Math 283 Linear Algebra Review

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1 Matrix Multiplication

Let A be an m-by-n matrix and B be an n-by-p matrix. Then the product of A and B is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example 1

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} ag + bh + ci \\ dg + eh + fi \end{bmatrix}$$

Problem 1 *Do the following matrix multiplication*

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

2 Determinant

The determinant of a 2-by-2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant of a 3-by-3 matrix is given by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Example 2
$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$

In matlab:

```
 det([1 2; 3 4])
ans =
    -2
```

In R:

```
 det(array(c(1,3,2,4), c(2,2)))
[1] -2
```

Problem 2 *Calculate the following determinant*
$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{vmatrix} =$

3 Matrix Inversion

A matrix is *invertible* if and only if its determinant is non-zero. The inverse matrix A^{-1} of the n-by-n matrix A is a unique n-by-n which satisfies

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the n-by-n identity matrix.

Example 3

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \cong \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - 3R_1$$

$$\cong \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 + R_2 \\ R_2 \leftarrow -\frac{1}{2}R_2 \end{array}$$

In matlab:

```
>> inv([1 2; 3 4])
ans =
   -2.0000    1.0000
    1.5000   -0.5000
```

In R:

```
> solve(array(c(1,3,2,4), c(2,2)))
     [,1] [,2]
[1,] -2.0  1.0
[2,]  1.5 -0.5
```

Problem 3 *Find the inverse of* $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

4 Eigenvalues and Eigenvectors

The vector $\vec{x}$ is an *eigenvector* for A if there exist a scalar λ such that $A\vec{x} = \lambda\vec{x}$. λ is the *eigenvalue* corresponding to $\vec{x}$. It can be shown that λ is an eigenvalue of A if and only if $det(A - \lambda I_n) = 0$.

Example 4 *We want to find the eigenvalues and eigenvectors of* $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

$$\begin{aligned} det(A - \lambda I_n) &= \begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda) \end{aligned}$$

The roots of $(2-\lambda)(1-\lambda)$ *are* 1 *and* 2. *These are the eigenvalues of* A.

Now we want to solve $A\vec{x} = \lambda\vec{x}$.

$$A\vec{x} = \lambda\vec{x} \Rightarrow \begin{cases} 2x_1 + x_2 &= \lambda x_1 \\ x_2 &= \lambda x_2 \end{cases}$$

For $\lambda_1 = 1$, $x_2 = -x_1$ *so the eigenvectors have the form* $\begin{bmatrix} c \\ -c \end{bmatrix}$.

For $\lambda_2 = 2$, $x_2 = 0$ *so the eigenvectors have the form* $\begin{bmatrix} c \\ 0 \end{bmatrix}$.

In matlab,

```
>> [V,D] = eig([2 1; 0 1])
V =
    1.0000   -0.7071
         0    0.7071
D =
     2     0
     0     1
```

In R:

```
> eig = eigen(array(c(2,0,1,1), c(2,2)))
> eig
$values
[1] 2 1
$vectors
     [,1]       [,2]
 [1,]    1 -0.7071068
 [2,]    0  0.7071068
```

Note that matlab and R will normalize the eigenvectors.

Problem 4 *Find the eigenvalues and eigenvectors of* $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

5 Matrix Diagonalization

A square matrix A is *diagonalizable* if there exists an invertible matrix V such that $A = VDV^{-1}$ is a diagonal matrix. Matrix diagonalization is the process of finding V and D. One can form D by putting the eigenvalues along the diagonal of a matrix, and V by having the corresponding eigenvectors in each column.

Example 5 *Let's look at the previous example. We had $\lambda_1 = 1$ and $\lambda_2 = 2$. Therefore, we find*

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

We found the corresponding eigenvectors to be of the forms $\begin{bmatrix} c \\ -c \end{bmatrix}$ and $\begin{bmatrix} c \\ 0 \end{bmatrix}$ respectively, so

$$V = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Now, we can show that

$$V^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$VDV^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = A$$

Problem 5 *Diagonalize $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.*

Hint: Use previous problem.

6 Spectral Decomposition

We can obtain the n^{th} power of a matrix by multiplying it with itself n times. So $A^n = A \cdot A \cdot \ldots \cdot A$. Now, if A is diagonalizable then we can find V and D such that $D = V^{-1}AV \Rightarrow A = VDV^{-1}$. Thus we can write the following expansion:

$$A^n = VDV^{-1} \cdot VDV^{-1} \ldots VDV^{-1} = VD^nV^{-1}.$$

Now, we have $V = [\vec{r_1}\ \vec{r_2}\ \ldots \vec{r_m}]$ where $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_m}$ are the columns of V so the right eigenvectors of A. We have $V^{-1} = \begin{bmatrix} \vec{l_1}' \\ \vdots \\ \vec{l_m}' \end{bmatrix}$ where $\vec{l_1}', \vec{l_2}', \ldots, \vec{l_m}'$ are the rows of V^{-1} so the tranposes of the left eigenvectors of A (solution to $\lambda\vec{x} = A^T\vec{x}$). Then,

$$\begin{aligned}
VD^nV^{-1} &= V \begin{bmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & \lambda_2^n & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_m^n \end{bmatrix} V^{-1} \\
&= V \begin{bmatrix} \lambda_1^n & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} V^{-1} + \ldots + V \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_m^n \end{bmatrix} V^{-1} \\
&= \vec{r_1}\lambda_1^n\vec{l_1}' + \vec{r_2}\lambda_2^n\vec{l_2}' + \ldots + \vec{r_m}\lambda_m^n\vec{l_m}' \\
&= \lambda_1^n\vec{r_1}\vec{l_1}' + \lambda_2^n\vec{r_2}\vec{l_2}' + \ldots + \lambda_m^n\vec{r_m}\vec{l_m}'.
\end{aligned}$$

This is the spectral decomposition of A^n.

Example 6 *Let's continue the previous example. We have*

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad V^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

We can calculate

$$A^3 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}$$

If we use the spectral decomposition, we get

$$\begin{aligned}
A^3 &= 1^3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \end{bmatrix} + 2^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 8 & 8 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Problem 6 *Calculate* A^3 *for* $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ *using both basic matrix multiplication and spectral decomposition.*