Linear Algebra review
Powers of a diagonalizable matrix
Spectral decomposition

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Also see the separate version of this with Matlab and R commands.

Matrices

- A matrix is a square or rectangular table of numbers.
- An $m \times n$ matrix has m rows and n columns. This is read "m by n".
- This matrix is 2×3:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
- The entry in row i, column j, is denoted $A_{i,j}$ or A_{ij}.

$$\begin{array}{lll} A_{1,1} = 1 & A_{1,2} = 2 & A_{1,3} = 3 \\ A_{2,1} = 4 & A_{2,2} = 5 & A_{2,3} = 6 \end{array}$$

Matrix multiplication

$$\underset{2\times 3}{\underbrace{\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}}}^{A}\ \underset{3\times 4}{\underbrace{\begin{bmatrix}5 & -2 & 3 & 2\\ 0 & 1 & 1 & -1\\ -1 & 6 & 4 & 3\end{bmatrix}}}^{B} \overset{=}{=} \underset{2\times 4}{\underbrace{\begin{bmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{bmatrix}}}^{C}$$

- Let A be $p \times q$ and B be $q \times r$.
- The product $AB = C$ is a certain $p \times r$ matrix of dot products:

$$C_{i,j} = \sum_{k=1}^{q} A_{i,k} B_{k,j} = \text{dot product } (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B)$$

- The number of columns in A must equal the number of rows in B (namely q) in order to be able to compute the dot products.

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C_{1,1} = 1(5) + 2(0) + 3(-1) = 5 + 0 - 3 = 2$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C_{1,2} = 1(-2) + 2(1) + 3(6) = -2 + 2 + 18 = 18$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C_{1,3} = 1(3) + 2(1) + 3(4) = 3 + 2 + 12 = 17$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & 9 \\ . & . & . & . \end{bmatrix}$$

$$C_{1,4} = 1(2) + 2(-1) + 3(3) = 2 - 2 + 9 = 9$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & 9 \\ 14 & \cdot & \cdot & \cdot \end{bmatrix}$$

$$C_{2,1} = 4(5) + 5(0) + 6(-1) = 20 + 0 - 6 = 14$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & 9 \\ 14 & 33 & \cdot & \cdot \end{bmatrix}$$

$$C_{2,2} = 4(-2) + 5(1) + 6(6) = -8 + 5 + 36 = 33$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & 9 \\ 14 & 33 & 41 & \cdot \end{bmatrix}$$

$$C_{2,3} = 4(3) + 5(1) + 6(4) = 12 + 5 + 24 = 41$$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & 9 \\ 14 & 33 & 41 & 21 \end{bmatrix}$$

$$C_{2,4} = 4(2) + 5(-1) + 6(3) = 8 - 5 + 18 = 21$$

Transpose of a matrix

- Given matrix A of dimensions $p \times q$, the transpose A' is $q \times p$, obtained by interchanging rows and columns: $(A')_{ij} = A_{ji}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- Transpose of a product reverses the order and transposes the factors: $(AB)' = B'A'$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ -1 & 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 18 & 17 & 9 \\ 14 & 33 & 41 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 1 \\ -2 & 1 & 6 \\ 3 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 18 & 33 \\ 17 & 41 \\ 9 & 21 \end{bmatrix}$$

Matrix multiplication is *not* commutative: usually, $AB \neq BA$

- For both AB and BA to be defined, need compatible dimensions:
$$A\colon m \times n, \quad B\colon n \times m$$
giving
$$AB\colon m \times m, \quad BA\colon n \times n$$
- The only chance for them to be equal would be if A and B are both square and of the same size, $n \times n$.
- Even then, they are usually not equal:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}$$

Multiplying several matricies

- Multiplication *is* associative: $(AB)C = A(BC)$
- Suppose A is $p_1 \times p_2$
  B is $p_2 \times p_3$
  C is $p_3 \times p_4$
  D is $p_4 \times p_5$

  Then $ABCD$ is $p_1 \times p_5$. By associativity, it may be computed in many ways, such as $A(B(CD))$, $(AB)(CD)$, ... or directly by:

$$(ABCD)_{i,j} = \sum_{k_2=1}^{p_2} \sum_{k_3=1}^{p_3} \sum_{k_4=1}^{p_4} A_{i,k_2} B_{k_2,k_3} C_{k_3,k_4} D_{k_4,j}$$

  This generalizes to any number of matrices.
- Powers $A^2 = AA$, $A^3 = AAA$, ... are defined for square matrices.

Identity matrix

- The $n \times n$ *identity matrix* I is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ (main diagonal);} \\ 0 & \text{if } i \neq j \text{ (elsewhere).} \end{cases}$$

- For any $n \times n$ matrix A,

$$IA = AI = A.$$

This plays the same role as 1 does in multiplication of numbers:

$$1 \cdot x = x \cdot 1 = x.$$

Inverse matrix

- The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} such that $AA^{-1} = I$ and $A^{-1}A = I$. It may or may not exist. This plays the role of reciprocals of ordinary numbers, $x^{-1} = 1/x$.
- For 2×2 matrices

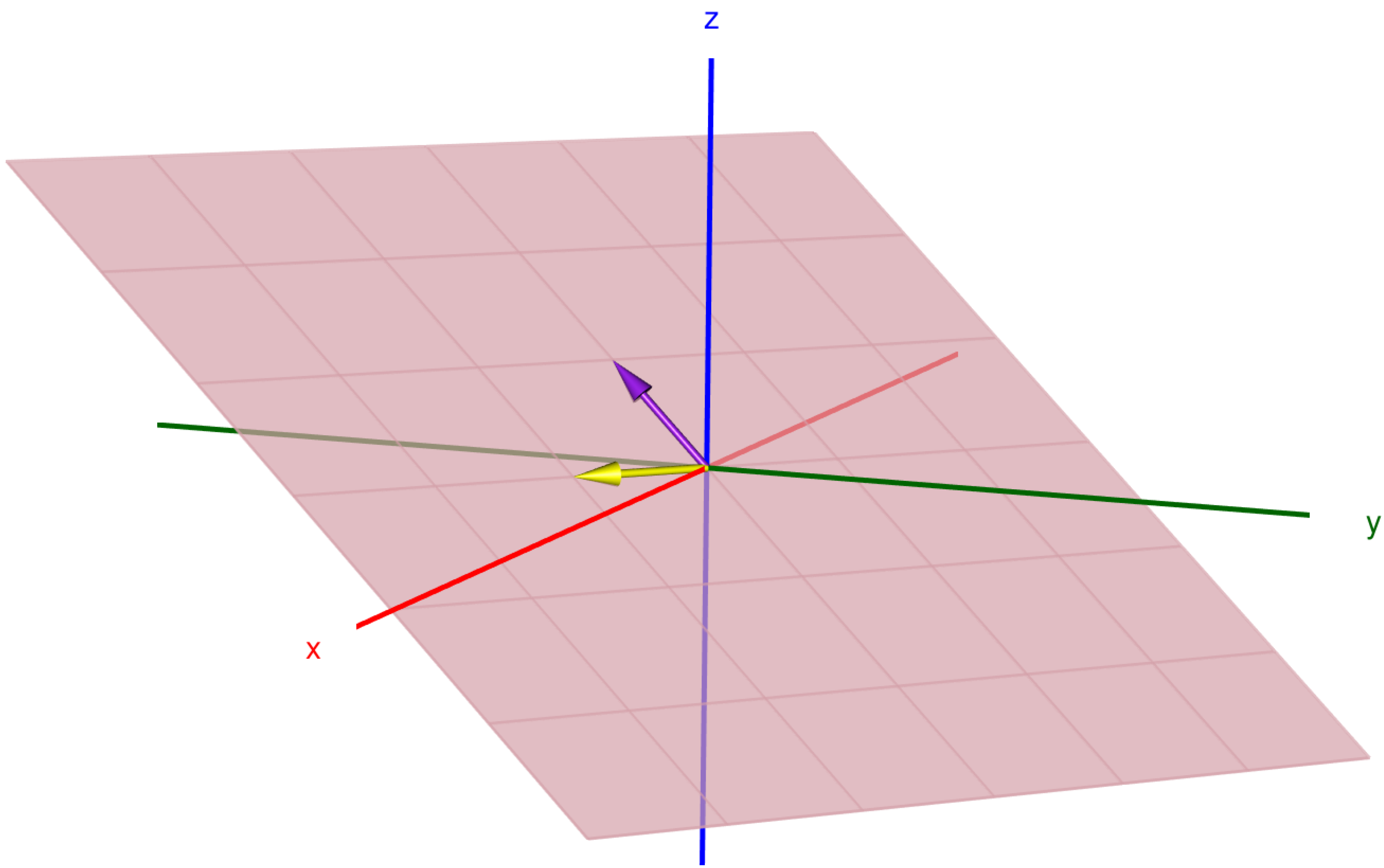
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  unless $\det(A) = ad - bc = 0$, in which case A^{-1} is undefined.
- For $n \times n$ matrices, use the *row reduction algorithm* (a.k.a. *Gaussian elimination*) in Linear Algebra.
- If A, B are invertible and the same size: $(AB)^{-1} = B^{-1}A^{-1}$
  The order is reversed and the factors are inverted.

Span, basis, and linear (in)dependence

The *span* of vectors $\vec{v}_1, \ldots, \vec{v}_k$ is the set of all *linear combinations*

$$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k \qquad \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$



Span, basis, and linear (in)dependence

Example 1

- In 3D,

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}=\left\{\begin{bmatrix}x\\0\\z\end{bmatrix}\;:\;x,z\in\mathbb{R}\right\}=xz\text{ plane}$$

- Here, the span of these two vectors is a 2-dimensional space. Every vector in it is generated by a unique linear combination.

Span, basis, and linear (in)dependence

Example 2

- In 3D,

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\-1/2\end{bmatrix}\right\}=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\;:\;x,y,z\in\mathbb{R}\right\}=\mathbb{R}^3.$$

- Note that

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=(x-y)\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}-2z\begin{bmatrix}0\\0\\-1/2\end{bmatrix}$$

- Here, the span of these three vectors is a 3-dimensional space. Every vector in $\mathbb{R}^3$ is generated by a unique linear combination.

Span, basis, and linear (in)dependence

Example 3

- In 3D,

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \left\{\begin{bmatrix}x\\0\\z\end{bmatrix} : x, z \in \mathbb{R}\right\} = xz \text{ plane}$$

- This is a plane (2D), even though it's a span of three vectors.
- Note that $\vec{v}_2 = \vec{v}_1 + \vec{v}_3$, or $\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$.
- There are multiple ways to generate each vector in the span: for all x, z, t,

$$\begin{bmatrix}x\\0\\z\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} + t\underbrace{(\vec{v}_1-\vec{v}_2+\vec{v}_3)}_{=\vec{0}} = (x+t)\begin{bmatrix}1\\0\\0\end{bmatrix} - t\begin{bmatrix}1\\0\\1\end{bmatrix} + (z+t)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Span, basis, and linear (in)dependence

- Given vectors $\vec{v}_1, \ldots, \vec{v}_k$, if there is a linear combination

$$\alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k = \vec{0}$$

  with at least one $\alpha_i \neq 0$, the vectors are *linearly dependent* (Ex. 3). Otherwise they are *linearly independent* (Ex. 1–2).

- Linearly independent vectors form a *basis* of the space S they span.
  - Any vector in S is a *unique* linear combination of basis vectors (vs. it's not unique if $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent).
  - One basis of $\mathbb{R}^n$ is a unit vector on each axis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

    but there are other possibilities, e.g., Example 2: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/2 \end{bmatrix}$

Eigenvectors

Eigenvalues and eigenvectors

Let A be a square matrix ($k \times k$) and $\vec{v} \neq \vec{0}$ be a column vector ($k \times 1$).
If $A\vec{v} = \lambda\vec{v}$ for a scalar λ, then $\vec{v}$ is an *eigenvector* of A with *eigenvalue* λ.

Example

$$\begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (8)(1) + (-1)(3) \\ (6)(1) + (3)(3) \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 5.

But this is just a verification. How do we find eigenvalues and eigenvectors?

Finding eigenvalues and eigenvectors

- We will work with the example

$$P = \begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix}$$

- Form the *identity matrix* of the same dimensions:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The formula for the *determinant* depends on the dimensions of the matrix. For a 2×2 matrix,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Finding eigenvalues and eigenvectors

- Compute the *determinant* of $P - \lambda I$:

$$\begin{aligned}
\det(P - \lambda I) &= \det \begin{bmatrix} 8-\lambda & -1 \\ 6 & 3-\lambda \end{bmatrix} \\
&= (8-\lambda)(3-\lambda) - (-1)(6) \\
&= 24 - 11\lambda + \lambda^2 + 6 \\
&= \lambda^2 - 11\lambda + 30
\end{aligned}$$

  This is the *characteristic polynomial* of P. It has degree k in λ.

- The *characteristic equation* is $\det(P - \lambda I) = 0$. Solve it for λ.
  For $k = 2$, use the quadratic formula:

$$\lambda = \frac{11 \pm \sqrt{(-11)^2 - 4(1)(30)}}{2} = 5,\ 6$$

- The eigenvalues are $\lambda = 5$ and $\lambda = 6$.

Finding the (right) eigenvector for $\lambda = 5$

- Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. We will solve for a, b.
- The equation $P\vec{v} = \lambda\vec{v}$ is equivalent to $(P - \lambda I)\vec{v} = \vec{0}$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (P - 5I)\vec{v} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a - b \\ 6a - 2b \end{bmatrix}$$

so $3a - b = 0$ and $6a - 2b = 0$ (which are equivalent).

- Solving gives $b = 3a$. Thus,

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 3a \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- Any nonzero scalar multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of P with eigenvalue 5.

Finding the (right) eigenvector for $\lambda = 6$

- Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. We will solve for a, b.
- The equation $P\vec{v} = \lambda\vec{v}$ is equivalent to $(P - \lambda I)\vec{v} = \vec{0}$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (P - 6I)\vec{v} = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ 6a - 3b \end{bmatrix}$$

so $2a - b = 0$ and $6a - 3b = 0$ (which are equivalent).

- Solving gives $b = 2a$. Thus,

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Any nonzero scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of P with eigenvalue 6.

Verify the eigenvectors

$$\begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8(1) - 1(3) \\ 6(1) + 3(3) \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8(2) - 1(4) \\ 6(2) + 3(4) \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Normalization: Which scalar multiple should we use?

In some applications, any nonzero multiple is fine.
In others, a particular scaling is required.

Markov chains / Stochastic matrices

Entries are probabilities of different cases. Scale the vector so that the entries sum up to 1.

For $\vec{v} = a \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, the sum is $a \cdot (1+3) = 4a = 1$, so $a = \frac{1}{4}$: $\quad \vec{v} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$

Principal component analysis

Scale it to be a *unit vector*, so that the sum of the squares of its entries equals 1:

$$1 = a^2(1^2 + 3^2) = 10a^2 \text{ so } a = \frac{\pm 1}{\sqrt{1^2+3^2}} = \frac{\pm 1}{\sqrt{10}}.$$

$$\vec{v} = \pm \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \qquad \text{(two possibilities)}$$

Finding the left eigenvector for $\lambda = 5$

- Let $\vec{v} = \begin{bmatrix} a & b \end{bmatrix}$. We will solve for a, b.
- The equation $\vec{v}P = \lambda\vec{v}$ is equivalent to $\vec{v}(P - \lambda I) = \vec{0}$.

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = \vec{v}(P - 5I) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3a + 6b & -a - 2b \end{bmatrix}$$

  so $3a + 6b = 0$ and $-a - 2b = 0$ (which are equivalent).
- Solving gives $b = -a/2$. Thus,

$$\vec{v} = \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & -a/2 \end{bmatrix} = a \begin{bmatrix} 1 & -1/2 \end{bmatrix}$$

- Any nonzero scalar multiple of $\begin{bmatrix} 1 & -1/2 \end{bmatrix}$ is a left eigenvector of P with eigenvalue 5.

Finding the left eigenvector for $\lambda = 6$

- Let $\vec{v} = \begin{bmatrix} a & b \end{bmatrix}$. We will solve for a, b.
- The equation $\vec{v}P = \lambda\vec{v}$ is equivalent to $\vec{v}(P - \lambda I) = \vec{0}$.

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = \vec{v}(P - 6I) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} 2a + 6b & -a - 3b \end{bmatrix}$$

  so $2a + 6b = 0$ and $-a - 3b = 0$ (which are equivalent).
- Solving gives $b = -a/3$. Thus,

$$\vec{v} = \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & -a/3 \end{bmatrix} = a \begin{bmatrix} 1 & -1/3 \end{bmatrix}$$

- Any nonzero scalar multiple of $\begin{bmatrix} 1 & -1/3 \end{bmatrix}$ is a left eigenvector of P with eigenvalue 6.

Verify the left eigenvectors

$$\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}8 & -1\\ 6 & 3\end{bmatrix} = \begin{bmatrix}-2(8)+1(6) & -2(-1)+1(3)\end{bmatrix}$$
$$= \begin{bmatrix}-10 & 5\end{bmatrix} = 5\begin{bmatrix}-2 & 1\end{bmatrix}$$

$$\begin{bmatrix}1.5 & -.5\end{bmatrix}\begin{bmatrix}8 & -1\\ 6 & 3\end{bmatrix} = \begin{bmatrix}1.5(8)-.5(6) & 1.5(-1)-.5(3)\end{bmatrix}$$
$$= \begin{bmatrix}9 & -3\end{bmatrix} = 6\begin{bmatrix}1.5 & -.5\end{bmatrix}$$

Diagonalizing a matrix

- This procedure assumes there are k *linearly independent* eigenvectors, where P is $k \times k$.
- If the characteristic polynomial has k *distinct* roots, then there are k such eigenvectors.
- But if roots are repeated, there may or may not be a full set of eigenvectors. We'll explore this complication later.

Diagonalizing a matrix

- Put the right eigenvectors $\vec{r}_1, \vec{r}_2, \ldots$ into the columns of a matrix V. Form diagonal matrix D with eigenvalues $\lambda_1, \lambda_2, \ldots$ in the same order:

$$V = \left[\vec{r}_1 \mid \vec{r}_2\right] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$$

- Compute $V^{-1} = \begin{bmatrix} \vec{\ell}_1 \\ \hline \vec{\ell}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

  Its rows are the left eigenvectors $\vec{\ell}_1, \vec{\ell}_2, \ldots$ of P, in the same order as the eigenvalues in D, scaled so that $\vec{\ell}_i \cdot \vec{r}_i = 1$.

- This gives the *diagonalization* $P = VDV^{-1}$:

$$\begin{array}{ccccc} P & = & V & D & V^{-1} \\ \begin{bmatrix} 8 & -1 \\ 6 & 3 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} & \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \end{array}$$

Matrix powers using the spectral decomposition

An expansion of P^n is $P^n = (VDV^{-1})(VDV^{-1})\cdots(VDV^{-1}) = VD^nV^{-1}$:

$$P^n = VD^nV^{-1} = V\begin{bmatrix}5^n & 0\\ 0 & 6^n\end{bmatrix}V^{-1} = V\begin{bmatrix}5^n & 0\\ 0 & 0\end{bmatrix}V^{-1} + V\begin{bmatrix}0 & 0\\ 0 & 6^n\end{bmatrix}V^{-1}$$

$$V\begin{bmatrix}5^n & 0\\ 0 & 0\end{bmatrix}V^{-1} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5^n & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-2 & 1\\ 1.5 & -.5\end{bmatrix} = \begin{bmatrix}(1)(5^n)(-2) & (1)(5^n)(1)\\ (3)(5^n)(-2) & (3)(5^n)(1)\end{bmatrix}$$

$$= 5^n\begin{bmatrix}1\\ 3\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix} = {\lambda_1}^n\,\vec{r}_1\,\vec{\ell}_1 = 5^n\begin{bmatrix}-2 & 1\\ -6 & 3\end{bmatrix}$$

$$V\begin{bmatrix}0 & 0\\ 0 & 6^n\end{bmatrix}V^{-1} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 6^n\end{bmatrix}\begin{bmatrix}-2 & 1\\ 1.5 & -.5\end{bmatrix} = \begin{bmatrix}2(6^n)(1.5) & 2(6^n)(-.5)\\ 4(6^n)(1.5) & 4(6^n)(-.5)\end{bmatrix}$$

$$= 6^n\begin{bmatrix}2\\ 4\end{bmatrix}\begin{bmatrix}1.5 & -.5\end{bmatrix} = {\lambda_2}^n\,\vec{r}_2\,\vec{\ell}_2 = 6^n\begin{bmatrix}3 & -1\\ 6 & -2\end{bmatrix}$$

Matrix powers using the spectral decomposition

- Continue computing P^n:

$$P^n = VD^nV^{-1} = V \begin{bmatrix} 5^n & 0 \\ 0 & 6^n \end{bmatrix} V^{-1} = V \begin{bmatrix} 5^n & 0 \\ 0 & 0 \end{bmatrix} V^{-1} + V \begin{bmatrix} 0 & 0 \\ 0 & 6^n \end{bmatrix} V^{-1}$$
$$= 5^n \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} + 6^n \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$$

- **General formula (with $k = 2$ and two distinct eigenvalues):**

$$P^n = VD^nV^{-1} = {\lambda_1}^n \, \vec{r}_1 \, \vec{\ell}_1 + {\lambda_2}^n \, \vec{r}_2 \, \vec{\ell}_2$$

- **General formula:** If P is $k \times k$ and is diagonalizable, this becomes:

$$P^n = VD^nV^{-1} = {\lambda_1}^n \, \vec{r}_1 \, \vec{\ell}_1 + {\lambda_2}^n \, \vec{r}_2 \, \vec{\ell}_2 + \cdots + {\lambda_k}^n \, \vec{r}_k \, \vec{\ell}_k$$

- **What if the matrix is not diagonalizable?**
  We will see a generalization called the *Jordan Canonical Form*.