MATH 150A HOMEWORK SOLUTIONS

Problem # 3.3.20

For this problem we'll assume a, b, c are all distinct because in the other cases o the calculations are less involved. Recall that the definition of umbilical point is exactly when the second fundamental form Π is a scalar multiple of the first fundamental form. That is:

$$e \;=\; kE \;, \qquad f \;=\; kF \;, \qquad g \;=\; kG \;, \tag{1}$$

where $k = k_1 = k_2$ is the repeated principle curvature. Since the surface is invariant with respect to reflections is suffices to find all such points in one of the half spaces. For now lets look at $z > 0$. To do this we parametrize the ellipsoid as the graph:

$$z \;=\; h(x, y) \;, \qquad \text{where} \quad h(x, y) \;=\; c\sqrt{1 - (x/a)^2 - (y/b)^2} \;.$$

First we need to compute all of E, F, G, e, f, g for a graph. The parametrization function if $\Phi(x, y) = (x, y, h)$, so:

$$E \;=\; 1 + h_x^2 \;, \qquad F \;=\; h_x h_y \;, \qquad G \;=\; 1 + h_y^2 \;.$$

In addition to this $N = (-h_x, -h_y, 1)/\sqrt{1 + |\nabla h|^2}$, so:

$$e \;=\; N \cdot \Phi_{xx} \;=\; \frac{h_{xx}}{\sqrt{1 + |\nabla h|^2}} \;, \qquad f \;=\; N \cdot \Phi_{yx} \;=\; \frac{h_{xy}}{\sqrt{1 + |\nabla h|^2}} \;, \qquad g \;=\; N \cdot \Phi_{yy} \;=\; \frac{h_{yy}}{\sqrt{1 + |\nabla h|^2}}$$

Next we need to substitute the specific form of h into these formulas. Differentiating we find:

$$h_x \;=\; -(\frac{c}{a})^2 \frac{x}{h} \;, \qquad h_y \;=\; -(\frac{c}{b})^2 \frac{y}{h} \;.$$

Therefore differentiating one more time we find:

$$h_{xx} \;=\; -(\frac{c^2}{a})^2 \frac{1 - (y/b)^2}{h^3} \;, \qquad h_{xy} \;=\; -(\frac{c^2}{ab})^2 \frac{xy}{h^3} \;, \qquad h_{yy} \;=\; -(\frac{c^2}{b})^2 \frac{1 - (x/a)^2}{h^3} \;.$$

The next thing we'll do is show there are no solutions to (1) unless one of $x = 0$ or $y = 0$ when $z > 0$. This shows that all the umbilical points must occur on one of the coordinate axes $x = 0$, $y = 0$, or $z = 0$. Now when both $x \neq 0$ and $y \neq 0$ the middle equation on line (1) gives us:

$$h\sqrt{1 + |\nabla h|^2} \;=\; -\frac{1}{k} \;.$$

Substituting this into the first and second formulas on line (1) and cleaning up a little bit we get the two relations:

$$1 + \big((\frac{c}{a})^2 - 1\big)(x/a)^2 - (y/b)^2 \;=\; (\frac{c}{a})^2(1 - (y/b)^2) \;, \qquad 1 + \big((\frac{c}{a})^2 - 1\big)(x/a)^2 - (y/b)^2 \;=\; (\frac{c}{a})^2(1 - (y/b)^2)$$

This may be written as a linear system:

$$\begin{pmatrix} (\frac{c}{a})^2 - 1 & (\frac{c}{a})^2 - 1 \\ (\frac{c}{b})^2 - 1 & (\frac{c}{b})^2 - 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \;=\; \begin{pmatrix} (\frac{c}{a})^2 - 1 \\ (\frac{c}{b})^2 - 1 \end{pmatrix} \;, \qquad \text{where} \quad \xi = (x/a)^2 \;, \quad \eta = (y/b)^2 \;.$$

Solutions to this system are all such that $\xi + \eta = 1$. But this can never happen when $z > 0$.

It remains to look for umbilical points on the coordinate axes. Without making any assumptions yet on the relative sizes of a, b, c, we may assume the coordinate axes under consideration is $x = 0$. Now the middle equation of (1) is automatic, and the other two equations give us the relations:

$$k \;=\; -(\frac{c}{a})^2 \frac{1}{h\sqrt{1 + h_y^2}} \;, \qquad k(1 + h_y^2) \;=\; -(\frac{c^2}{b})^2 \frac{1}{h^3\sqrt{1 + h_y^2}} \;.$$

Eliminating k from these identities we find:

$$1 + \left(\left(\frac{c}{b}\right)^2 - 1\right)(y/b)^2 \;=\; \left(\frac{a}{b}\right)^2 .$$

The key observation here it that the LHS varies between 1 and $\left(\frac{c}{b}\right)^2$ as $0 \leq y \leq b$. Thus, the only possible solutions are in the configurations $c < a < b$ and $b < a < c$. In particular this shows that the coordinate which vanishes at an umbilical point must have a denominator coefficient which occupies the middle position in the ordering of the denominators. In particular when a, b, c are all distinct there can be umbilical points only along one of the planes $x = 0$, $y = 0$, or $z = 0$ intersection with S. Letting this be $x = 0$ as above, we find there are exactly four umbilical points which are given by one of the four combinations of the coordinate choices:

$$x \;=\; 0 , \qquad y \;=\; \pm b\sqrt{\frac{b^2 - a^2}{b^2 - c^2}} , \qquad z \;=\; \pm c\sqrt{\frac{c^2 - a^2}{c^2 - b^2}} .$$

PROBLEM # 3.3.23

a) Let $\Phi : \Omega \to S$ be a local parametrization of S. Then for $(u, v) \in S$ we can write $h_r(u, v) = \| \Phi(u, v) - r \|$. A point (u_0, v_0) is critical iff $\partial_u h_r(u_0, v_0) = \partial_v h_r(u_0, v_0) = 0$. Computing this condition we find that:

$$\frac{\Phi_u(u_0, v_0) \cdot (\Phi(u_0, v_0) - r)}{\| \Phi(u_0, v_0) - r \|} \;=\; \frac{\Phi_v(u_0, v_0) \cdot (\Phi(u_0, v_0) - r)}{\| \Phi(u_0, v_0) - r \|} \;=\; 0 .$$

In other words the displacement vector $X = \Phi(u_0, v_0) - r$ is such that $X \perp T_p(S)$ at all critical points $p \in S$.

b) The next thing is to compute the Hessian of h_r at the critical points. Since this quadratic form on $T_p(S)$ is an invariant we can choose our system of coordinates to our favor. To simplify things we may assume S is the graph of $z = f(x, y)$ and the critical point is where $x = y = z = 0$, and $r = (0, 0, \lambda)$ for some $\lambda > 0$. With this setup $f_x(0, 0) = f_y(0, 0)$ as well, and after a rotation we can make one further assumption which is that the x and y axes are principle directions. Then:

$$f(x, y) \;=\; \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 + R(x, y) ,$$

where k_1 and k_2 are the principle curvatures at $(0, 0)$ and where the remainder R vanishes to cubic order at $(0, 0)$ (i.e. all derivatives up to order two of R vanish at $(0, 0)$). With these assumptions we then have:

$$h_r(x, y) \;=\; \sqrt{x^2 + y^2 + (f(x, y) - \lambda)^2} \;=\; \sqrt{\lambda^2 + (1 - \lambda k_1)x^2 + (1 - \lambda k_2)y^2 + \widetilde{R}(x, y)} ,$$

where $\widetilde{R}(x, y)$ again vanishes to order 3. Taylor expanding the radical we then find that close to the origin:

$$h_r(x, y) \;=\; \lambda + \frac{1}{2}\left(\frac{1}{\lambda} - k_1\right)x^2 + \frac{1}{2}\left(\frac{1}{\lambda} - k_2\right)y^2 + \widetilde{\widetilde{R}}(x, y) ,$$

where $\widetilde{\widetilde{R}}(x, y)$ is some other cubic expression. In particular:

$$H_{h_r}|_{(0,0)} \;=\; \begin{pmatrix} \frac{1}{\lambda} - k_1 & 0 \\ 0 & \frac{1}{\lambda} - k_2 \end{pmatrix} .$$

In this shows $H_{h_r}|_{(0,0)}(e_1, e_1) = \frac{1}{h_r(0,0)} - k_1$ and $H_{h_r}|_{(0,0)}(e_2, e_2) = \frac{1}{h_r(0,0)} - k_2$ for the two principle directions which shows $H_{h_r}|_{(0,0)}(w, w) = \frac{1}{h_r(0,0)} - k_n$ more generally. Also one can see that the only way the above Hessian can be degenerate is if one of the diagonal elements is zero, that is $h_r(0, 0) = \frac{1}{k_i}$ for one of the principle curvatures k_1 or k_2.

c) This part requires a bit of real analysis. It suffices to prove the statement for a closed patch of a surface which is the graph of $z = f(x, y)$ over some closed and bounded domain $\overline{\Omega} \subseteq \mathbb{R}^2$. Since the surface S may be assumed to consist of finitely many closed patches we then would just need to show that the intersection of finitely many open dense sets is both open and dense which is a standard result of point set topology (don't worry if you didn't get this point of you have not taken 140A).

Let $N(x,y) = \frac{1}{\sqrt{1+|\nabla f|^2}}(-f_x, -f_y, 1)$. Then by part b) the only points $r \in \mathbb{R}^3$ that can lead to degenerate critical points must lie along the traces of:

$$\Phi^{\pm}(x,y) \;=\; (x,y,f(x,y)) \pm \frac{1}{k_1(x,y)} N(x,y) \;, \qquad \Psi^{\pm}(x,y) \;=\; (x,y,f(x,y)) \pm \frac{1}{k_2(x,y)} N(x,y) \;.$$

where the domains are restricted to the portion of $\overline{\Omega}$ where $k_i(x,y) \neq 0$ respectively. It suffices to show these traces are closed and their compliments are dense.

To see that the traces of $\Phi^{\pm}(x,y)$ (say) is closed let $(a_n, b_n, c_n) \to (a_0, b_0, c_0)$ where the sequence is such that $\Phi^{\pm}(x_n, y_n) = (a_n, b_n, c_n)$. WLOG loss of generality, possibly by considering a subsequence, we may also assume $(x_n, y_n) \to (x_0, y_0) \in \overline{\Omega}$ (here I used compactness of $\overline{\Omega}$). Since (a_0, b_0, c_0) is a fixed (bounded) point we must have (x_0, y_0) is in the domain of $\Phi^{\pm}(x,y)$. Then by continuity of $\Phi^{\pm}$ on its domain we must have $\Phi^{\pm}(x_0, y_0) = (a_0, b_0, c_0)$. This means (a_0, b_0, c_0) is also on the trace of $\Phi^{\pm}$ which shows this trace is closed.

To show that the compliment of the traces are dense seems to require us to go a bit further beyond the confines of the course. The key point here (proved in a more advanced analysis class) is that if $\Phi : U \to \mathbb{R}^n$ is differentiable and $U \subseteq \mathbb{R}^k$ with $k < n$ then the image $\Phi(U)$ has dense compliment (i.e. differentiable maps cannot be "space filling surfaces"). A quick way one can prove this is to show the image $\Phi(U)$ has Lebesgue measure zero by using Lipschitz bounds for Φ. Do not worry at all if this last statement makes no sense to you at this point!

Problem # 3.5.14

To simplify the notation a bit we introduce complex coordinates:

$$\zeta \;=\; u + iv \;, \qquad \overline{\zeta} \;=\; u - iv \;,$$

and the corresponding derivatives:

$$\partial \;=\; \frac{1}{2}(\partial_u - i\partial_v) \;, \qquad \overline{\partial} \;=\; \frac{1}{2}(\partial_u + i\partial_v) \;.$$

Then a complex function $h = f(u,v) + ig(u,v)$, where f, g are real valued, satisfies the Cauchy-Riemann equations iff $\overline{\partial} h = 0$. With this setup a function g is the *harmonic conjugate* of f iff $h = f + ig$ satisfies the Cauchy-Riemann equations. This implies by the way that $\Delta f = \Delta g = 0$ where $\Delta = \partial_u^2 + \partial_v^2$ is Laplace's equation (although not every pair of harmonic functions are conjugate). Thus, another way to write the condition that f, g are harmonic conjugates is that:

$$f \;=\; \Re(h) \;, \qquad g \;=\; \Im(h) \;,$$

for some fixed h which is complex analytic, that is $\overline{\partial} h = 0$. Now we'll use this setup to solve the problems.

a) In this case we need to write the isothermal parametrizations as:

$$\Phi(u,v) \;=\; (\cosh(u)\sin(v), \cosh(u)\cos(v), u) \;, \qquad \Psi(u,v) \;=\; (-\sinh(u)\cos(v), \sinh(u)\sin(v), v) \;.$$

Note that Φ still parametrizes the catenoid and Ψ the helicoid although we've rotated and reflected things a bit in the (x,y) variables. To show these parametrizations are harmonic conjugates we simply need to show:

$$\Phi(u,v) + i\Psi(u,v) \;=\; F(\zeta) \;=\; (F_1(\zeta), F_2(\zeta), F_3(\zeta)) \;,$$

where each F_i is complex analytic. To so it we use the Euler formulas:

$$\cos(v) \;=\; \frac{e^{iv} + e^{-iv}}{2} \;, \qquad \sin(v) \;=\; \frac{e^{iv} - e^{-iv}}{2i} \;.$$

Then:

$$\cosh(u)\sin(v) \;=\; \frac{(e^u + e^{-u})(e^{iv} - e^{-iv})}{4i} \;=\; \frac{1}{4i}(e^{\zeta} - e^{-\zeta} - e^{\overline{\zeta}} + e^{-\overline{\zeta}}) \;=\; \frac{1}{2i}(\sinh(\zeta) - \sinh(\overline{\zeta})) \;.$$

Similar calculations reveal the other equations:

$$\cosh(u)\cos(v) \;=\; \frac{1}{2}(\cosh(\zeta) + \cosh(\overline{\zeta})) \;, \quad \sinh(u)\cos(v) \;=\; \frac{1}{2}(\sinh(\zeta) + \sinh(\overline{\zeta})) \;, \quad \sinh(u)\sin(v) \;=\; \frac{1}{2i}(\cosh(\zeta) - \cosh(\overline{\zeta})) \;.$$

Now if $F = \Phi + i\Psi$ we get:

$$F = (-i\sinh(\zeta), \cosh(\zeta), \zeta) \ ,$$

and all three functions are easily seen to be complex analytic.

b) Let Φ and Ψ be two isothermal parametrizations of minimal surfaces S_1 and S_2. Then as we have shown we have the conditions:

$$\overline{\partial}\partial\Phi \ = \ 0 \ , \quad \partial\Phi \cdot \partial\Phi \ = \ 0 \ , \qquad \qquad \overline{\partial}\partial\Psi \ = \ 0 \ , \quad \partial\Psi \cdot \partial\Psi \ = \ 0 \ .$$

In addition if Φ and Ψ are harmonic conjugates we also have $F = \Phi + i\Psi$ satisfies $\overline{\partial}F = 0$ (componentwise). Since $\Phi = \Re(F)$ and $\partial\overline{F} = 0$ this forces:

$$\partial\Phi \ = \ \frac{1}{2}\partial F \ = \ \frac{1}{2}F' \ , \qquad \text{so} \quad F' \cdot F' \ = \ 0 \ .$$

Now consider:

$$\Phi_t \ = \ \cos(t)\Phi + \sin(t)\Psi \ = \ \frac{1}{2}e^{-it}F(\zeta) + \frac{1}{2}e^{it}\overline{F(\zeta)} \ = \ \Re(e^{-it}F)$$

Thus, since Φ_t is the real part of a complex analytic function we automatically have $\overline{\partial}\partial\Phi = 0$. On the other hand:

$$\partial\Phi_t \ = \ \frac{1}{2}e^{-it}F'(\zeta) \ , \qquad \text{so} \quad \partial\Phi_t \cdot \partial\Phi_t \ = \ \frac{1}{4}e^{-2it}F' \cdot F' \ = \ 0 \ .$$

This shows that Φ_t are isothermally parametrized minimal surfaces.

c) Finally, for any parametrized surface note that:

$$4\partial\Phi \cdot \overline{\partial}\Phi \ = \ E + G.$$

In particular since Φ_t is isothermal we have:

$$4\partial\Phi_t \cdot \overline{\partial}\Phi_t \ = \ 2E_t \ = \ 2G_t \ .$$

On the other hand

$$4\partial\Phi_t \cdot \overline{\partial}\Phi_t \ = \ F' \cdot \overline{F'} \ ,$$

which does not depend on t. Thus $E_t = G_t = E = G$ for all t, in particular for Φ and Ψ by taking $t = 0$ and $t = \frac{\pi}{2}$.

Hurray for complex notation!