Math 142B Homework Assignment 2
Due 11:00pm Thursday, April 25, 2024

1. Suppose f is differentiable on $\mathbb{R}$, $1 \le f'(x) \le 2$ for all $x \in \mathbb{R}$, and $f(0) = 0$. Show that $x \le f(x) \le 2x$ for all $x \ge 0$.

2. Let f be differentiable on some interval (c, ∞) such that $\lim_{x\to\infty} \left[f(x) + f'(x)\right] = L$, with L finite.

   Show that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$. $\left[\text{Hint: Write } f(x) = \frac{f(x)\ e^x}{e^x}.\right]$

3. Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

   (a) Compute the upper and lower Darboux integrals for f on the interval $[0, b]$.

   (b) Is f integrable on $[0, b]$? Be sure to justify your answer.

4. Let f be a bounded function on $[a, b]$. Suppose there exist sequences (L_n) and (U_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$.

   Show that f is integrable on $[a, b]$ and that $\displaystyle\int_a^b f = \lim L_n = \lim U_n$.

5. Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many $x \in [a, b]$.

   Show that g is integrable on $[a, b]$ and that $\displaystyle\int_a^b g = \int_a^b f$.

6. Show that a decreasing function f on $[a, b]$ is integrable.

7. Let f be a bounded function on $[a, b]$ so that there is $B > 0$ for which $|f(x)| \le B$ for all $x \in [a, b]$.

   (a) Show that
   $$U(f^2, P) - L(f^2, P) \le 2B\,[U(f, P) - L(f, P)]$$
   for all partitions P of $[a, b]$.

   (b) Show that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

8. Let f and g be integrable functions on $[a, b]$.

   (a) Show that fg is integrable on $[a, b]$.

   (b) Show that $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$.

9. Suppose f and g are continuous functions on $[a, b]$ such that $\displaystyle\int_a^b f = \int_a^b g$. Prove that there exists $x \in (a, b)$ at which $f(x) = g(x)$.

10. (a) Prove that if f and g are continuous functions on $[a, b]$ with $g(t) \ge 0$ for all $t \in [a, b]$, then there exists $x \in (a, b)$ such that
    $$\int_a^b f(t)g(t)\,dt = f(x)\int_a^b g(t)\,dt.$$

    (b) Show that the *Intermediate Value Theorem for Integrals* is a special case of part (a).

    (c) Does the conclusion in part (a) hold if $[a, b] = [-1, 1]$ and $f(t) = g(t) = t$ for all $t \in [a, b]$? Be sure to justify your answer.