1. Let $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. By the definition of the cross product, $\mathbf{C}$ is perpendicular to $\mathbf{A}$ and $\mathbf{C} \times \mathbf{A}$ is perpendicular to $\mathbf{C}$ and $\mathbf{A}$. Since the length of a cross product of two perpendicular vectors is the product of their lengths, $|\mathbf{C} \times \mathbf{A}| = |\mathbf{C}|\,|\mathbf{A}|$.

2. Since $\partial\mathbf{R}/\partial u = \cos v\mathbf{i} - \sin v\mathbf{j} - 2u\mathbf{k}$ and $\partial\mathbf{R}/\partial v = -u\sin v\mathbf{i} - u\cos\mathbf{j}$, we have the normal

$$\mathbf{N} \;=\; \frac{\partial\mathbf{R}}{\partial u} \times \frac{\partial\mathbf{R}}{\partial v} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & -\sin v & -2u \\ -u\sin v & -u\cos v & 0 \end{vmatrix} \;=\; -2u^2\cos v\mathbf{i} + 2u^2\sin v\mathbf{j} - u\mathbf{k}.$$

Since $|\mathbf{N}| = \sqrt{4u^4\cos^2 v + 4u^4\sin^2 v + u^2} = |u|\sqrt{4u^2+1}$ and $u \geq 0$, the desired answer is

$$\frac{-\mathbf{N}}{|\mathbf{N}|} \;=\; \frac{2u\cos v\mathbf{i}}{\sqrt{4u^2+1}} - \frac{2u\sin v\mathbf{j}}{\sqrt{4u^2+1}} + \frac{\mathbf{k}}{\sqrt{4u^2+1}}.$$

3. This is assigned homework problem Section 4.1 number 4.

4. Since div curl is zero, $0 = \nabla\cdot(\nabla\times\mathbf{H}) = \nabla\cdot(\mathbf{F}\times\mathbf{R})$. Use the identity $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$ with $\mathbf{H}$ replaced by $\mathbf{R}$ to get

$$0 \;=\; \mathbf{R}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{R}) \;=\; \mathbf{R}\cdot(\nabla\times\mathbf{F}),$$

where the last equality comes from $\nabla\times\mathbf{R} = \mathbf{0}$.
You may know or have on your sheet $\nabla\times\mathbf{R} = \mathbf{0}$, or you could compute $\nabla\times\mathbf{R}$ to get $\mathbf{0}$, or you might note that $\mathbf{R}$ is the gradient of $(x^2+y^2+z^2)/2$ and that curl grad is zero.

5. Let V be the region enclosed by S. By the Divergence Theorem

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} \;=\; \iiint_V \nabla\cdot(\nabla\times\mathbf{F})\,dV,$$

which is zero since div curl is zero.

6. By the Divergence Theorem, the integral over the closed surface consisting of S and the disk D in the xy-plane with boundary $x^2+y^2=4$ is zero:

$$\iint_S \mathbf{F}(\mathbf{R})\cdot\mathbf{n}\,dS + \iint_D \mathbf{F}(\mathbf{R})\cdot(-\mathbf{k})\,dx\,dy \;=\; 0.$$

(In the second integral, the normal is $-\mathbf{k}$, not $\mathbf{k}$ since it must point outward.) Rearranging and changing to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) gives the answer since

$$\frac{\partial(x,y)}{\partial(r,\theta)} \;=\; \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \;=\; r.$$

Alternatively, since $\nabla\cdot\mathbf{F} = 0$ for all $\mathbf{R}$, we can write $\mathbf{F} = \nabla\times\mathbf{G}$ for some $\mathbf{G}$. Now Stokes' Theorem tells us that integrating $\nabla\times\mathbf{G}\cdot d\mathbf{S}$ over a surface depends only on the boundary, not the surface. Hence we can change the surface to the interior of $x^2 + y^2 = 4$ in the xy-plane. This gives

$$\iint_S \mathbf{F}(\mathbf{R})\cdot\mathbf{n}\,dS \;=\; \iint_D \mathbf{F}(\mathbf{R})\cdot\mathbf{k}\,dx\,dy.$$

Change coordinates as before.

7. We have

$$\mathbf{F} \;=\; \nabla\phi + \nabla\times\mathbf{G} \qquad \text{and} \mathbf{F} \;=\; \nabla(\phi - h) + \nabla\times\mathbf{H}.$$

Equating and rearranging, we have $\nabla\times(\mathbf{H}-\mathbf{G}) = \nabla h$. Since h is harmonic, the divergence of ∇h is zero. Thus this equation has a solution in a star-shaped domain. In fact, we can write the solution as

$$\mathbf{H}(\mathbf{R}) - \mathbf{G}(\mathbf{R}) \;=\; \int_0^1 t(\nabla h(\mathbf{r}))\times\frac{d\mathbf{r}}{dt}\,dt,$$

where $\mathbf{r} = t\mathbf{R} + (1-t)\mathbf{R}_0$ and $\mathbf{R}_0$ is any point you wish with $|\mathbf{R}_0| < 1$, for example, $\mathbf{R}_0 = \mathbf{0}$. Add $\mathbf{G}$ to both sides to obtain $\mathbf{H}$.

8. By the given the fact, we have

$$\int_C \mathbf{F}(\mathbf{R})\cdot d\mathbf{R} \;=\; \int_{C_1}\mathbf{F}(\mathbf{R})\cdot d\mathbf{R} + \int_{C_2}\mathbf{F}(\mathbf{R})\cdot d\mathbf{R}, \tag{1}$$

where we must be careful which directions we traverse C_1 and C_2.

(a) Now $\mathbf{0}\in C$. Start traversing C_1 from $\mathbf{0}$. If we replace $\mathbf{R}$ with $-\mathbf{R}$, this will cause us to traverse C_2 starting at $\mathbf{0}$. One of these two curves is being traversed in the wrong direction, say C_2. Thus

$$\begin{aligned}\int_{C_2}\mathbf{F}(\mathbf{R})\cdot d\mathbf{R} \;&=\; -\int_{C_1}\mathbf{F}(-\mathbf{R})\cdot d(-\mathbf{R}) \qquad (2)\\ &=\; \int_{C_1}\mathbf{F}(-\mathbf{R})\cdot d\mathbf{R} \;=\; -\int_{C_1}\mathbf{F}(\mathbf{R})\cdot d\mathbf{R},\end{aligned}$$

where the first equality uses the relationship between C_1 and C_2 and the last equality uses the fact that $\mathbf{F}$ is odd on C. Thus (1) reduces to zero.

(b) Now $\mathbf{0}\notin C$. In this case you should be able to convince yourself that, as $\mathbf{R}$ moves along C_1, $-\mathbf{R}$ moves along C_2 in the same direction—the points $\mathbf{R}$ and $-\mathbf{R}$ are on opposite sides of the origin. The argument now proceeds as in (a) except that the first $-\int_{C_1}$ in (2) should be $\int_{C_1}$.