

Exercises for Section 2

The following exercises will give you additional practice with lists with restricted repetitions.

In each exercise, indicate how you are using the Rules of Sum and Product.

It is instructive to first do these exercises using only the techniques introduced so far and then, after reading the next section, to return to these exercises and look for other ways of doing them.

2.1. We want to know how many ways 3 boys and 4 girls can sit in a row.

(a) How many ways can this be done if there are no restrictions?

(b) How many ways can this be done if the boys sit together and the girls sit together?

(c) How many ways can this be done if the boys and girls must alternate?

2.2. Repeat the previous exercise when there are 3 boys and 3 girls.

2.3. What are the answers to the previous two exercises if the table is circular?

2.4. How many ways are there to form a list of two distinct letters from the set of letters in the word COMBINATORICS? three distinct letters? four distinct letters?

2.5. How many ways are there to form a list of two letters from the set of letters in the word COMBINATORICS if the letters cannot be used more often than they appear in COMBINATORICS? three letters?

2.6. We are interested in forming 3 letter words (“3-words”) using the letters in LITTLEST. For the purposes of the problem, a “word” is any list of letters.

(a) How many words can be made with no repeated letters?

(b) How many words can be made with unlimited repetition allowed?

(c) How many words can be made if repeats are allowed but no letter can be used more often than it appears in LITTLEST?

2.7. By 2050 spelling has deteriorated considerably. The dictionary defines the spelling of "relief" to be any combination (with repetition allowed) of the letters R, L, F, I and E subject to certain constraints:

- The number of letters must not exceed 6.
- The word must contain at least one L.
- The word must begin with an R and end with an F.
- There is just one R and one F.

(a) How many spellings are possible?

(b) The most popular spelling is the one that, in dictionary order, is five before the spelling RELIEF. What is it?

***2.8.** By the year 2075, further deterioration in spelling has occurred. The dictionary now defines the spelling of "relief" to be any combination (with repetition allowed) of the letters R, L, F, I and E subject to these constraints:

- The number of letters must not exceed 6.
- The word must contain at least one L.
- The word must begin with a nonempty string of R's and end with a nonempty string of F's, and there are no other R's and F's.

(a) How many spellings are possible?

(b) The most popular spelling is the one that, in dictionary order, is five before the spelling RELIEF. What is it?

***2.9.** Prove that the number of lists without repeats that can be constructed from an n-set is very nearly $n!e$. Your count should include lists of *all* lengths from 0 to n. *Hint*: Recall that from Taylor's Theorem in calculus $e^x = 1+x+x^2/2!+x^3/3!+\cdots$.

Section 3: Sets

We first review some standard terminology and notation associated with sets. When we discuss sets, we usually have a "universal set" U in mind, and the sets we discuss are subsets of U. For example, $U = \mathbb{Z}$ might be the integers. We then speak of the natural numbers

$\mathbb{N} = \{0, 1, 2, \ldots\}$, the positive integers $\mathbb{N}^+$, the odd integers $\mathbb{N}_o$, etc., thinking of these sets as subsets of the "universal set" $\mathbb{Z}$.

Definition 3 (Set notation) *A set is an unordered collection of distinct objects. We use the notation $x \in S$ to mean "x is an element of S" and $x \notin S$ to mean "x is not an element of S." Given two subsets (subcollections) of U, X and Y, we say "X is a subset of Y," written $X \subseteq Y$, if $x \in X$ implies that $x \in Y$. Alternatively, we may say that "Y is a superset of X." $X \subseteq Y$ and $Y \supseteq X$ mean the same thing. We say that two subsets X and Y of U are equal if $X \subseteq Y$ and $Y \subseteq X$. We use braces to designate sets when we wish to specify or describe them in terms of their elements: $A = \{a, b, c\}$, $B = \{2, 4, 6, \ldots\}$. A set with k elements is called a k-set* or *set with* cardinality *k. The cardinality of a set A is denoted by $|A|$.*

Since a set is an unordered collection of distinct objects, the following all describe the same 3-element set

$$\{a, b, c\} = \{b, a, c\} = \{c, b, a\} = \{a, b, b, c, b\}.$$

The first three are simply listing the elements in a different order. The last happens to mention some elements more than once. But, since a set consists of distinct objects, the elements of the set are still just a, b, c. Another way to think of this is:

> Two sets A and B are equal if and only if every element of A is an element of B and every element of B is an element of A.

Thus, with $A = \{a, b, c\}$ and $B = \{a, b, b, c, b\}$, we can see that everything in A is in B and everything in B is in A. You might think "When we write a set, the elements are in the order written, so why do you say a set is not ordered?" When we write something down we're stuck — we have to list them in some order. You can think of a set differently: Write each element on a separate slip of paper and put the slips in a paper bag. No matter how you shake the bag, it's still the same set.

For the most part, we shall be dealing with finite sets. Let U be a set and let A and B be subsets of U.

- The sets $A \cap B$ and $A \cup B$ are the *intersection* and *union* of A and B.
- The set $A \setminus B = \{x : x \in A, x \notin B\}$ is the *set difference* of A and B. It is also written $A - B$.
- The set $U \setminus A$ or A^c is the *complement* of A (relative to U). The complement of A is also written A' and $\sim A$.
- The set $A \oplus B = (A \setminus B) \cup (B \setminus A)$ is the *symmetric difference* of A and B.
- The set $A \times B = \{(x, y) : x \in A, y \in B\}$ is the *product* or *Cartesian product* of A and B.

Example 11 (Cardinality of various sets) Recall that $|S|$, the cardinality of the set S is its size; that is, the number of elements in the set.

By the Rule of Product, $|A \times B| = |A| \times |B|$. (The first multiplication is Cartesian product; the second is multiplication of numbers.) Also, by the Rule of Product, the number of subsets of A is $2^{|A|}$. To see this, notice that for each element of A we have two choices — include the element in the subset or not include it.

What about things like $|A \cup B|$ and $|A \oplus B|$? They can't be expressed just in terms of $|A|$ and $|B|$. To see this, note that if $A = B$, then $|A \cup B| = |A|$ and $|A \oplus B| = |\emptyset| = 0$. On the other hand, if A and B have no common elements, $|A \cup B| = |A| + |B|$ and $|A \oplus B| = |A| + |B|$ as well. Can we say anything in general? Yes. We'll return to this later. $\square$

The algebraic rules for operating with sets are also familiar to most beginning university students. Here is such a list of the basic rules. In each case the standard name of the rule is given first, followed by the rule as applied first to $\cap$ and then to $\cup$.

Theorem 6 (Algebraic rules for sets) *The universal set U is not mentioned explicitly but is implicit when we use the notation $\sim X = U - X$ for the complement of X. An alternative notation is $X^c = \sim X$.*

Associative:	$(P \cap Q) \cap R = P \cap (Q \cap R)$	$(P \cup Q) \cup R = P \cup (Q \cup R)$
Distributive:	$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$	$P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$
Idempotent:	$P \cap P = P$	$P \cup P = P$
Double Negation:	$\sim\sim P = P$	
DeMorgan:	$\sim(P \cap Q) = \sim P \cup \sim Q$	$\sim(P \cup Q) = \sim P \cap \sim Q$
Absorption:	$P \cup (P \cap Q) = P$	$P \cap (P \cup Q) = P$
Commutative:	$P \cap Q = Q \cap P$	$P \cup Q = Q \cup P$

These rules are "algebraic" rules for working with $\cap$, $\cup$, and $\sim$. You should memorize them as you use them. They are used just like rules in ordinary algebra: whenever you see an expression on one side of the equal sign, you can replace it by the expression on the other side.

We use the notation $\mathcal{P}(A)$ to denote the set of all subsets of A and $\mathcal{P}_k(A)$ the set of all subsets of A of size (or cardinality) k. (In the previous example, we saw that $|\mathcal{P}| = 2^{|A|}$.) Let $C(n,k) = |\mathcal{P}_k(A)|$ denote the number of different k-subsets that can be formed from an n-set. The notation $\binom{n}{k}$ is also frequently used. These are called *binomial coefficients* and are read "n choose k." How do we compute $C(n,k)$?

Can we rephrase the problem in a way that converts it to a list problem, since we know how to solve those? In other words, can we relate this problem, where order does not matter, to a problem where order matters?

Let's consider all possible orderings of each of our k-sets. This gives us a way to construct all lists with distinct elements in two steps: First construct a k-set, then order it.[4] We can order a k-set by forming a k-list without repeats from the k-set. By Theorem 4

[4] We used an idea like this in Example 9 when we counted circular lists with distinct elements.

of Section 2, we know that this can be done in $k!$ ways. By the Rule of Product, there are $C(n,k)\,k!$ distinct k-lists with no repeats. By Theorem 4 again, this number is $n(n-1)\cdots(n-k+1) = n!/(n-k)!$. Dividing by $k!$, we have

Theorem 7 (Binomial coefficient formula) *The value of the binomial coefficient is*

$$\binom{n}{k} = C(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}.$$

Furthermore $\binom{n}{k} = \binom{n}{n-k}$.

Example 12 (Computing binomial coefficients) Let's compute some binomial coefficients for practice.

$$\binom{7}{3} = \frac{7\times 6\times 5}{3!} = 35,$$

because $n = 7$, $k = 3$ and so $n - k + 1 = 5$. Alternatively,

$$\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{1\times 2\times 3\times 4\times 5\times 6\times 7}{(1\times 2\times 3)(1\times 2\times 3\times 4)},$$

which again gives 35 after some work.

How about computing $\binom{12}{10}$? Using the formula $\frac{12(11)\cdots(3)}{10!}$ involves a lot of writing and then a lot of cancellation (there are common factors in the numerator and denominator). There is a quicker way. By the last sentence in the theorem, $\binom{12}{10} = \binom{12}{2}$. Now we have $\binom{12}{2} = \frac{12\times 11}{2!} = 66$. $\square$

***Example 13 (A generating function for binomial coefficients)** We'll now approach the problem of evaluating $C(n,k)$ in another way. In other words, we'll "forget" the formula we just derived and start over with a new approach. You may ask "Why waste time using another approach when we've already gotten what we want?" We gave a partial answer to this earlier. Here is a more complete response.

- By looking at a problem from different viewpoints, we may come to understand it better and so be more comfortable working similar problems in the future.
- By looking at a problem from different viewpoints, we may discover that things we previously thought were unrelated have interesting connections. These connections might open up easier ways to solve some types of problems and may make it possible for us to solve problems we couldn't do before.
- A different point of view may lead us to a whole new approach to problems, putting powerful new tools at our disposal.

In the approach we are about to take, we'll begin to see a powerful tool for solving counting problems. It's called "generating functions" and it lets us put calculus and related subjects to work in combinatorics.

Suppose that $S = \{x_1, \ldots, x_n\}$ where x_1, x_2, ... and x_n are variables as in high school algebra. Let $P(S) = (1+x_1)\cdots(1+x_n)$. The first three values of $P(S)$ are

$$\begin{aligned} n=1: &\quad 1+x_1 \\ n=2: &\quad 1+x_1+x_2+x_1x_2 \\ n=3: &\quad 1+x_1+x_2+x_3+x_1x_2+x_1x_3+x_2x_3+x_1x_2x_3. \end{aligned}$$

From this you should be able to convince yourself that $P(S)$ consists of a sum of terms where each term represents one of the subsets of S as a product of its elements. Can we reach some understanding of why this is so? Yes, but we'll only explore it briefly now. The understanding relates to the Rules of Sum and Product. Interpret plus as OR, times as AND and 1 as "nothing." Then $(1+x_1)(1+x_2)(1+x_3)$ can be read as

- include the factor 1 in the term OR include the factor x_1 AND
- include the factor 1 in the term OR include the factor x_2 AND
- include the factor 1 in the term OR include the factor x_3.

In other words

- omit x_1 OR include x_1 AND
- omit x_2 OR include x_2 AND
- omit x_3 OR include x_3.

This is simply a description of how to form an arbitrary subset of $\{x_1, x_2, x_3\}$. On the other hand we can form an arbitrary subset by the rule

- include nothing in the subset OR
- include x_1 in the subset OR
- include x_2 in the subset OR
- include x_3 in the subset OR
- include x_1 AND x_2 in the subset OR
- include x_1 AND x_3 in the subset OR
- include x_2 AND x_3 in the subset OR
- include x_1 AND x_2 AND x_3 in the subset.

If we drop the subscripts on the x_i's, then a product representing a k-subset becomes x^k. We get one such term for each subset and so it follows that the coefficient of x^k in the polynomial $f(x) = (1+x)^n$ is $C(n,k)$; that is,

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k.$$

This expression is called a *generating function* for the binomial coefficients $C(n,k)$.

Can this help us evaluate $C(n,k)$? Calculus comes to the rescue through Taylor's Theorem! Taylor's Theorem tells us that the coefficient of x^k in $f(x)$ is $f^{(k)}(0)/k!$. Let $f(x) = (1+x)^n$. Taking the k-th derivative of f gives

$$f^{(k)}(x) = n(n-1)\cdots(n-k+1)\ (1+x)^{n-k}.$$

Thus $C(n,k)$, the coefficient of x^k in $(1+x)^n$, is

$$C(n,k) = \frac{f^{(k)}(0)}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

We conclude this example with

Theorem 8 (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

This follows from the identity $(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$: Since $(x+y)^n = x^n(1+(y/x))^n$, the coefficient of $x^n(y/x)^k$ in $(x+y)^n$ is $C(n,k)$. $\square$

To illustrate, $(x+y)^3 = \binom{3}{3}x^3y^0 + \binom{3}{2}x^2y^1 + \binom{3}{1}x^1y^2 + \binom{3}{0}x^0y^3$, which equals $x^3 + 3x^2y + 3xy^2 + y^3$.

Example 14 (Smorgasbord College programs) Smorgasbord College allows students to study in three principal areas: (a) Swiss naval history, (b) elementary theory and (c) computer science. The number of upper division courses offered in these fields are 2, 92, and 15 respectively. To graduate, a student must choose a major and take 6 upper division courses in it, and also choose a minor and take 2 upper division courses in it. Swiss naval history cannot be a major because only 2 upper division courses are offered in it. How many programs are possible?

The possible major-minor pairs are b-a, b-c, c-a, and c-b. By the Rule of Sum we can simply add up the number of programs in each combination. Those programs can be found by the Rule of Product. The number of major programs in (b) is $C(92,6)$ and in (c) is $C(15,6)$. For minor programs: (a) is $C(2,2) = 1$, (b) is $C(92,2) = 4186$ and (c) is $C(15,2) = 105$. Since the possible programs are constructed by

$$\begin{aligned} &\Big(\text{major (b) AND } \big(\text{minor (a) OR minor (c)}\big)\Big) \\ \text{OR } &\Big(\text{major (c) AND } \big(\text{minor (a) OR minor (b)}\big)\Big), \end{aligned}$$

the number of possible programs is

$$\binom{92}{6}(1+105) + \binom{15}{6}(1+4186) = 75{,}606{,}201{,}671,$$

a rather large number. $\square$

Example 15 (Card hands: Full house) Card hands provide a source of some simple sounding but tricky set counting problems. A standard deck of cards contains 52 cards, each of which is marked with two labels. The first label, called the "suit," belongs to the set

$$\text{suits} = \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\},$$

called club, heart, diamond and spade, respectively. (On the blackboard, we will use C, H, D and S rather than drawing the symbols.) The second label, called the "value" belongs to the set

$$\text{values} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J}, \text{Q}, \text{K}, \text{A}\},$$

where J, Q, K and A are jack, queen, king and ace, respectively. Each pair of labels occurs exactly once in the deck. A hand is a subset of a deck. Two cards are a pair if they have the same values.

How many 5 card hands consist of a pair and a triple? (In poker, such a hand is called a "full house.")

To calculate this we describe how to construct such a hand:

- Choose the value for the pair AND
- Choose the value for the triple different from the pair AND
- Choose the 2 suits for the pair AND
- Choose the 3 suits for the triple.

This produces each full house exactly once, so the number is the product of the answers for the four steps, namely

$$13 \times 12 \times C(4,2) \times C(4,3) = 3{,}744. \qquad \square$$

Example 16 (Card hands: Two pairs) We'll continue with our poker hands. How many 5 card hands consist of two pairs? A description of a hand always means that there is nothing better in the hand, so "two pairs" means we don't have a full house or four of a kind.

The obvious thing to do is replace "triple" by "second pair" in the description for constructing a full house and add a choice for the card that belongs to no pair. This is not correct! Each hand is constructed twice, depending on which pair is the "second pair." Try it! What happened? Before choosing the cards for a pair and a triple, we can distinguish a pair from a triple because a pair contains 2 cards and a triple 3. We can't distinguish the two pairs, though, until the values are specified. This is an example of a situation where we can easily make mistakes if we forget that "AND" means "AND then." Here's a correct description, with "then" put in for emphasis.

- Choose the values for the two pairs AND then
- Choose the 2 suits for the pair with the larger value AND then
- Choose the 2 suits for the pair with the smaller value AND then
- Choose the remaining card from the 4×11 cards that have different values from the pairs.

The answer is

$$\binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times 44 = 123{,}552. \qquad \square$$

Example 17 (Rearranging MISSISSIPPI) We are going to count the ways to "rearrange" the letters in the word MISSISSIPPI. Before "rearranging" them, we should be precise about what we mean by "arranging" them. The distinct letters in the word MISSISSIPPI are I, M, S, and P. There are eleven letter positions in the word MISSISSIPPI which we can explicitly label as follows:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \mathrm{M} & \mathrm{I} & \mathrm{S} & \mathrm{S} & \mathrm{I} & \mathrm{S} & \mathrm{S} & \mathrm{I} & \mathrm{P} & \mathrm{P} & \mathrm{I} \end{array}$$

We can describe this placement of letters by a rule such as

$$\mathrm{I} \leftarrow \{2,5,8,11\}, \quad \mathrm{M} \leftarrow \{1\}, \quad \mathrm{P} \leftarrow \{9,10\}, \quad \text{and} \quad \mathrm{S} \leftarrow \{3,4,6,7\}.$$

If we remember the ordering (alphabetic in this case), I, M, P, S, then we can specify this arrangement by the *ordered partition*

$$\big(\{2,5,8,11\}, \{1\}, \{9,10\}, \{3,4,6,7\}\big)$$

of the set $\{1, 2, \ldots, 11\}$.[5] We say that this ordered partition is of *type* $(4, 1, 2, 4)$, referring to the sizes of the sets, in order, that make up the ordered partition. Each of these sets is called a *block* or, in statistics, a *cell*. In general, an ordered partition of a set T of type $(m_1, m_2, \ldots, m_k)$ is a sequence of *disjoint* sets $(B_1, B_2, \ldots, B_k)$ such that $|B_i| = m_i$, $i = 1, 2, \ldots, k$, and $\cup_{i=1}^k B_i = T$. Empty sets are allowed in ordered partitions. The set of all *rearrangements* of the letters in the word MISSISSIPPI corresponds to the set of all ordered partitions (B_1, B_2, B_3, B_4) of $\{1, 2, \ldots, 11\}$ of type $(4, 1, 2, 4)$. For example, the ordered partition $(\{1,5,7,10\}, \{2\}, \{9,11\}, \{3,4,6,8\})$ corresponds to the placement

$$\mathrm{I} \leftarrow \{1,5,7,10\},\ \mathrm{M} \leftarrow \{2\},\ \mathrm{P} \leftarrow \{9,11\}, \text{ and } \mathrm{S} \leftarrow \{3,4,6,8\}$$

and leads to the "word"

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \mathrm{I} & \mathrm{M} & \mathrm{S} & \mathrm{S} & \mathrm{I} & \mathrm{S} & \mathrm{I} & \mathrm{S} & \mathrm{P} & \mathrm{I} & \mathrm{P} \end{array}$$

Another, somewhat picturesque, way of describing ordered partitions of a set T is to think of ordered (i.e., labeled) boxes $(B_1, B_2, \ldots, B_k)$ into which we distribute the elements of T, m_i elements to box B_i, $i = 1, \ldots, k$. The next example takes that point of view and concludes that the number of such distributions of elements into boxes (i.e., the number of ordered partitions) is the *multinomial coefficient*

$$\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1!\, m_2! \cdots m_k!}.$$

As a result, the number of rearrangements of the word MISSISSIPPI is the multinomial coefficient

$$\binom{11}{4,1,2,4} = \frac{11!}{4!\ 1!\ 2!\ 4!} = 34{,}650. \qquad \square$$

[5] Note the use of $(\ldots)$ and $\{\ldots\}$ here: We have a list, indicated by $(\ldots)$. Each element of the list is a set, indicated by $\{\ldots\}$.

Example 18 (Multinomial coefficients) Suppose we are given k boxes labeled 1 through k and an n-set S and we are told to distribute the elements of S among the boxes so that the ith box contains exactly m_i elements. How many ways can this be done?

Let $n = |S|$. Unless $m_1 + \ldots + m_k = n$, the answer is zero because we don't have the right number of objects. Therefore, we assume from now on that

$$m_1 + \ldots + m_k = n.$$

Here's a way to describe filling the boxes.

- Fill the first box (There are $C(n, m_1)$ ways.[6]) AND
- Fill the second box (There are $C(n - m_1, m_2)$ ways.) AND

  • • •

- Fill the kth box. (There are $C(n - (m_1 + \ldots + m_{k-1}), m_k) = C(m_k, m_k) = 1$ ways.)

Now apply the Rule of Product, use the formula $C(p, q) = p!/q!\,(p-q)!$ everywhere, and cancel common factors in numerator and denominator to obtain $n!/m_1!\,m_2! \cdots m_k!$. To illustrate

$$\binom{12}{4}\binom{12-4}{3}\binom{12-4-3}{3} = \frac{12!}{4!\,8!}\,\frac{8!}{3!\,5!}\,\frac{5!}{3!\,2!} = \frac{12!}{4!\,3!\,3!\,2!},$$

which we write $\binom{12}{4,3,3,2}$. In general, this expression is written

$$\binom{n}{m_1, m_2, \ldots, m_k} = \frac{n!}{m_1!\; m_2! \cdots m_k!}$$

where $n = m_1 + m_2 + \ldots + m_k$ and is called a *multinomial coefficient*. In multinomial notation, the binomial coefficient $\binom{n}{k}$ would be written $\binom{n}{k,(n-k)}$. You can think of the first box as the k things that are chosen and the second box as the $n - k$ things that are not chosen.

As in the previous example (Example 17), we can think of the correspondence

$$\begin{aligned} \text{objects being distributed} &\Longleftrightarrow \text{positions in a word} \\ \text{boxes} &\Longleftrightarrow \text{letters.} \end{aligned}$$

If the object "position 3" is placed in the box "D," then the letter D appears as the third letter in the word. The multinomial coefficient is then the number of words that can be made so that letter i appears exactly m_i times. A word can be thought of as a list of its letters. □

[6] Since m_1 things went into the first box, we have only $n - m_1$ left, from which we must choose m_2 for the second box.

Example 19 (Distributing toys) Eleven toys are to be distributed among 4 children. How many ways can this be done if the oldest child is to receive only 2 toys and each of the other children is to receive 3 toys?

We can do this directly if we are used to thinking in terms of multinomial coefficients. We could also do it by converting the problem into one of our previous interpretations.

Here is the first: We want an ordered partition of 11 toys into 4 piles ("blocks") such that the first pile (for the oldest child) contains 2 and each of the 3 remaining piles contain 3 toys. This is an ordered partition of type (2,3,3,3). The number of them is $\binom{11}{2,3,3,3} = 92,400$.

Here is the second: Think of each child as a box into which we place toys. The number of ways to fill the boxes is, again, $\binom{11}{2,3,3,3}$. □

Example 20 (Words from a collection of letters — second try) Using the idea at the end of the previous example, we can more easily count the words that can be made from ERROR, a problem discussed in Example 10. Suppose we want to make words of length k. Let m_1 be the number of E's, m_2 the number of O's and m_3 the number of R's. By considering all possible cases for the number of each letter, you should be able to see that the answer is the sum of $\binom{k}{m_1,m_2,m_3}$ over all m_1, m_2, m_3 such that

$$m_1 + m_2 + m_3 = k, \quad 0 \le m_1 \le 1, \quad 0 \le m_2 \le 1, \quad 0 \le m_3 \le 3.$$

Thus we obtain

$$\begin{aligned}
k=1: &\quad \binom{1}{0,0,1} + \binom{1}{0,1,0} + \binom{1}{1,0,0} = 3 \\
k=2: &\quad \binom{2}{0,0,2} + \binom{2}{0,1,1} + \binom{2}{1,0,1} + \binom{2}{1,1,0} = 7 \\
k=3: &\quad \binom{3}{0,0,3} + \binom{3}{0,1,2} + \binom{3}{1,0,2} + \binom{3}{1,1,1} = 13 \\
k=4: &\quad \binom{4}{0,1,3} + \binom{4}{1,0,3} + \binom{4}{1,1,2} = 20 \\
k=5: &\quad \binom{5}{1,1,3} = 20.
\end{aligned}$$

This is better than in Example 10. Instead of having to list words, we have to list triples of numbers and each triple generally corresponds to more than one word. Here are the lists of triples for the preceding computations

$$\begin{aligned}
k=1: &\quad (0,0,1)\ \ (0,1,0)\ \ (1,0,0) \\
k=2: &\quad (0,0,2)\ \ (0,1,1)\ \ (1,0,1)\ \ (1,1,0) \\
k=3: &\quad (0,0,3)\ \ (0,1,2)\ \ (1,0,2)\ \ (1,1,1) \\
k=4: &\quad (0,1,3)\ \ (1,0,3)\ \ (1,1,2) \\
k=5: &\quad (1,1,3)
\end{aligned}$$

□

Example 21 (Forming teams) How many ways can we form 4 teams from 12 people so that each team has 3 members? This is another multinomial coefficient (ordered set partition) problem and the answer is $\binom{12}{3,3,3,3} = 554,400$.

Wait! We forgot to tell you that the teams don't have names or any other distinguishing features except who the team members are. The solution that gave 554,400 created a list of teams, so there was a Team 1, Team 2, Team 3 and Team 4. We can deal with this the same way we got the formula for counting subsets: To form a list of 4 teams, first form a set and then order it. Since 4 distinct things can be ordered in $4! = 24$ ways, we have $554,400 = 24x$ where x is our answer. We obtain 23,100.

If we told you in the first place that the teams were not ordered, you may not have thought of multinomial coefficients. This leads to two points.

- It may be helpful to impose order and then divide it out.
- We have found a way to count unordered partitions when all the blocks are the same size. This can be extended to the general case of blocks of various sizes but we will not do so.

Wait! We forgot to tell you that we are going to form 4 teams, pair them up to play each other in a contest, say the team with Alice plays the team with Bob, and the other two teams play each other. The winners then play each other. Now we have to form the teams and divide them into pairs that play each other. Let's do that. Suppose we have formed 4 unordered teams. Now we must pair them off. This is another unordered partition: The four teams must be partitioned into two blocks each of size 2. From what we learned in the previous paragraph, we compute $\binom{4}{2,2}$ and divide by 2!, obtaining 3. Thus the answer is $23,100 \times 3 = 69,300$. □

Example 22 (Card hands and multinomial coefficients) To form a full house, we must choose a face value for the triple, choose a face value for the pair, and leave eleven face values unused. This can be done in $\binom{13}{1,1,11}$ ways. We then choose the suits for the triple in $\binom{4}{3}$ ways and the suits for the pair in $\binom{4}{2}$ ways. Note that we choose suits only for the cards in the hand, not for the "unused face values."

To form two pair, we must choose two face values for the pairs, choose a face value for the single card, and leave ten face values unused. This can be done in $\binom{13}{2,1,10}$ ways. We then choose suits for each of the face values in turn, so we must multiply by $\binom{4}{2}\binom{4}{2}\binom{4}{1}$.

Let's imagine an eleven card hand containing two triples, a pair and three single cards. You should be able to see that the number of ways to do this is

$$\binom{13}{2,1,3,7}\binom{4}{3}\binom{4}{3}\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1}. \qquad \square$$

We conclude this section with an introduction to recursions. Let's explore yet another approach to evaluating the binomial coefficient $C(n,k) = \binom{n}{k}$. Let $S = \{x_1, \ldots, x_n\}$. We'll think of $C(n,k)$ as counting k-subsets of S. Either the element x_n is in our subset or it is not. The cases where it is in the subset are all formed by taking the various $(k-1)$-subsets of $S - \{x_n\}$ and adding x_n to them. The cases where it is not in the subset are all formed by taking the various k-subsets of $S - \{x_n\}$. What we've done is describe how to build k-subsets of S from certain subsets of $S - \{x_n\}$. Since this gives each subset exactly once,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

by the Rule of Sum.

The equation $C(n,k) = C(n-1,k-1) + C(n-1,k)$ is called a *recursion* because it tells how to compute $C(n,k)$ from values of the function with smaller arguments. This is a common approach which we can state in general form as follows.

Example 23 (Deriving recursions) To count things, you might ask and answer the question

> How can I construct the things I want to count of a given size by using the same type of things of a smaller size?

This process usually gives rise to a recursion.

Actually, we've cheated a bit in all of this because the recursion only works when we have some values to start with. The correct statement of the recursion is either

$$\begin{aligned}
&C(0,0) = 1,\\
&C(0,k) = 0 \quad \text{for } k \neq 0 \quad \text{and}\\
&C(n,k) = C(n-1,k-1) + C(n-1,k) \quad \text{for } n > 0;
\end{aligned}$$

or

$$\begin{aligned}
&C(1,0) = C(1,1) = 1,\\
&C(1,k) = 0 \quad \text{for } k \neq 0,1 \quad \text{and}\\
&C(n,k) = C(n-1,k-1) + C(n-1,k) \quad \text{for } n > 1;
\end{aligned}$$

depending on how we want to start the computations based on this recursion. Below we have made a table of values for $C(n,k)$. Sometimes this tabular representation of $C(n,k)$ is called "Pascal's Triangle."

$C(n,k)$

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Sometimes it is easier to think in terms of "breaking down" rather than "constructing." That is, ask the question

> How can I break down the things I want to count into smaller things of the same type?

Let's look at the binomial coefficients again. What happens to the k-subsets of the set $S = \{x_1, \ldots, x_n\}$ if we throw away x_n? We then have subsets of $S \setminus \{x_n\} = \{x_1, \ldots, x_{n-1}\}$. The k-subsets of S that did not contain x_n are still k-subsets, but those that contained x_n have become $(k-1)$-subsets. We get all k-subsets and all $(k-1)$-subsets of $S \setminus \{x_n\}$ exactly once when we do this. Thus $C(n,k) = C(n-1,k) + C(n-1,k-1)$ by the Rule of Sum. $\square$

Example 24 (Set partitions) A *partition of a set B* is a collection of nonempty subsets of B such that each element of B appears in exactly one subset. Each subset is called a *block* of the partition. The 15 partitions of $\{1,2,3,4\}$ by number of blocks are

1 block:	$\{\{1,2,3,4\}\}$			
2 blocks:	$\{\{1,2,3\},\{4\}\}$	$\{\{1,2,4\},\{3\}\}$	$\{\{1,2\},\{3,4\}\}$	$\{\{1,3,4\},\{2\}\}$
	$\{\{1,3\},\{2,4\}\}$	$\{\{1,4\},\{2,3\}\}$	$\{\{1\},\{2,3,4\}\}$	
3 blocks:	$\{\{1,2\},\{3\},\{4\}\}$	$\{\{1,3\},\{2\},\{4\}\}$	$\{\{1,4\},\{2\},\{3\}\}$	$\{\{1\},\{2,3\},\{4\}\}$
	$\{\{1\},\{2,4\},\{3\}\}$	$\{\{1\},\{2\},\{3,4\}\}$		
4 blocks:	$\{\{1\},\{2\},\{3\},\{4\}\}$			

Let $S(n,k)$ be the number of partitions of an n-set having exactly k blocks. These are called *Stirling numbers of the second kind*. Do not confuse $S(n,k)$ with $C(n,k) = \binom{n}{k}$. In both cases we have an n-set. For $C(n,k)$ we want to *choose a subset* containing k elements and for $S(n,k)$ we want to *partition the set* into k blocks.

What is the value of $S(n,k)$? Let's try to get a recursion. How can we build partitions of $\{1,2,\ldots,n\}$ with k blocks out of smaller cases? If we take partitions of $\{1,2,\ldots,n-1\}$ with $k-1$ blocks, we can simply add the block $\{n\}$. If we take partitions of $\{1,2,\ldots,n-1\}$ with k blocks, we can add the element n to one of the k blocks. You should convince yourself that all k block partitions of $\{1,2,\ldots,n\}$ arise in exactly one way when we do this. This gives us a recursion for $S(n,k)$. Putting n in a block by itself contributes $S(n-1,k-1)$. Putting n in a block with other elements contributes $S(n-1,k) \times k$ by the Rule of Product. By the Rule of Sum

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k).$$

Let's take a tearing down view. If we remove n from the set $\{1,\ldots,n\}$ and from the block of the partition in which it occurs:

- We get a partition counted by $S(n-1,k-1)$ if n was in a block by itself because that block disappears.
- We get a partition counted by $S(n-1,k)$ if n was in a block with other things. In fact, we get each of these partitions k times since n could have been in any of the k blocks.

This gives us our recursion $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ again.

To illustrate, let's look at what happens when we remove 4 from our earlier list of 3-block partitions:

3 blocks:	$\{\{1,2\},\{3\},\{4\}\}$	$\{\{1,3\},\{2\},\{4\}\}$	$\{\{1,4\},\{2\},\{3\}\}$	$\{\{1\},\{2,3\},\{4\}\}$
	$\{\{1\},\{2,4\},\{3\}\}$	$\{\{1\},\{2\},\{3,4\}\}$		

The partitions with singleton blocks $\{4\}$ removed give us the partitions

$$\{\{1,2\},\{3\}\} \qquad \{\{1,3\},\{2\}\} \qquad \{\{1\},\{2,3\}\}.$$

Thus the partitions counted by $S(3,2)$ each occur once. The partitions in which 4 is not in a singleton block, with 4 removed, give us the partitions

$$\{\{1\},\{2\},\{3\}\} \qquad \{\{1\},\{2\},\{3\}\} \qquad \{\{1\},\{2\},\{3\}\}.$$

Thus the partitions counted by $S(3,3)$ (there's only one) each occur 3 times. Hence $S(4,3) = S(3,2) + 3S(3,3)$.

Below is the tabular form for $S(n,k)$ analogous to the tabular form for $C(n,k)$.

$n \backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1		$S(n,k)$			
3	1	3	1				
4	1	7	6	1			
5	1	15	25	10	1		
6	1	31	90	65	15	1	
7	1	--	--	--	--	--	1

Notice that the starting conditions for this table are that $S(n,1) = 1$ for all $n \geq 1$ and $S(n,n) = 1$ for all $n \geq 1$. The values for $n = 7$ are omitted from the table. You should fill them in to test your understanding of this computational process. For each n, the total number of partitions of a set of size n is equal to the sum $S(n,1) + S(n,2) + \ldots S(n,n)$. These numbers, gotten by summing the entries in the rows of the above table, are called the *Bell numbers*, B_n. For example, $B_4 = 1 + 7 + 6 + 1 = 15$. $\square$

Exercises for Section 3

3.1. How many 6 card hands contain 3 pairs?

3.2. How many 5 card hands contain a straight? A straight is 5 consecutive cards from the sequence A,2,3,4,5,6,7,8,9,10,J,Q,K,A without regard to suit.

3.3. How many compositions of n (sequences of positive integers called "parts" that add to n) are there that have exactly k parts? A composition of 5, for example, corresponds to a placement of either a "+" or a "," in the four spaces between a sequence of five ones: 1 1 1 1 1. Thus, the placement $1\,,1+1\,,1+1$ corresponds to the composition $(1,2,2)$ of 5 which has 3 parts.

3.4. How many rearrangements of the letters in EXERCISES are there?

3.5. In some card games only the values of the cards matter and their suits are irrelevant. Thus there are effectively only 13 distinct cards among 52 total. How many different ways can a deck of 52 cards be arranged in this case? The answer is a multinomial coefficient.

3.6. In a distant land, their names are spelled using the letters A, I, L, S, and T. Each name consists of seven letters. Each name begins and ends with a consonant, contains no adjacent vowels and never contains three adjacent consonants. If two consonants are adjacent, they cannot be the same. An example of a name is LASLASS.

(a) List the first 4 names in dictionary order.

(b) List the last 4 names in dictionary order.

(c) How many names are possible?

3.7. Prove $\binom{n}{k} = \binom{n}{n-k}$ and $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

3.8. For $n > 0$, prove the following formulas for $S(n,k)$:

$$S(n,n) = 1, \qquad S(n,n-1) = \binom{n}{2}, \qquad S(n,1) = 1, \qquad S(n,2) = (2^n-2)/2 = 2^{n-1}-1.$$

3.9. Let B_n be the total number of partitions of an n element set. Thus

$$B_n = S(n,0) + S(n,1) + \cdots + S(n,n).$$

These numbers are called the *Bell numbers*.

(a) Prove that

$$B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_{n-i},$$

where B_0 is defined to be 1.
Hint: Construct the block containing $n+1$ and then construct the rest of the partition. If you prefer tearing down instead of building up, remove the block containing $n+1$.

(b) Calculate B_n for $n \le 5$.

3.10. We consider permutations $a_1, \ldots, a_9$ of 1,2,3,4,5,6,7,8,9.

(a) How many have the property that $a_i < a_{i+1}$ for all $i \le 8$?

(b) How many have the property that $a_i < a_{i+1}$ for all $i \le 8$ except $i = 5$?

Section 4: Probability and Basic Counting

Techniques of counting are very important in probability theory. In this section, we take a look at some of the basic ideas in probability theory and relate these ideas to our counting techniques. This requires, for the most part, a minor change of viewpoint and of terminology.

Let U be a set and suppose for now that U is finite. We think of U as a "universal set" in the sense that we are going to be concerned with various subsets of U and their relationship with each other. In probability theory, the term "universal set" is replaced by *sample space*. Thus, let U be a sample space. We say that we "choose an element of U uniformly at random" if we have a method of selecting an element of U such that all elements of U have the same chance of being selected. This definition is, of course, self referential and pretty sloppy, but it has intuitive appeal to anyone who has selected people for a sports team, or for a favored task at camp, and attempted to be fair about it. We leave it at this intuitive level.

The quantitative way that we say that we are selecting uniformly at random from a sample space U is to say that each element of U has *probability* $1/|U|$ of being selected.

A subset $E \subseteq U$ is called an *event* in probability theory. If we are selecting uniformly at random from U, the probability that our selection belongs to the set E is $|E|/|U|$. At this point, basic probability theory involves nothing more than counting (i.e., we need to count to get $|E|$ and $|U|$).

A more general situation arises when the method of choosing is not "fair" or "uniform." Suppose $U = \{H, T\}$ is a set of two letters, H and T. We select either H or T by taking a coin and flipping it. If "heads" comes up, we choose H, otherwise we choose T. The coin, typically, will be dirty, have scratches in it, etc., so the "chance" of H being chosen might be different from the chance of T being chosen. If we wanted to do a bit of work, we could flip the coin 1000 times and keep some records. Interpreting these records might be a bit tricky in general, but if we came out with 400 heads and 600 tails, we might suspect that tails was more likely. It is possible to be very precise about these sort of experiments (the subject of statistics is all about this sort of thing). But for now, let's just suppose that the "probability" of choosing H is 0.4 and the probability of choosing T is 0.6. Intuitively, we mean by this that if you toss the coin a large number N of times, about $0.4N$ will be heads and $0.6N$ will be tails. The function P with domain $U = \{H, T\}$ and values $P(H) = 0.4$ and $P(T) = 0.6$ is an example of a "probability function" on a sample space U.

The more general definition is as follows:

Definition 4 (Probability function and probability space) *Let U be a finite sample space and let P be a function from U to $\mathbb{R}$ (the real numbers) such that $P(t) \geq 0$ for all t and $\sum_{t \in U} P(t) = 1$.*

- *P is called a probability function on U.*
- *The pair (U, P) is called a probability space.*

- *We extend P to events $E \subseteq U$ by defining $P(E) = \sum_{t \in E} P(t)$. $P(E)$ is called the probability of the event E. (If $t \in U$, we write $P(t)$ and $P(\{t\})$ interchangeably.)*

An element $t \in U$ is called an elementary event or a simple event.

Note that since $P(t) \geq 0$ for all t, it follows from $\sum P(t) = 1$ that $P(t) \leq 1$.

Think of U as a set of elementary events that can occur. Each time we do an experiment or observe something, *exactly one* of the elementary events in U occurs. Imagine repeating this many times. Think of $P(t)$ as the fraction of the cases where the elementary event t occurs. The equation $\sum_{t \in U} P(t) = 1$ follows from the fact that exactly one elementary event occurs each time we do our experiment. Think of $P(E)$ as the fraction of time an elementary event in E occurs.

Theorem 9 (Disjoint events) *Suppose that (U, P) is a probability space and that X and Y are disjoint subsets of U; that is, $X \cap Y = \emptyset$. Then $P(X \cup Y) = P(X) + P(Y)$.*

Proof: By definition, $P(X \cup Y)$ is the sum of $P(t)$ over all $t \in X \cup Y$. If $t \in X \cup Y$, then either $t \in X$ or $t \in Y$, but not both because $X \cap Y = \emptyset$. Thus we can split the sum into two sums, one over $t \in X$ and the other over $t \in Y$. These two sums are $P(X)$ and $P(Y)$, respectively. Thus $P(X \cup Y) = P(X) + P(Y)$.

We could rephrase this using summation notation:

$$P(X \cup Y) = \sum_{t \in X \cup Y} P(t) = \sum_{t \in X} P(t) + \sum_{t \in Y} P(t) = P(X) + P(Y),$$

where we could split the sum into two sums because $t \in X \cup Y$ means that either $t \in X$ or $t \in Y$, but not both because $X \cap Y = \emptyset$. $\square$

Example 25 (Dealing a full house) What is the probability of being dealt a full house? There are $\binom{52}{5}$ distinct hands of cards so we could simply divide the answer 3,744 from Example 15 by this number. That gives the correct answer, but there is another way to think about the problem.

When a hand of cards is dealt, the *order* in which you receive the cards matters: Thus receiving 3♠ 6♢ 2♡ in that order is a different dealing of the cards than receiving 2♡ 3♠ 6♢ in that order. Thus, we regard each of the $52 \times 51 \times 50 \times 49 \times 48$ ways of dealing five cards from 52 as equally likely. Thus each hand has probability $1/52 \times 51 \times 50 \times 49 \times 48$. Since all the cards in a hand of five cards are different, they can be ordered in 5! ways. Hence the probability of being dealt a full house is $\frac{3{,}774 \times 5!}{52 \times 51 \times 50 \times 49 \times 48}$, which does indeed equal 3,744 divided by $\binom{52}{5}$.

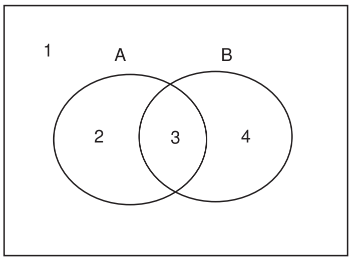
If cards are not all distinct and if we are not careful, the two approaches give different answers. The first approach gives the wrong answer. We now explain why. Be prepared to think carefully, because this is a difficult concept for beginning students.

To illustrate consider a deck of 4 cards that contains two aces of spades and two jacks of diamonds. There are 3 possible two card hands: 2 aces, 1 ace and 1 jack, or 2 jacks, but the probability of getting two aces is only 1/6. Can you see how to calculate that correctly?

We can look at this in at least two ways. Suppose we are being dealt the top two cards. The probability of getting two aces equals the fraction of ways to assign positions to cards so that the top two are given to aces. There are $\binom{4}{2}$ ways to assign positions to aces and only one of those results in the aces being in the top two positions.

Here's the other way to look at it: Mark the cards so that the aces can be told apart, and the jacks can be told apart, say A_1, A_2, J_1, and J_2. Since the cards are distinct each hand can be ordered in the same number of ways, namely 2!, and so we can count ordered or unordered hands. There are now $\binom{4}{2}$ unordered hands (or 4×3 ordered ones) and only one of these (or 2×1 ordered ones) contain A_1 and A_2. $\square$

Example 26 (Venn diagrams and probability) A "Venn diagram" shows the relationship between elements of sets. The interior of the rectangle in the following figure represents the sample space U. The interior of each of the circular regions represents the events A and B.



Let's list what each of the regions in the figure are:

$$1 \text{ is } (A \cup B)^c \qquad 2 \text{ is } A - B \qquad 3 \text{ is } A \cap B \qquad 4 \text{ is } B - A.$$

We can compute either set cardinalities or probabilities. For example, $U \setminus A$ is all of U except what is in the region labeled A. Thus $|U \setminus A| = |U| - |A|$. On the other hand, A and A^c partition the sample space and so $P(A) + P(A^c) = 1$. Rewriting this as

$$P(A^c) = 1 - P(A)$$

puts it in the same form as $|U \setminus A| = |U| - |A|$ since $U \setminus A = A^c$. Notice that the only difference between the set and probability equations is the presence of the function P and the fact that $P(U) = 1$. Also notice that the probability form did *not* assume that the probability was uniformly at random.

What about $A \cup B$? It corresponds to the union of the disjoint regions labeled 2, 3 and 4 in the Venn diagram. Thus

$$P(A \cup B) = P(A - B) + P(A \cap B) + P(B - A)$$

by Theorem 9. We can express $P(A - B)$ in terms of $P(A)$ and $P(A \cap B)$ because A is the disjoint union of $A - B$ and $A \cap B$: $P(A) = P(A - B) + P(A \cap B)$. Solving for $P(A - B)$ and writing a similar expression for $P(B - A)$:

$$P(A - B) = P(A) - P(A \cap B) \qquad\qquad P(B - A) = P(B) - P(A \cap B).$$

Combining our previous results

$$P(A \cup B) = P(A - B) + P(A \cap B) + P(B - A) = P(A) + P(B) - P(A \cap B).$$

There is a less formal way of saying this. If we take A and B we get the region labeled 3 twice — once in A and once in B. The region labeled 3 corresponds to $A \cap B$ since it is the region that belongs to both A and B. Thus $|A| + |B|$ gives us regions 2, 3 and 4 (which is $|A \cup B|$) and a second copy of 3, (which is $|A \cap B|$). We have shown that

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

The probability form is $P(A) + P(B) = P(A \cup B) + P(A \cap B)$. We can rewrite this as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

(This is the two set case of the *Principle of Inclusion and Exclusion*.)

One more example: Using DeMorgan's Rule from Theorem 6, $(A \cup B)^c = A^c \cap B^c$. (Check this out in the Venn diagram.) Combining the results of the two previous paragraphs,

$$\begin{aligned} P(A^c \cap B^c) = 1 - P(A \cup B) &= 1 - \Big(P(A) + P(B) - P(A \cap B)\Big) \\ &= 1 - P(A) - P(B) + P(A \cap B). \end{aligned}$$

This is another version of the Principle of Inclusion and Exclusion. □

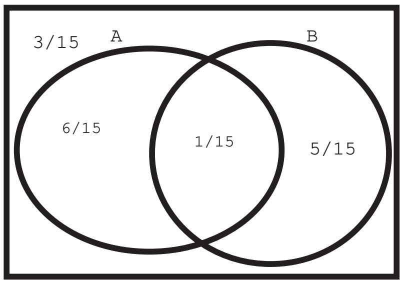
Example 27 (Combining events) Let U be a sample space with probability function P. Let A and B be events. Suppose we know that

- A occurs with probability 7/15,
- B occurs with probability 6/15, and
- the probability that neither of the events occurs is 3/15.

What is the probability that both of the events occur?

Let's translate the given information into mathematical notation. The first two data are easy: $P(A) = 7/15$ and $P(B) = 6/15$. What about the last? What is the event corresponding to neither of A and B occurring? One person might say $A^c \cap B^c$; another might say $(A \cup B)^c$. Both are correct by DeMorgan's Rule. Thus the third datum can be written $P((A \cup B)^c) = P(A^c \cap B^c) = 3/15$. We are asked to find $P(A \cap B)$.

What do we do now? A Venn diagram can help. The situation is shown in the following Venn diagram for A and B. The rectangle stands for U, the whole sample space. (We've put in some numbers that we haven't computed yet, so you should ignore them.)



We have been given just partial information, namely $P(A) = 7/15$, $P(B) = 6/15$, and $P((A \cup B)^c) = 3/15$. The best way to work such problems is to use the information given to, if possible, find the probabilities of the four fundamental regions associated with A and B, namely the regions

$$(A \cup B)^c \qquad A - B \qquad B - A \qquad A \cap B.$$

(You should identify and label the regions in the figure.) Recall that $P(E^c) = 1 - P(E)$ for any event E. Thus

$$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - 3/15 = 12/15.$$

From this we get (check the Venn diagram)

$$P(A - B) = P(A \cup B) - P(B) = 12/15 - 6/15 = 6/15.$$

Similarly, $P(B - A) = 12/15 - 7/15 = 5/15$. Finally,

$$P(A \cap B) = P(A) - P(A - B) = 7/15 - 6/15 = 1/15.$$

The answer to the question we were asked at the beginning is that $P(A \cap B) = 1/15$. $\square$

Example 28 (Odds and combining events) Let U be a sample space and let A and B be events where the odds of A occurring are 1:2, the odds of B occurring are 5:4 and the odds of both A and B occurring are 1:8. Find the odds of neither A nor B occurring. In popular culture, probabilities are often expressed as *odds*. If an event E occurs with odds $a : b$, then it occurs with $P(E) = a/(a + b)$. Thus, $P(A) = 1/3$, $P(B) = 5/9$, and $P(A \cap B) = 1/9$. From the equation $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ in Example 26, $P(A \cup B) = 7/9$. From the equation $P(A^c) = 1 - P(A)$ in that example, with $A \cup B$ replacing A, we have $P((A \cup B)^c) = 2/9$. The odds of neither A nor B occurring are 2:7.

Caution: It is not always clear what odds mean. If someone says that the odds on Beatlebomb in a horse race are 100:1, this means that the probability is $100/(100 + 1)$ that Beatlebomb will *lose*. The probability that he will win is $1/(100 + 1)$. $\square$

Example 29 (Hypergeometric probabilities) Six light bulbs are chosen at random from 18 light bulbs of which 8 are defective. What is the probability that exactly two of the chosen bulbs are defective? We'll do the general situation. Let

- B denote the total number of bulbs,
- D the total number of defective bulbs, and
- b the number of bulbs chosen.

Let the probability space be the set of all $\binom{B}{b}$ ways to choose b bulbs from B and let the probability be uniform. Let $E(B, D, b, d)$ be the event consisting of all selections of b from B when a total of D bulbs are defective and d of the selected bulbs are defective. We want $P(E(B, D, b, d))$. The total number of ways to choose b, of which exactly d are defective, is $\binom{D}{d}\binom{B-D}{b-d}$. To see this, first choose d bulbs from the D defective bulbs and then choose $b-d$ bulbs from the $B-D$ good bulbs. Thus,

$$P(E(B,D,b,d)) = \frac{\binom{D}{d}\binom{B-D}{b-d}}{\binom{B}{b}}.$$

Substituting $B = 18$, $b = 6$, $D = 8$, and $d = 2$ gives $P(E(18,8,6,2)) = 0.32$, the answer to our original question.

The function $P(E(B,D,b,d))$ occurs frequently. It is called the *hypergeometric probability distribution.* $\square$

Example 30 (Sampling with replacement from six cards) First one card and then a second card are selected at random, with replacement, from 6 cards numbered 1 to 6. What is the probability that the sum of the values on the cards equals 7? That the sum of the values of the cards is divisible by 5? Since both cards are selected from the same set of cards numbered one to six, this process is called "sampling with replacement." The idea is that one can choose the first card, write down its number, *replace* it and repeat the process a second (or more) times. The basic sample space is $S \times S = \{(i,j) \mid 1 \le i \le 6,\ 1 \le j \le 6\}$. Every point in this sample space is viewed as equally likely. Call the two events of interest E_7 (sum equals 7) and D_5 (sum divisible by 5). It is helpful to have a way of visualizing $S \times S$. This can be done as follows:

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The 6×6 rectangular array has 36 squares. The square with row label i and column label j corresponds to $(i,j) \in S \times S$. The rectangular array on the right has the sum $i+j$ in square (i,j). Thus

$$E_7 = \{(i,j) : 1 \le i \le 6,\ 1 \le j \le 6,\ i+j = 7\}$$

corresponds to six points in the sample space and so $P(E_7) = |E_7|/36 = 6/36$.

A number k is divisible by 5 if $k = 5j$ for some integer j. In that case, we write $5|k$. Thus

$$D_5 = \{(i,j) : 1 \le i \le 6,\ 1 \le j \le 6,\ 5|(i+j)\}$$

and so $D_5 = E_5 \cup E_{10}$. Finally, $|D_5| = 4 + 3 = 7$ and $P(D_5) = 7/36$. $\square$

Example 31 (Girls and boys sit in a row) Four girls and two boys sit in a row. Find the probability that each boy has a girl to his left and to his right. Suppose that the girls are g_1, g_2, g_3, g_4 and the boys are b_1, b_2. There are $6! = 720$ ways of putting these six people in a row. This set of 720 such permutations is the sample space S, and we assume each permutation is equally likely. Let S_g denote the set of such permutations where each boy has a girl on his left and one on his right. There are three patterns where each boy has a girl on both his left and his right: *gbgbgg*, *gbggbg*, and *ggbgbg*. For each pattern, there are $2!\,4! = 48$ ways of placing the girls and boys into that pattern. Thus, $(3 \times 48)/6! = 144/720 = 1/5$ is the required probability. Note that we could have also taken the sample space to be the set of $\binom{6}{2}$ patterns. Each pattern is equally likely since each arises from the same number of arrangements of the 6 children. The probability would then be computed as $3/\binom{6}{2} = 3/15 = 1/5$. □

Example 32 (Dealing cards from a standard deck of 52 cards) A man is dealt 4 spade cards from an ordinary deck of 52 cards. If he is given five more cards, what is the probability that three of them are spades? This is another example of the hypergeometric probability distribution. There are $B = 48$ cards remaining, $D = 9$ of them spades. We ask for the probability that, from $b = 5$ cards selected, $d = 3$ are spades.

$$P(E(B,D,b,d)) = \frac{\binom{D}{d}\binom{B-D}{b-d}}{\binom{B}{b}} = \frac{\binom{9}{3}\binom{39}{2}}{\binom{48}{5}} = 0.036\,. \qquad \square$$

Example 33 (Selecting points at random from a square) Suppose we have a square with side s and inside it is a circle of diameter $d \le 1$. A point is selected uniformly at random from the square. What is the probability that the point selected lies inside the circle?

We haven't defined probability for infinite sample spaces. The intuition is that probability is proportional to area — a "geometric probability" problem. Thus we have

$$P(E) = \frac{\text{area}(E)}{\text{area}(U)},$$

where U is the sample space, which is the set of points in the square. Computing areas, we obtain $P = \pi d^2/(4s^2)$. This is the correct answer. Clearly, this answer doesn't depend on the figure being a circle. It could be any figure of area $\pi d^2/4$ that fits inside the square. □

The next example deals with the following question: If k items are randomly put one at a time into n boxes what are the chances that no box contains more than one item? A related problem that can be dealt with in a similar manner is the following: If I choose k_1 items at random from n items and you choose k_2 items from the same n, what are the chances that our choices contain no items in common? These problems arise in the analysis of some algorithms.

***Example 34 (The birthday problem)** Assume that all days of the year are equally likely to be birthdays and ignore leap years. If k people are chosen at random, what is the probability that they all have different birthdays? While we're at it, let's replace the number of days in a year with n.

Here's one way we can think about this. Arrange the people in a line. Their birthdays, listed in the same order as the people, are $b_1, b_2, \ldots, b_k$. The probability space is $\underline{n} \times \underline{n} \times \cdots \times \underline{n}$, where there are k copies of $\underline{n}$. Each of the n^k possible k-long lists are equally likely. We are interested in $P(A)$, where A consists of those lists without repeats. Thus $|A| = n(n-1)\cdots(n-(k-1))$ and so

$$P(A) = \frac{n(n-1)\cdots(n-(k-1))}{n^k} = \prod_{i=1}^{k-1} \frac{n-i}{n} = \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right).$$

While this answer is perfectly correct, it does not give us any idea how large $P(A)$ is. Of course, if k is very small, $P(A)$ will be nearly 1, and, if k is very large, $P(A)$ will be nearly 0. (In fact $P(A) = 0$ if $k > n$. This can be proved by using our formula. You should do it.) Where does this transition from near 1 to near 0 occur and how does $P(A)$ behave during the transition? Our goal is to answer this question.

We now suppose that $k \le n^{3/5}$. Why 3/5? Just accept that this is a good choice. We need the following fact which will be proved at the end of the example:

$$\text{If } 0 \le x \le 1/2, \text{ then } e^{-x-x^2} \le 1 - x \le e^{-x}.$$

First, we get an upper bound for $P(A)$. Using $1 - x \le e^{-x}$ with $x = i/n$,

$$P(A) = \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \le \prod_{i=1}^{k-1} e^{-i/n} = \exp\left(-\sum_{i=1}^{k-1} i/n\right).$$

Using the formula[7] $1 + 2 + \cdots + N = N(N+1)/2$ with $N = k-1$, we have

$$\sum_{i=1}^{k-1} \frac{i}{n} = \frac{(k-1)k}{2n} = \frac{k^2}{2n} - \frac{k}{2n}.$$

Thus $P(A) \le e^{-k^2/2n} e^{k/2n}$. Since $0 \le k/n \le n^{3/5}/n = n^{-2/5}$, which is small when n is large, $e^{k/2n}$ is close to 1. Thus, we have an upper bound for $P(A)$ that is close to $e^{-k^2/2n}$.

Next, we get our lower bound for $P(A)$ From the other inequality in our fact, namely $1 - x \ge e^{-x-x^2}$, we have

$$P(A) = \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \ge \prod_{i=1}^{k-1} e^{-i/n-(i/n)^2} = \left(\prod_{i=1}^{k-1} e^{-i/n}\right)\left(\prod_{i=1}^{k-1} e^{-(i/n)^2}\right).$$

Let's look at the last product. It is less than 1. Since $i < k$, all of the factors in the product are greater than $e^{-(k/n)^2}$. Since there are less than k factors, the product is greater than

[7] This is a formula you should have learned in a previous class.

e^{-k^3/n^2}. Since $k \le n^{3/5}$, $k^3/n^2 \le n^{9/5}/n^2 = n^{-1/5}$, which is small when n is large. Thus the last product is close to 1 when n is large. This shows that $P(A)$ has a lower bound which is close to $\prod_{i=1}^{k-1} e^{-i/n}$, which is the upper bound estimate we got in the previous paragraph. Since our upper and lower bounds are close together, they are both close to $P(A)$. In the previous paragraph, we showed that the upper bound is close to $e^{-k^2/2n}$.

To summarize, we have shown that

If n is large and $k \le n^{3/5}$, then $P(A)$ is close to $e^{-k^2/2n}$.

What happens when $k > n^{3/5}$?

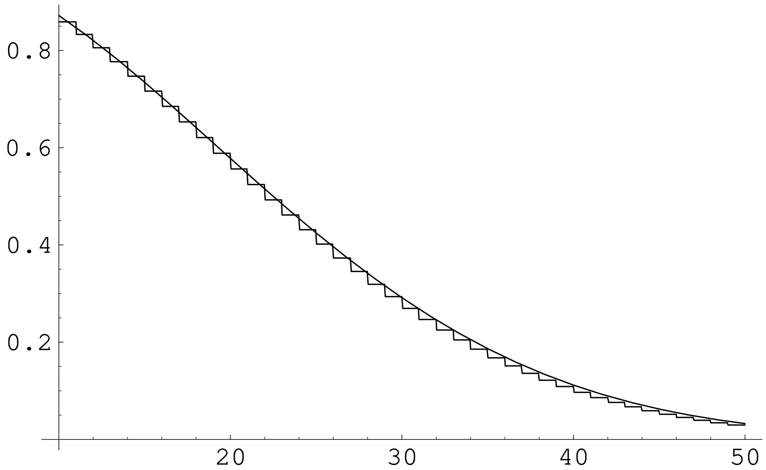
- First note that $P(A)$ decreases as k increases. You can see this by thinking about the original problem. You can also see it by looking at the product we obtained for $P(A)$, noting that each factor is less than 1 and noting that we get more factors as k increases.
- Second note that, when k is near $n^{3/5}$ but smaller than $n^{3/5}$, then $k^2/2n$ is large and so $P(A)$ is near 0 since e to a large negative power is near 0.

Putting these together, we see that $P(A)$ must be near 0 when $k \ge n^{3/5}$.

How does $e^{-k^2/2n}$ behave? When k is much smaller than $\sqrt{n}$, $k^2/2n$ is close to 0 and so $e^{-k^2/2n}$ is close to 1. When k is much larger than $\sqrt{n}$, $k^2/2n$ is large and so $e^{-k^2/2n}$ is close to 0. Put in terms of birthdays, for which $n = 365$ and $\sqrt{365} \approx 19$:

- When k is much smaller than 19, the probability of distinct birthdays is nearly 1.
- When k is much larger than 19, the probability of distinct birthdays is nearly 0.
- In between, the probability of distinct birthdays is close to $e^{-k^2/(2\times 365)}$.

Here's a graph of $P(A)$ ("staircase" curve) and the approximation function $e^{-k^2/2n}$ (smooth curve) for various values of k when $n = 365$.[8] As you can see, the approximation is quite accurate.



[8] Since $P(A)$ is defined only for k an integer, it should be a series of dots. To make it more visible, we've plotted it as a step function (staircase). The approximation is given by the function $e^{-x^2/2n}$, which is a smooth curve.

We now prove our fact. It requires calculus. By Taylor's theorem, the series for the natural logarithm is

$$\ln(1-x) = -\sum_{i=1}^{\infty} \frac{x^k}{k}\,.$$

Since $x > 0$ all the terms in the sum are negative. Throwing away all but the first term, $\ln(1-x) < -x$. Exponentiating, we have $1 - x < e^{-x}$, which is half of our fact.

Note that

$$\sum_{k=1}^{\infty} x^k/k = x + x^2 \sum_{k=2}^{\infty} x^{k-2}/k.$$

Since $0 \le x \le 1/2$ and $k \ge 2$ in the second sum, we have $x^{k-2}/k \le (1/2)^{k-2}/2$. By the formula for the sum of a geometric series,[9]

$$\sum_{k=2}^{\infty} (1/2)^{k-2}/2 = 1/2 + (1/2)^2 + (1/2)^3 + \cdots = \frac{1/2}{1-1/2} = 1.$$

Thus

$$\sum_{k=1}^{\infty} x^k/k = x + x^2 \sum_{k=2}^{\infty} x^{k-2}/k \le x + x^2,$$

and so $\ln(1-x) \ge -x - x^2$, which gives us the other half of our fact. $\square$

Exercises for Section 4

4.1. Six horses are in a race. You pick two of them at random and bet on them both. Find the probability that you picked the winner. State clearly what your probability space is.

4.2. A roulette wheel consists of 38 containers numbered 0 to 36 and 00. In a fair wheel the ball is equally likely to fall into each container. A special wheel is designed in which all containers are the same size except that 00 is 5% larger than any of the others so that 00 has a 5% greater chance of occurring than any of the other values. What is the probability that 00 will occur on a spin of the wheel?

4.3. Alice and Bob have lost a key at the beach. They each get out their metal detectors and hunt until the key is found. If Alice can search 20% faster than Bob, what are the odds that she finds the key? What is the probability that Alice finds the key?

[9] Recall that the sum of the geometric series $a + ar + ar^2 + \cdots$ is $a/(1-r)$. You should be able to see that here $a = 1/2$ and $r = 1/2$.

Basic Counting and Listing

4.4. Six horses are in a race. You pick two of them at random and bet on them both. Find the probability that you picked a horse that won or placed (came in second). This should include the possibility that one of your picks won and the other placed.

4.5. Suppose 4 different balls are placed into 4 labeled boxes at random. (This can be done in 4^4 ways.)

(a) What is the probability that no box is empty?

(b) What is the probability that exactly one box is empty?

(c) What is the probability that at least one box is empty?

(d) Repeat (a)–(c) if there are 5 balls and 4 boxes.

4.6. For each event E determine $P(E)$.

(a) Suppose a fair die is thrown k times and the values shown are recorded. What is the sample space? What is the probability of the event E that the sum of the values is even?

(b) A card is drawn uniformly at random from a regular deck of cards. This process is repeated n times, with replacement. What is the sample space? What is the probability that a king, K, doesn't appear on any of the draws? What is the probability that at least one K appears in n draws?

(c) An urn contains 3 white, 4 red, and 5 blue marbles. Two marbles are drawn without replacement. What is the sample space? What is the probability that both marbles are red?

4.7. Six light bulbs are chosen at random from 15 bulbs of which 5 are defective. What is the probability that exactly 3 are defective?

4.8. An urn contains ten labeled balls, labels $1, 2, \ldots, 10$.

(a) Two balls are drawn together. What is the sample space? What is the probability that the sum of the labels on the balls is odd?

(b) Two balls are drawn one after the other without replacement. What is the sample space? What is the probability that the sum is odd?

(c) Two balls are drawn one after the other with replacement. What is the sample space? What is the probability that the sum is odd?

4.9. Let A and B be events with $P(A) = 3/8$, $P(B) = 1/2$, and $P((A \cup B)^c) = 3/8$. What is $P(A \cap B)$?

4.10. Of the students at a college, 20% are computer science majors and 58% of the entire student body are women. 430 of the 5,000 students at the college are women majoring in computer science.

(a) How many women are not computer science majors?

(b) How many men are not computer science majors?

(c) What is the probability that a student selected at random is a woman computer science major?

(d) What is the probability that a female student selected at random is a computer science major?

4.11. The odds on the horse Beatlebomb in the Kentucky Derby are 100 to 1. A man at the races tells his wife that he is going to flip a coin. If it comes up heads he will bet on Beatlebomb, otherwise he will skip this race and not bet. What is the probability that he bets on Beatlebomb and wins?

4.12. Four persons, called North, South, East, and West, are each dealt 13 cards from an ordinary deck of 52 cards. If South has exactly two aces, what is the probability that North has the other two aces?

4.13. You have been dealt 4 cards and discover that you have 3 of a kind; that is, 3 cards have the same face value and the fourth is different. For example, you may have been dealt 4♠ 4♡ 10♠ 4♣. The other three players each receive four cards, but you do not know what they have been dealt. What is the probability that the fifth card will improve your hand by making it 4 of a kind or a full house (3 of a kind and a pair)?

4.14. Three boys and three girls are lined up in a row.

(a) What is the probability of all three girls being together?

(b) Suppose they are then seated around a circular table with six seats in the same order they were lined up. What is the probability that all three girls sit together?

4.15. Prove the *principle of inclusion exclusion*, for three sets namely that

$$P(A^c \cap B^c \cap C^c) = 1 - P(A) - P(B) - P(C) + P(A \cap B) + P(A \cap C) + P(B \cap C) - P(A \cap B \cap C).$$

(The formula extends in a fairly obvious way to any number of sets.)
Hint: Recall that, that for two sets, $P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B)$.

4.16. A point is selected uniformly at random on a stick. This stick is broken at this point. What is the probability that the longer piece is at least twice the length of the shorter piece?

4.17. Two points are selected uniformly at random on a stick of unit length. The stick is broken at these two points. What is the probability that the three pieces form a triangle?

***4.18.** What is the probability that a coin of diameter $d \leq 1$ when tossed onto the Euclidean plane (i.e., $\mathbb{R} \times \mathbb{R}$, $\mathbb{R}$ the real numbers) covers a lattice point of the plane

(i.e., a point (p, q), where p and q are integers)?
Hint: Compare this problem with Example 33.

***4.19.** Three points are selected at random on a circle C. What is the probability that all three points lie on a common semicircle of C? What if 3 is replaced by k?

Multiple Choice Questions for Review

1. Suppose there are 12 students, among whom are three students, M, B, C (a Math Major, a Biology Major, a Computer Science Major). We want to send a delegation of four students (chosen from the 12 students) to a convention. How many ways can this be done so that the delegation includes *exactly* two (not more, not less) students from $\{M, B, C\}$?

(a) 32 (b) 64 (c) 88 (d) 108 (e) 144

2. The permutations of $\{a, b, c, d, e, f, g\}$ are listed in lex order. What permutations are just before and just after $bacdefg$?

(a) Before: $agfedbc$, After: $bacdfge$

(b) Before: $agfedcb$, After: $badcefg$

(c) Before: $agfebcd$, After: $bacedgf$

(d) Before: $agfedcb$, After: $bacdfge$

(e) Before: $agfedcb$, After: $bacdegf$

3. Teams A and B play in a basketball tournament. The first team to win two games in a row or a total of three games wins the tournament. What is the number of ways the tournament can occur?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

4. The number of four letter words that can be formed from the letters in BUBBLE (each letter occurring at most as many times as it occurs in BUBBLE) is

(a) 72 (b) 74 (c) 76 (d) 78 (e) 80

5. The number of ways to seat 3 boys and 2 girls in a row if each boy must sit next to at least one girl is

(a) 36 (b) 48 (c) 148 (d) 184 (e) 248

6. Suppose there are ten balls in an urn, four blue, four red, and two green. The balls are also numbered 1 to 10. How many ways are there to select an ordered sample of four balls without replacement such that there are two blue balls and two red balls in the sample?

(a) 144 (b) 256 (c) 446 (d) 664 (e) 864

7. How many different rearrangements are there of the letters in the word BUBBLE?

(a) 40 (b) 50 (c) 70 (d) 80 (e) 120

8. The English alphabet has 26 letters of which 5 are vowels (A,E,I,O,U). How many seven letter words, with all letters distinct, can be formed that start with B, end with the letters ES, and have exactly three vowels? The "words" for this problem are just strings of letters and need not have linguistic meaning.

(a) $2^3 \times 3^4 \times 17$

(b) $2^3 \times 3^4 \times 19$

(c) $2^4 \times 3^4 \times 19$

(d) $2^4 \times 3^3 \times 19$

(e) $2^4 \times 3^3 \times 17$

9. The permutations on $\{a, b, c, d, e, f, g\}$ are listed in lex order. All permutations $x_1x_2x_3x_4x_5x_6x_7$ with $x_4 = a$ or $x_4 = c$ are kept. All others are discarded. In this reduced list what permutation is just after $dagcfeb$?

(a) $dbacefg$

(b) $dbcaefg$

(c) $dbacgfe$

(d) $dagcfbe$

(e) $dcbaefg$

10. The number of four letter words that can be formed from the letters in SASSABY (each letter occurring at most as many times as it occurs in SASSABY) is

(a) 78 (b) 90 (c) 108 (d) 114 (e) 120

11. How many different rearrangements are there of the letters in the word TATARS if the two A's are never adjacent?

(a) 24 (b) 120 (c) 144 (d) 180 (e) 220

12. Suppose there are ten balls in an urn, four blue, four red, and two green. The balls are also numbered 1 to 10. How many ways are there to select an *ordered* sample of four balls without replacement such that the number $B \geq 0$ of blue balls, the number $R \geq 0$ of red balls, and the number $G \geq 0$ of green balls are all different?

(a) 256 (b) 864 (c) 1152 (d) 1446 (e) 2144

13. Suppose there are ten balls in an urn, four blue, four red, and two green. The balls are also numbered 1 to 10. You are asked to select an *ordered* sample of four balls without replacement. Let $B \geq 0$ be the number of blue balls, $R \geq 0$ be the number of red balls, and $G \geq 0$ be the number of green balls in your sample. How many ways are there to select such a sample if *exactly* one of B, R, or G must be zero?

(a) 256 (b) 1152 (c) 1446 (d) 2144 (e) 2304

14. The number of partitions of $X = \{a, b, c, d\}$ with a and b in the same block is

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

15. Let W_{ab} and W_{ac} denote the set of partitions of $X = \{a, b, c, d, e\}$ with a and b belonging to the same block and with a and c belonging to the same block, respectively. Similarly, let W_{abc} denote the set of partitions of $X = \{a, b, c, d, e\}$ with a, b, and c belonging to the same block. What is $|W_{ab} \cup W_{ac}|$? (Note: $B(3) = 5$, $B(4) = 15$, $B(5) = 52$, where $B(n)$ is the number of partitions of an n-element set).

(a) 25 (b) 30 (c) 35 (d) 40 (e) 45

16. The number of partitions of $X = \{a, b, c, d, e, f, g\}$ with a, b, and c in the same block and c, d, and e in the same block is

(a) 2 (b) 5 (c) 10 (d) 15 (e) 52

17. Three boys and four girls sit in a row with all arrangements equally likely. Let x be the probability that no two boys sit next to each other. What is x?

(a) 1/7 (b) 2/7 (c) 3/7 (d) 4/7 (e) 5/7

18. A man is dealt 4 spade cards from an ordinary deck of 52 cards. He is given 2 more cards. Let x be the probability that they both are the same suit. Which is true?

(a) $.2 < x \leq .3$

(b) $0 < x \leq .1$

(c) $.1 < x \leq .2$

(d) $.3 < x \leq .4$

(e) $.4 < x \leq .5$

19. Six light bulbs are chosen at random from 15 bulbs of which 5 are defective. What is the probability that exactly one is defective?

(a) $C(5,1)C(10,6)/C(15,6)$

(b) $C(5,1)C(10,5)/C(15,6)$

(c) $C(5,1)C(10,1)/C(15,6)$

(d) $C(5,0)C(10,6)/C(15,6)$

(e) $C(5,0)C(10,5)/C(15,6)$

20. A small deck of five cards are numbered 1 to 5. First one card and then a second card are selected at random, with replacement. What is the probability that the sum of the values on the cards is a prime number?

(a) 10/25 (b) 11/25 (c) 12/25 (d) 13/25 (e) 14/25

21. Let A and B be events with $P(A) = 6/15$, $P(B) = 8/15$, and $P((A \cup B)^c) = 3/15$. What is $P(A \cap B)$?

(a) 1/15 (b) 2/15 (c) 3/15 (d) 4/15 (e) 5/15

22. Suppose the odds of A occurring are 1:2, the odds of B occurring are 5:4, and the odds of both A and B occurring are 1:8. The odds of $(A \cap B^c) \cup (B \cap A^c)$ occurring are

(a) 2:3 (b) 4:3 (c) 5:3 (d) 6:3 (e) 7:3

23. A pair of fair dice is tossed. Find the probability that the greatest common divisor of the two numbers is one.

(a) 12/36 (b) 15/36 (c) 17/36 (d) 19/36 (e) 23/36

24. Three boys and three girls sit in a row. Find the probability that exactly two of the girls are sitting next to each other (the remaining girl separated from them by at least one boy).

(a) 4/20 (b) 6/20 (c) 10/20 (d) 12/20 (e) 13/20

25. A man is dealt 4 spade cards from an ordinary deck of 52 cards. If he is given five more, what is the probability that none of them are spades?

Basic Counting and Listing

(a) $\binom{39}{1}/\binom{48}{5}$ (b) $\binom{39}{2}/\binom{48}{5}$ (c) $\binom{39}{3}/\binom{48}{5}$ (d) $\binom{39}{5}/\binom{48}{5}$ (e) $\binom{39}{6}/\binom{48}{5}$

Answers: **1** (d), **2** (e), **3** (c), **4** (a), **5** (a), **6** (e), **7** (e), **8** (c), **9** (a), **10** (d), **11** (b), **12** (c), **13** (e), **14** (b), **15** (a), **16** (b), **17** (b), **18** (a), **19** (b), **20** (b), **21** (b), **22** (d), **23** (e), **24** (d), **25** (d).