## MATH 150A MIDTERM

**Important: Do not separate these sheets.** Please put your name on each sheet, and your student ID number on the first sheet. Please show *all* your work on the pages provided. These will be uploaded to GradeScope. You may use both sides of each sheet.

Part I (25 pts. total)

Classify all regular curves $\alpha : \mathbb{R} \to \mathbb{R}^3$ with constant curvature $k(s) = k_0$ and constant torsion $\tau(s) = \tau_0$ according to the following breakdown:

i) (5 pts.) When $k_0 = 0$ the the image of $\alpha$ is contained in a straight line.

ii) (5 pts.) When $k_0 > 0$ and $\tau_0 = 0$ the image of $\alpha$ is contained in a circle inside of some plane.

iii) (15 pts.) When $k_0 > 0$ and $\tau_0 \neq 0$, then after a translation and rotation the image of $\alpha$ is a contained in the trace of a standard cylindrical helix of the form:

$$\beta(s) \;=\; (R\cos(s), R\sin(s), \lambda s) \;,$$

where $R > 0$ and $\lambda \neq 0$ are some constants.

**Solution:** For part i) assume $\alpha$ is parametrized by arc length. Then its equation is $\alpha'' = 0$, which has general solution $\alpha(s) = sv_0 + z_0$ for two vectors $v_0, z_0 \in \mathbb{R}^3$.

We will solve parts ii) and iii) simultaneously. Since curves are determined uniquely up to a rigid motion by their curvature and torsion $k$ and $\tau$, it suffices to show that all the curves of the form:

$$\beta(t) \;=\; (R\cos(t), R\sin(t), \lambda t) \;,$$

exhaust all possible cases with constant $k > 0$ and $\tau$ by simply adjusting $R > 0$ and $\lambda \in \mathbb{R}$. In addition to this we need to show $\tau = 0$ iff $\lambda = 0$.

To compute the curvature and torsion of $\beta$ we use Problem 1.5.12 and $k > 0$ which gives:

$$k \;=\; \frac{\|\,\beta' \wedge \beta''\,\|}{\|\,\beta'\,\|^3} \;, \qquad \tau \;=\; -\frac{(\beta' \wedge \beta'')\cdot \beta'''}{k^2 \|\,\beta'\,\|^6} \;.$$

None that these more general formulas are necessary because $\beta$ is not parametrized by arc length. We have:

$$\beta' \;=\; (-R\sin(t), R\cos(t), \lambda) \;, \qquad \beta'' \;=\; (-R\cos(t), -R\sin(t), 0) \;,$$
$$\beta''' \;=\; (R\sin(t), -R\cos(t), 0) \;, \qquad \beta' \wedge \beta'' \;=\; (\lambda R\sin(t), -\lambda R\cos(t), R^2) \;.$$

From this and the previous formulas we find:

$$k \;=\; \frac{R}{\lambda^2 + R^2} \;, \qquad \tau \;=\; -\frac{\lambda}{\lambda^2 + R^2} \;.$$

Notice that the map $(R, \lambda) \mapsto (k, \tau)$ involves an inversion through the unit circle followed by a reflection in the second coordinate. Therefore this map is its own inverse so (one can also check this by hand):

$$R \;=\; \frac{k}{k^2 + \tau^2} \;, \qquad \lambda \;=\; -\frac{\tau}{k^2 + \tau^2} \;.$$

In particular for every fixed pair $k > 0$ and $\tau \in \mathbb{R}$ one can find a curve of the form $\beta$ above with this curvature and torsion, and the curve is a planar circle iff $\tau = \lambda = 0$.

## Part II (25 pts. total)

A *great circle* on the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$ is the intersection of $\mathbb{S}^2$ with a plane $\mathcal{P}$ which passes through the origin $(0, 0, 0) \in \mathbb{R}^3$.

Let $\alpha : I \to \mathbb{S}^2$ be a regular curve parametrized by arc length. Show that the image of $\alpha$ is contained in a great circle iff the binormal vector of $\alpha$ is such that $b(s) \in T_{\alpha(s)}(\mathbb{S}^2)$ for all $s \in I$, that is iff the binormal to $\alpha$ lies in the tangent plane to $\mathbb{S}^2$ at all points along $\alpha$.

**Solution:** To begin with we show that $\alpha$ is (part of a) great circle iff $\alpha = -n$ where $n$ is the normal along $\alpha$. To see first assume $\alpha$ is a great circle. Then after a rotation $\alpha = (\cos(s), \sin(s), 0)$ so $n = -\alpha'' = -\alpha$ is immediate. On the other hand if $\alpha = -n$ then differentiation with respect to the arc length parameter gives $t = -n' = kt + \tau b$. Thus $\tau = 0$ and $k = 1$ since $\{t, b\}$ are linearly independent. By the previous problem this means $\alpha$ is a circle in some plane.

It remains to show $\alpha = -n$ iff $b \in T_p(\mathbb{S}^2)$ at all points along $\alpha$. The last condition happens iff $\{t, b\}$ span $T_p(\mathbb{S}^2)$, which is equivalent to $n \perp T_p(\mathbb{S}^2)$. Since $\alpha \perp T_p(\mathbb{S}^2)$ this means $b \in T_p(\mathbb{S}^2)$ iff $\alpha = \lambda n$ for some scalar $\lambda$. Since $\alpha, n$ are unit vectors $\lambda = \pm 1$. Since $n$ must point towards the inside of $\alpha$ we get $\lambda = -1$.

## Part III (25 pts. total)

Consider the closed parametrized curve in $\mathbb{R}^2$ that is given implicitly by the formula:

$$(x^2+y^2)^2 \;=\; a^2(x^2-y^2)\;, \qquad a>0\;.$$

i) (5 pts.) Show that in polar coordinates this curve is given by:

$$r^2 \;=\; a^2\cos(2\theta)\;, \qquad \text{where} \quad |\theta|\le\frac{\pi}{4}\;, \quad \text{or} \quad |\theta-\pi|\le\frac{\pi}{4}\;.$$

ii) (10 pts.) Compute the signed curvature function $k(s)$ along this curve (as in Chapter 1.7B).

iii) (10 pts.) Show that the rotation index of this closed curve is 0. Why does this not contradict the theorem on turning tangents?

**Solution:** For i) the polar substitution $x=r\cos(\theta)$ and $y=r\sin(\theta)$ gives:

$$r^4 \;=\; a^2r^2(\cos^2(\theta)-\sin^2(\theta)) \;=\; a^2r^2\cos(2\theta)\;.$$

Then divide through by $r^2$. It is important to note here the restriction $\cos(2\theta)\ge 0$ because the LHS is always nonnegative. We have $\cos(\psi)\ge 0$ for $\psi$ in the range $|\psi|\le\frac{\pi}{2}$ and $|\psi-2\pi|\le\frac{\pi}{2}$. Dividing these by two gives the restrictions $|\theta|\le\frac{\pi}{4}$ or $|\theta-\pi|\le\frac{\pi}{4}$ which correspond to the right and left hand "petals" of the curve.

For part ii) we use the formula from part i) and Problem 1.5.11 of the text which gives:

$$k(\theta) \;=\; \frac{2(\rho')^2-\rho\rho''+\rho^2}{((\rho')^2+\rho^2)^{\frac{3}{2}}}\;, \qquad \text{where} \quad \rho \;=\; a\sqrt{\cos(2\theta)}\;.$$

We compute:

$$\rho' \;=\; -\frac{a\sin(2\theta)}{\sqrt{\cos(2\theta)}} \;=\; -\tan(2\theta)\rho\;, \qquad \rho'' \;=\; -2\sec^2(2\theta)\rho+\tan^2(2\theta)\rho\;.$$

Plugging this into the above formula we have:

$$k \;=\; \frac{2\tan^2(2\theta)\rho^2+2\sec^2(2\theta)\rho^2-\tan^2(2\theta)\rho^2+\rho^2}{(\tan^2(2\theta)\rho^2+\rho^2)^{\frac{3}{2}}} \;=\; \frac{3}{\rho\sec(2\theta)} \;=\; \frac{3}{a^2}\rho \;=\; \frac{3}{a^2}\sqrt{x^2+y^2}\;.$$

Notice that this is only the signed curvature for $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$ also traces the right hand petal counterclockwise. Thus to be consistent with the orientation coming from the right hand petal we actually need to go backwards in $\theta$ from $\frac{5\pi}{4}$ to $\frac{3\pi}{4}$. In other words the signed curvature on the LHS petal going clockwise (which is how one ends up by coming counterclockwise from the RHS) is actually $k=-\frac{3}{a^2}\sqrt{x^2+y^2}$.

For part iii) its ok to use symmetry to conclude:

$$\int_C kds \;=\; 0\;,$$

because the curve is symmetric with respect to $(x,y)\mapsto(-x,y)$, while $k\mapsto -k$ under this transformation. On the other hand $k$ is simple enough function, so using the formulas from Problem 1.5.11 we can in fact compute the integral explicitly using:

$$ds=\sqrt{\rho^2+(\rho')^2}d\theta \;=\; \rho\sec(2\theta)d\theta \;=\; \frac{3}{|k|}d\theta\;.$$

Using this we find:

$$\int_C kds \;=\; \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 3d\theta - \int_{-\frac{3\pi}{4}}^{\frac{5\pi}{4}} 3d\theta \;=\; 0\;.$$

## Part IV (25 pts. total)

A conformally flat parametrization of a surface $S \subseteq \mathbb{R}^3$ is a parametrization $\Phi : \Omega \to S$ such that $E = G$ and $F = 0$. That is:

$$\| \Phi_u \| \ = \ \| \Phi_v \| \ , \qquad \text{and} \quad \Phi_u \cdot \Phi_v \ = \ 0 \ .$$

Prove that every surface of revolution has a conformally flat parametrization locally about each of its points. (Note: It turns out that *every* surface in $\mathbb{R}^3$ has a conformally flat parametrization about each of its points, but you may not use this fact here.)

**Solution:** Without loss of generality we may assume $S$ is revolved around the $z$-axis. In this case the surface is given by 2D curve $s \mapsto (r(s), z(s))$ in the $(r, z)$ plane where $r = \sqrt{x^2 + y^2}$ is the polar coordinate. The parametrization for the surface becomes:

$$\Phi(s, \theta) \ = \ (r(s)\cos(\theta), r(s)\sin(\theta), z(s)) \ .$$

A simple calculation shows that in this parametrization:

$$E \ = \ (r')^2 + (z')^2 \ , \qquad F \ = \ 0 \ , \qquad G \ = \ r^2 \ .$$

There is still the freedom to reparametrize the curve $(r(s), z(s))$. First of all we could have assumed $s$ was arc length, so $(r')^2 + (z')^2 = 1$. Thus, in terms of line element notation:

$$metric \ = \ ds^2 + r^2(s) d\theta^2 \ = \ r^2(s)\Big( \big(\frac{ds}{r}\big)^2 + d\theta^2 \Big) \ .$$

Thus, to get the conformally flat parametrization we need to find a function $s = s(t)$ so that:

$$\frac{ds}{dt} \ = \ r(s(t)) \ .$$

In this case our metric becomes:

$$metric \ = \ r^2(s(t))\Big( dt^2 + d\theta^2 \Big) \ .$$

The last ODE can be solved via separation of variables by integrating both sides of:

$$\frac{ds}{r(s)} \ = \ dt \ ,$$

and then finding the inverse function of $Q(s) = \int \frac{ds}{r}$. Note that since the integrand is positive $Q$ is monotone increasing so $s(t) = Q^{-1}(t)$ exists.