I. Some properties of the FT on L^p.

<u>Definition</u>: If $u \in S'(\mathbb{R}^n)$ is a tempered distribution we define $u^\vee$ via the formula $u^\vee(f) = \frac{1}{(2\pi)^n} u(R\hat{f})$, for all $f \in S(\mathbb{R}^n)$. We call this the "inverse Fourier transform".

A general, but somewhat abstract, resut about the computation of Fourier transforms and the Fourier inversion formula is the following:

<u>Proposition</u>: Let $u \in S'(\mathbb{R}^n)$. Then $(\hat{u})^\vee = u$. In addition, if $u \in L^p(\mathbb{R}^n)$ for $1 \le p \le 2$ and $f_k \in S(\mathbb{R}^n)$ are such that $f_k \to u$ in $L^p(\mathbb{R}^n)$ then $\hat{f}_k \to \hat{u}$ in $L^{p'}(\mathbb{R}^n)$.
In addition, if we write $\hat{u}_R(\xi) = \int_{|x| \le R} e^{-ix\cdot\xi} u(x)\,dx$, then $\hat{f}_R \to \hat{u}$ in $L^{p'}(\mathbb{R}^n)$.

<u>pf</u>: The abstract inversion formula follows from computing $(\hat{u})^\vee(f) = \frac{1}{(2\pi)^n}\hat{u}(RFf) = \frac{1}{(2\pi)^n} u(FRFf) = u(\frac{1}{(2\pi)^n} RF^2 f)$ all $f \in S(\mathbb{R}^n)$, so $(\hat{u})^\vee(f) = u(f)$ thanks to inversion on $S(\mathbb{R}^n)$. Suppose $f_k \in S(\mathbb{R}^n)$ with $f_k \to u$, then by Hausdorff-Young $\|\hat{f}_k - \hat{u}\|_{L^{p'}} \le (2\pi)^{n/p'} \|f_k - u\|_{L^p}$ so the result follows.
The same result for $\hat{f}_k$ replaced by $\widehat{\chi_{|x| \le R}\, u}$ follows similarly.

Just like before we have the following list of identities for the FT on L^p.

<u>Proposition</u>: Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ be functions and let $\hat{f}, \hat{g}$ be their Fourier transforms in the sense of distributions. Then:

1) $\widehat{\tau_y f} = e^{-iy\cdot\xi}\hat{f}$, $\widehat{e^{ix\cdot\eta} f} = \tau_\eta \hat{f}$, $\widehat{f\circ A} = \frac{1}{|A|}\hat{f}\circ(A^{-1})^t$ all $A \in Gl_n$ for $1 \le p \le 2$.
2) $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$ when $1 \le p = q \le 2$.
3) $\widehat{f * g} = \hat{f}\hat{g}$ and $(f*g)^\vee = (2\pi)^n f^\vee \cdot g^\vee$ if $3/2 \le 1/p + 1/q$.

4) If $f \in L^p(\mathbb{R}^n)$ for $1 \le p \le 2$ and $\varphi \in S$ then $(\hat{f} \cdot \varphi)^\vee = \varphi^\vee * f$.

5) If $p=q=2$ then $(f,g) = \frac{1}{(2\pi)^n}(\hat{f}, \hat{g})$.

pf: The proofs follow easily from the corresponding statements for $S(\mathbb{R}^n)$ and density. Note that the conditions on p, q in 3)-4) are so Hausdorff-Young is satisfied. The constants in 3) comes from $(f*g)^\vee = \frac{1}{(2\pi)^n} R\widehat{(f*g)} = \frac{1}{(2\pi)^n} R\hat{f} \cdot R\hat{g} = (2\pi)^n f^\vee \cdot g^\vee$. Part 4) follows by taking the FT of both sides.

Now we can prove the following thing about Fourier inversion.

Thm: Let $f \in L^p(\mathbb{R}^n)$ with $1 \le p \le 2$. Let $\varphi \in S(\mathbb{R}^n)$ such that $\varphi(0) = 1$. Set $f_\epsilon(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \varphi(\epsilon\xi) \hat{f}(\xi) d\xi$. Then as $\epsilon \to 0$ one has $f_\epsilon \to f$ in L^p and $f_\epsilon(x) \to f(x)$ for all $x \in \mathcal{L}_f$.

pf: $\varphi(\epsilon\xi) \in S(\mathbb{R}^n)$, so $f_\epsilon = (\varphi(\epsilon\cdot)\hat{f})^\vee = (\varphi(\epsilon\cdot))^\vee * f$. We have $\varphi(\epsilon\cdot)^\vee(x) = \epsilon^{-n} \varphi^\vee(\epsilon^{-1}x)$. Since $\varphi^\vee \in S(\mathbb{R}^n)$, it is easy to check $|\varphi(\epsilon\cdot)^\vee(x)| \le \frac{C\epsilon^{-n}}{(1+|\epsilon x|)^{n+1}}$, and $\int \varphi(\epsilon\cdot)^\vee(x) dx = 1$. Thus $\varphi(\epsilon\cdot)^\vee$ is an "approximate identity" and the result follows.

Remark: By splitting functions, the previous theorem also shows that for $f \in L^1 + L^2$ one has $\frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \varphi(\epsilon\xi) \hat{f}(\xi) d\xi \longrightarrow f(x)$ pointwise a.e. for any $\varphi \in S(\mathbb{R}^n)$ with $\varphi(0)=1$. A typical example is $f(x) \overset{a.e.}{=} \lim_{\epsilon \to 0} \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} e^{-\epsilon|\xi|^2} \hat{f}(\xi) d\xi$ all $f \in L^1 + L^2$.

II. The Uncertainty Principle

We start with the most basic uncertainty principle which states that if

$\hat{f}$ is concentrated on a box of size $\lambda_1 \times \dots \times \lambda_n$ then the support of f cannot be compressed further than boxes of size $\lambda_1^{-1} \times \dots \times \lambda_n^{-1}$:

<u>Thm:</u> (Heisenberg's Uncertainty Relation) Let $f \in S(\mathbb{R}^n)$, then for $(x_0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^n$ one has $\|f\|_{L^2}^2 \leq \frac{2}{n}(2\pi)^{-n/2} \sum_{k=1}^{n} \|(x-x_0)^k f\|_{L^2} \|(\xi-\xi^0)_k \hat{f}\|_{L^2}$. The constant is sharp in the sense that there is equality for $f = |A|^{1/2} e^{ix \cdot \xi^0} e^{-\frac{1}{2}|A(x-x_0)|^2}$ where $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ is a positive definite diagonal matrix.

<u>Remark:</u> By the Cauchy-Schwarz inequality one has $\|f\|_{L^2}^2 \leq \frac{2}{n}(2\pi)^{-n/2} \| |x-x_0| f\|_{L^2} \| |\xi-\xi^0| \hat{f}\|_{L^2}$ which in general is a less precise form.

<u>Pf:</u> First compute $\operatorname{Im}(D_k f, x^k f) = \frac{1}{2i}\left((D_k f, x^k f) - (x^k f, D_k f)\right) = \frac{1}{2i}([x^k, D_k] f, f) = \frac{1}{2}\|f\|_{L^2}^2$

Thus $\|f\|_{L^2}^2 \leq 2\|x^k f\|_{L^2} \|D_k f\|_{L^2} = 2(2\pi)^{-n/2} \|x^k f\|_{L^2} \|\xi_k \hat{f}\|_{L^2}$. Suming on $k = 1, \dots, n$ yields $\|f\|_{L^2}^2 \leq \frac{2}{n}(2\pi)^{-n/2} \sum_{k=1}^{n} \|x^k f\|_{L^2} \|\xi_k \hat{f}\|_{L^2}$. Applying this estimate to $e^{-ix \cdot \xi^0} \tau_{-x_0} f$ yeilds the desired result by noting $\widehat{e^{-ix\cdot\xi_0}\tau_{-x_0} f} = e^{i(\xi+\xi_0)\cdot x_0} \hat{f}(\xi+\xi_0)$.

To see the optimality of the Gaussian note that $\widehat{e^{i(\cdot)\cdot\xi^0}\tau_{x_0} f} = e^{-ix_0\cdot(\xi-\xi^0)} \hat{f}(\xi-\xi^0)$,
so with $f(x) = e^{ix\cdot\xi_0} e^{-\frac{1}{2}|A(x-x_0)|^2}$ then $\hat{f} = (2\pi)^{n/2}|A^{-1}|^{1/2} e^{-ix_0\cdot(\xi-\xi^0)} e^{-\frac{1}{2}|A^{-1}(\xi-\xi_0)|^2}$.

By change of variables we compute $\|f\|_{L^2}^2 = \int e^{-|x|^2} dx = \pi^{n/2}$.

Likewise $\|(x-x_0)^k f\|_{L^2}^2 = \lambda_k^{-2} \int (x^k)^2 e^{-|x|^2} dx = \frac{1}{2}\lambda_k^{-2} \int e^{-|x|^2} dx = \frac{1}{2}\pi^{n/2} \lambda_k^{-2}$.

And $\|(\xi-\xi^0)_k \hat{f}\|_{L^2}^2 = (2\pi)^n \lambda_k^2 \int (\xi_k)^2 e^{-|\xi|^2} d\xi = 2^{n-1} \pi^{3n/2} \lambda_k^2$.

Thus $\sum_{k=1}^{n} \|(x-x_0)^k f\|_{L^2} \|(\xi-\xi^0)_k \hat{f}\|_{L^2} = \frac{n}{2}(2\pi)^{n/2} \cdot \pi^{n/2} = \frac{n}{2}(2\pi)^{n/2} \|f\|_{L^2}^2$

Next, we express another version of the uncertainty principle in terms of L^p norms. This will also allow us to introduce a few more basic concepts about distributions.

Defn: Let $u \in S'(\mathbb{R}^n)$ be a tempered distribution. We define $\mathrm{supp}(u)$ to be the smallest closed subset of $\mathbb{R}^n$ such that $u(f) = 0$ for all $f \in S(\mathbb{R}^n)$ with $\mathrm{supp}(f) \subseteq \mathbb{R}^n \setminus \mathrm{supp}(u)$. We say a distribution $u \in S'(\mathbb{R}^n)$ is of compact support if $\mathrm{supp}(u)$ is bounded. We define $\mathcal{E}'(\mathbb{R}^n)$ to be all distributions of compact support.

Lemma: Let $u \in S'(\mathbb{R}^n)$ with compact support. Then u has a natural extension to a map $u: C^\infty(\mathbb{R}^n) \to \mathbb{C}$ which is continuous with respect to the seminorms $\|\cdot\|_{\alpha,K}$, $\alpha \in \mathbb{N}^n$ and $K \subseteq \mathbb{R}^n$ compact, where $\|f\|_{\alpha,K} = \sup_{x \in K} |\partial^\alpha f(x)|$.

pf: Let $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ on some open set $U \supseteq \mathrm{supp}(u)$. Then for $f \in S(\mathbb{R}^n)$ we have $u(f) = u((1-\psi)f + \psi f) = u(\psi f)$ because $\mathrm{supp}((1-\psi)f) \subseteq \mathbb{R}^n \setminus \mathrm{supp}(u)$.

Thus $|u(f)| \leq C \sum_{|\alpha| \leq N} \|x^\alpha \partial^\beta (\psi f)\|_u \leq \tilde{C} \sum_{|\beta| \leq m} \|\partial^\beta f\|_{L^\infty(\mathrm{supp}(\psi))}$. Now this makes sense even for $f \in C^\infty(\mathbb{R}^n)$ so the extension property an continuity wrt $\|\cdot\|_{\alpha,K}$ follows at once.

The Fourier transform of compactly supported distributions is easy to describe "classically":

Prop: If $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\hat{u} \in C^\infty(\mathbb{R}^n)$ and is given by the formula $\hat{u}(\xi) = u(e^{-i\xi \cdot (\,)})$ and satisfies an estimate $|\partial_\xi^\alpha \hat{u}(\xi)| \leq C_\alpha (1+|\xi|)^N$ some fixed N. On the other hand if $\hat{u} \in \mathcal{E}'(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$, is given by $u(x) = \frac{1}{(2\pi)^n} \hat{u}(e^{ix\cdot(\,)})$, and satisfies an estimate of the form $|\partial_x^\alpha u(x)| \leq C_\alpha (1+|x|)^N$ for some fixed N.

pf: First define the function $\hat{u}(\xi) = u(e^{-i\xi\cdot(\,)}) = u_x(e^{-ix\cdot\xi})$. Since the difference quotients $\Delta_{k,h} e^{-ix\cdot\xi^0} = \frac{1}{h} e^{-i\xi\cdot x}\Big|_{\xi=\xi^0}^{\xi=\xi^0+he_k} \to \partial_{\xi_k^0} e^{-ix\cdot\xi}\Big|_{\xi=\xi^0}$ in $C^\infty(\mathbb{R}^n)$ wrt x variable (uniform on compact sets) we get $\Delta_{k,h}\hat{u}(\xi) = u_x(\Delta_{k,h} e^{-i\xi\cdot x}) \to u_x(-ix^k e^{-i\xi\cdot x})$. Induction gives $\partial_\xi^\alpha \hat{u}$ exists and $\partial^\alpha \hat{u}(\xi) = (-i)^{|\alpha|} u_x(x^\alpha e^{-i\xi\cdot x})$, thus $|\partial^\alpha \hat{u}(\xi)| \le C \sum_{|\beta|\le N}' \|\partial^\beta (x^\alpha e^{-i\xi\cdot x})\|_{L^\infty(K)}$ some compact $K \subseteq \mathbb{R}^n$. Thus $|\partial^\alpha \hat{u}| \le C_\alpha (1+|\xi|)^N$ follows. Finally, to show $\hat{u}(\xi)$ is the FT of u in the sense of distributions we need to show $\langle u, \int e^{-ix\cdot\xi} f(\xi) d\xi\rangle = \int \langle u, e^{-ix\cdot\xi}\rangle f(\xi) d\xi$ for all $f \in S(\mathbb{R}^n)$. By density & continuity of FT, it suffices to consider $f \in C_c^\infty(\mathbb{R}^n)$. Then the result follows from linearity and Riemann integration because for any sequence of meshes $|\Delta_k^{(j)}(\xi)| \to 0$ in j, we have $\sum_k' e^{-ix\cdot\xi} f(\xi) \Delta_k^{(j)}(\xi) \xrightarrow{C^\infty(\mathbb{R}^n)} \int e^{-ix\cdot\xi} f(\xi) d\xi$, and by smoothness of $\langle u, e^{-ix\cdot\xi}\rangle$ we have $\sum_k' \langle u, e^{-ix\cdot\xi}\rangle f(\xi) \Delta_k^{(j)}(\xi) \to \int \langle u, e^{-ix\cdot\xi}\rangle f(\xi) d\xi$ as well.

Corollary: Let $u \in S'(\mathbb{R}^n)$, then $\exists$ a sequence $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that $u_\varepsilon \rightharpoonup u$ (weakly in sense of distributions). Moreover, the convergence is "uniformly bounded" in the sense sense that $|\langle u_\varepsilon, f\rangle| \le C \sum_{\substack{|\alpha|\le N \\ |\beta|\le M}}' \|f\|_{\alpha,\beta}$ for C, N, M uniform in $\varepsilon \to 0$.

pf: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ on $|x| \le 1$. Define $u_\varepsilon(x) = \varphi(\varepsilon x) w_\varepsilon(x)$ where $\widehat{w_\varepsilon} = \varphi(\varepsilon x)\cdot \hat{u}$, i.e. $\hat{w}_\varepsilon(f) = \hat{u}(\varphi(\varepsilon\cdot) f)$. Since $\hat{w}_\varepsilon \in \mathcal{E}'(\mathbb{R}^n)$ we have $w_\varepsilon(x) \in C^\infty(\mathbb{R}^n)$, so $\varphi(\varepsilon x)\cdot w_\varepsilon(x) \in C_c^\infty(\mathbb{R}^n)$. Fix $f \in S(\mathbb{R}^n)$, then $\langle u_\varepsilon, f\rangle = \langle w_\varepsilon, f\rangle + \langle w_\varepsilon, (\varphi(\varepsilon\cdot) - 1) f\rangle$. Thus, to show $u_\varepsilon \rightharpoonup u$ uniformly in our sense we need $w_\varepsilon \rightharpoonup u$ and $|\langle w_\varepsilon, g\rangle| \le C \sum_{\substack{|\alpha|\le N \\ |\beta|\le M}}' \|g\|_{\alpha,\beta}$ uniform as $\varepsilon \to 0$. Both of these follow at once because $\langle w_\varepsilon, f\rangle = \langle \hat{w}_\varepsilon, f^\vee\rangle = \langle \hat{u}, \varphi(\varepsilon\cdot) f^\vee\rangle$. And $\varphi(\varepsilon\cdot) f^\vee \to f^\vee$ in S, while $\hat{u} \in S'(\mathbb{R}^n)$ so $|\langle \hat{u}, \varphi(\varepsilon\cdot) g^\vee\rangle| \le C \sum_{\substack{|\alpha|\le N \\ |\beta|\le M}}' \|g^\vee\|_{\alpha,\beta} \le \tilde{C} \sum_{\substack{|\alpha|\le N' \\ |\beta|\le M'}}' \|g\|_{\alpha,\beta}$ uniform as $\varepsilon \to 0$.

We now use these regularizations to set up the basic convolution identities for distributions.

<u>Defn</u>: Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then we define $u * f$ to be the function $x \mapsto \langle u, \tau_x R f \rangle$.

<u>Theorem</u>: Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then:

1) $u * f \in C^\infty(\mathbb{R}^n)$, and there exists an $N > 0$ depending only on u, and $C_\alpha > 0$ depending on on both u and f such that $|\partial^\alpha (u*f)(x)| \leq C_\alpha (1+|x|)^N$. One has $\partial^\alpha (u*f) = u * \partial^\alpha f$.

2) One has $(u*f)*g = u*(f*g)$, and $\langle u*f, g\rangle = \langle u, Rf*g\rangle$.

3) $\widehat{u*f} = \hat{u}\hat{f}$ and $(u*f)^\vee = (2\pi)^n u^\vee \cdot f^\vee$ where multiplication is defined by $\langle f\cdot u, g\rangle = \langle u, f\cdot g\rangle$.

4) $\widehat{uf} = \frac{1}{(2\pi)^n} \hat{u} * \hat{f}$ and $(u\cdot f)^\vee = u^\vee * f^\vee$.

<u>pf</u>: 1) To show $u*f \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(u*f) = u*(\partial^\alpha f)$ we use induction and the fact that the difference quotients in x for x fixed, $\Delta_{k,h} f(x-y) = \frac{1}{h}(f(x+he_k-y) - f(y))$ converge to $\partial_k f(x-y)$ in $\mathcal{S}(\mathbb{R}^n)$.

To get the bound note $|u*f(x)| \leq C \sum_{\substack{|\alpha| \leq N \\ |\beta| \leq m}} \|\tau_x R f\|_{\alpha,\beta}$, and $\|\tau_x R f\|_{\alpha,\beta} = \| y^\alpha \partial^\beta f(x-y)\|_{L^\infty(dy)}$
$\leq C_\alpha (1+|x|)^{|\alpha|} \|(1+|x-y|)^{|\alpha|} \partial^\beta f(x-y)\|_{L^\infty(dy)} \leq C_\alpha (1+|x|)^{|\alpha|} \sum_{|\alpha'| \leq |\alpha|} \|f\|_{\alpha',\beta}$.

To show the identities 2)-4) we use the same idea repeatedly: If $u_\varepsilon, u \in \mathcal{S}'(\mathbb{R}^n)$ and $u_\varepsilon \rightharpoonup u$, then for $f \in \mathcal{S}(\mathbb{R}^n)$ $u_\varepsilon * f \rightharpoonup u*f$. This is a simple consequence of the formula $\int \langle u, \tau_x R f\rangle g(x)\, dx = \langle u, \int \tau_x R f g(x) dx\rangle$ for $g \in C_c^\infty(\mathbb{R}^n)$ which can be shown using the Riemann sum idea of the FT proof for $\mathcal{E}'(\mathbb{R}^n)$ above. By density this gives $\langle u*f, g\rangle = \langle u, Rf*g\rangle$ all $u \in \mathcal{S}'(\mathbb{R}^n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $u_\varepsilon * f \rightharpoonup u*f$ follows. Now 2)-4) all follow from their counterparts in the case $u \in \mathcal{S}(\mathbb{R}^n)$ as well.

Now we return to uncertainty principles for the Fourier transform. The following estimate is often useful in the theory of PDE:

Thm: (Bernstein's Inequality) Let $u \in S'(\mathbb{R}^n)$ be a tempered distribution such that $\hat{u}$ is supported in a rectangle $R = \{ \xi \in \mathbb{R}^n \mid |\xi_k - \xi^0_k| \leq \lambda_k \}$ for some $\xi^0 \in \mathbb{R}^n$ and $\lambda_k > 0$. Then if $u \in L^p(\mathbb{R}^n)$ one has $u \in L^s(\mathbb{R}^n)$ all $s \geq p$ and there is a fixed $C > 0$ (not depending on p, s) such that: $\|u\|_{L^s} \leq C |R|^{1/p - 1/s} \|u\|_{L^p}$ where $|R| = \prod_{k=1}^n \lambda_k$ is the measure of R.

Pf: By multiplying u by $e^{-ix\cdot\xi^0}$ we can assume $\xi^0 = 0$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a function with $\varphi \equiv 1$ on the box $|x^k| \leq 2$ for $k = 1, \dots, n$, and set $\varphi^R(\xi) = \varphi(A^{-1}x)$ where $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then $\hat{u} \cdot \varphi^R = \hat{u}$, so by the Fourier inversion formula we have $u = u * (\varphi^R)^\vee$. By interpolating we only need to show both $\|(\varphi^R)^\vee\|_{L^1} \leq C$ and $\|(\varphi^R)^\vee\|_{L^\infty} \leq C|R|$. The second bound follows at once from $|\varphi^\vee(x)| \leq \frac{1}{(2\pi)^n} \|\varphi\|_{L^1(d\xi)}$, while the second follows from the formula $(\varphi^R)^\vee(x) = |A| \varphi^\vee(Ax)$ and $\varphi^\vee \in S(\mathbb{R}^n)$.