NOTES FOR 240C: THE HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

JACOB STERBENZ

Abstract. We give an account of HLS using dyadic methods.

1. Dyadic Rearrangements of Functions

First we discuss some general tools that are useful for proving estimates in L^p spaces. These tools are often used when inequalities can be had by consideration of rough information on the the bulk "size" of functions (e.g. estimates that don't involve delicate cancellation properties).

We let $(X, \mathcal{M}, \mu)$ be a general measure space, and for an $\mathcal{M}$-measureable function $f : X \to \mathbb{C}$ we denote by $\lambda_f(t) : [0,\infty) \to [0,\infty]$ its distribution function. Recall that in class (see Section 1.13 of [3]) we showed that if ν is any Radon measure on $[0,\infty)$ then one has the formula:

$$\int_X \Phi(|f|)d\mu \ = \ \int_0^\infty \lambda_f(t)d\nu(t) \ , \qquad \text{where} \quad \Phi(t) \ = \ \nu\big([0,t)\big) \ . \tag{1}$$

In particular this gives a convenient formula for things like the L^p norms of f when $1 \leqslant p < \infty$. Another useful quantity that is associated with f that can be used as a proxy for λ_f (and in particular records all useful bulk "size" information) is the following:

Definition 1.1 (Decreasing Rearrangements). Let $(X, \mathcal{M}, \mu)$ and $f : X \to \mathbb{C}$ be given as above. Then for $s \in (0,\infty)$ we define the function $f^*(s) = \inf\{t \,\big|\, \lambda_f(t) \leqslant s\}$ which is called the "decreasing rearrangement" of f. Note that we define $f^*(s) = \infty$ whenever $\lambda_f(t) > s$ for all $t \in [0,\infty)$.

It is not hard to prove that f^* is monotone decreasing and right continuous, and hence it is a Borel measureable function from $(0,\infty)$ to $[0,\infty]$. One of the key properties of f^* is that it is equidistributed with f, that is:

Lemma 1.2. *One has $\lambda_{f^*}(t) = \lambda_f(t)$ for all $t \in [0,\infty)$. In particular for any monotone increasing left continuous function $\Phi : [0,\infty] \to [0,\infty]$ as in formula (1) above one has $\int_X \Phi(|f|)d\mu = \int_0^\infty \Phi(f^*)ds$, where ds denotes Lebesgue measure.*

Proof. To show $\lambda_{f^*}(t) = \lambda_f(t)$ it suffices to compute:

$$\{f^* > t\} \ = \ (0, \lambda_f(t)) \ , \tag{2}$$

with the understanding that $\{f^* > t\} = \emptyset$ when $\lambda_f(t) = 0$. First, the containment $\{f^* > t\} \subseteq (0, \lambda_f(t))$ is immediate because if $s \in (0,\infty)$ is such that $f^*(s) = \inf\{t' \,\big|\, \lambda_f(t') \leqslant s\} > t$, we cannot have $s \geq \lambda_f(t)$ or the infimum would be $\leq t$. On the other hand if $\lambda_f(t) > s$ then by right continuity of λ_f one has $\lambda_f(t+\epsilon) > s$ for some $\epsilon > 0$. In this case $f^*(s) = \inf\{t' \,\big|\, \lambda_f(t') \leqslant s\} \geqslant t + \epsilon > t$. □

Before continuing, it is useful a few inequalities associated with f^* and λ_f which are closely related to (2). These are:

Lemma 1.3. *For the functions f^* and λ_f defined above one has the following identities for all finite values:*

$$\lambda_f(f^*(s)) \leqslant s \ , \qquad\qquad f^*(\lambda_f(t)) \leqslant t \ . \tag{3}$$

In addition whenever $0 < \epsilon < f^*(s), \lambda_f(t) < \infty$ one has:

$$\lambda_f(f^*(s) - \epsilon) > s \ , \qquad\qquad f^*(\lambda_f(t) - \epsilon) > t \ . \tag{4}$$

Proof. Note that these identities (3) follow immediately from the definition of f^*. Indeed by right continuity of $\lambda_f(t)$ we actually have $f^*(s) = \min\{t \,\big|\, \lambda_f(t) \le s\}$, so $\lambda_f(f^*(s)) \leqslant s$ follows at once. On the other hand $s \le \lambda_f(t)$ when $s = \lambda_f(t)$, so by definition $f^*(s) \le t$ for this value of s which gives $f^*(\lambda_f(t)) \leqslant t$.

Likewise, the identities on line (4) follow from (2), and we'll leave further work here to the reader. $\square$

From the previous Lemma and (1) we have:

Proposition 1.4. *Let $(X, \mathcal{M}, \mu)$ be given and $f : X \to \mathbb{C}$ a $\mathcal{M}$-measureable function. Then for $0 < p < \infty$ one has:*

$$(5) \qquad \| f \|_{L^p(d\mu)} \ = \ \| f^* \|_{L^p(dm)} \ , \qquad \sup_{s>0} s^{\frac{1}{p}} f^*(s) \ = \ \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}} \ ,$$

where m denotes Lebesgue measure on $(0, \infty)$.

Proof. We just need to prove the second identity. We'll use line (4) for this. First, for $\lambda_f(t) > 0$ and $\epsilon > 0$ sufficiently small we have:

$$\sup_{s>0} s^{\frac{1}{p}} f^*(s) \ \geqslant \ (\lambda_f(t) - \epsilon)^{\frac{1}{p}} f^*(\lambda_f(t) - \epsilon) \ \geqslant \ (\lambda_f(t) - \epsilon)^{\frac{1}{p}} t$$

so taking $\epsilon \to 0$ and supping over t with $\lambda_f(t) > 0$ gives $\sup_{s>0} s^{\frac{1}{p}} f^*(s) \geqslant \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}}$. The opposite inequality follows from $\sup_{t>0} t\lambda_f(t)^{\frac{1}{p}} \geqslant (f^*(s) - \epsilon)\lambda_f(f^*(s) - \epsilon)^{\frac{1}{p}} \geqslant (f^*(s) - \epsilon)s^{\frac{1}{p}}$ for $f^*(s) > \epsilon > 0$ and then taking $\epsilon \to 0$ and $\sup_{s>0}$. $\square$

Based on this we construct a function from f which gives its (weak) L^p norms and is somewhat easier to work with in estimates.

Theorem 1.5. *Let $(X, \mathcal{M}, \mu)$ and $f : X \to \mathbb{C}$ be a measureable function with $\lambda_f(t) < \infty$ for all $t > 0$. Then there exists a collection of constants $c_n > 0$ and disjoint measureable sets E_n, with the property that $\mu(E_n) \leqslant 2^n$, $c_{n+1} < |f_n(x)| \leqslant c_n$, and $f = 0$ a.e. on the set $X \setminus \cup E_n$. Finally, if we have the following equivalence:*

$$(6) \qquad (\sum_n c_n^p 2^n)^{\frac{1}{p}} \ \approx_p \ \| f \|_{L^p} \ , \qquad \sup_n c_n 2^{\frac{n}{p}} \ \approx_p \ \| f \|_{L^{p,\infty}} \ ,$$

where $\| f \|_{L^{p,\infty}} = \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}}$.

Proof of Theorem 1.5. Define $c_n = f^*(2^{n-1})$, and set $E_n = \{x \in X \,\big|\, c_{n+1} < |f(x)| \le c_n\}$. Then the bounds of f on E_n are immediate from definition, and $\mu(E_n) \le \lambda_f(f^*(2^n)) \le 2^n$ follows from line (3).

Next we show that $f = 0$ a.e. on the set $X \setminus \cup E_n$. First note that the condition $\lambda_f(t) < \infty$ and continuity of measures implies the level set identity:

$$\mu(\{|f| = t\}) \ = \ m(\{f^* = t\}) \ , \qquad \text{for all} \quad t \ > \ 0 \ ,$$

which follows by writing $\mu(\{|f| = t\}) = \mu(\cap_k \{t \ge |f| > t - \frac{1}{k}\}) = \lim_k \lambda_f(t - \frac{1}{k}) - \lambda_f(t)$. In addition to this we also compute for any $s > 0$:

$$f^*(s) \ \le \ f^*(0^+) \ = \ \lim_{k\to\infty} \inf\{t \,\big|\, \lambda_f(t) \le 1/k\} \ = \ \inf\{t \,\big|\, \lambda_f(t) = 0\} \ = \ \| f \|_{L^\infty} \ , \qquad \text{where} \quad \inf(\emptyset) = \infty \ .$$

Thus, if $\mu(\{|f| = \| f \|_{L^\infty}\}) = 0$ we have $c_n < \| f \|_{L^\infty}$ for all n and $\lim_{n\to-\infty} c_n = \| f \|_{L^\infty}$. And if $\mu(\{|f| = \| f \|_{L^\infty}\}) > 0$ we have $c_n = c_{n_0}$ for all $n \le n_0$. In either case $\{|f| > 0\} \subseteq \cup_n \{|f| \le c_n\}$ away from a set of measure zero. In addition $\lim_{s\to\infty} f^*(s) = 0$ again by $\lambda_f(t) < \infty$, so we also have $\{|f| > 0\} = \cup_n \{c_n < |f|\}$.

To get the identities on line (6) we use (5) and the monotonicity of f^* which gives:

$$\int_0^\infty (f^*)^p(s)ds \ \approx \ \sum_n (f^*)^p(2^n)2^n \ , \qquad \sup_{s>0} s^{\frac{1}{p}} f^*(s) \ \approx_p \ \sup_n c_n 2^{\frac{n}{p}} \ .$$

$\square$

2. The Weak Young's Inequality and the Hardy-Littlewood-Sobolev Fractional Integration Theorem

We now use the dyadic setup of the previous section to prove a basic inequality in analysis which often comes up in applications (PDE, mathematical physics, etc). This is:

Theorem 2.1 (The Hardy-Littlewood-Sobolev estimate)**.** *Let $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{n}=2$ where $0<\lambda<n$ and $1<p,q<\infty$. Then there exists a constant $C=C(n,p,\lambda)$ such that for $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$ one has $f(x)|x-y|^{-\lambda}g(y)\in L^1(R^{2n})$ and:*

$$|\iint_{\mathbb{R}^{2n}} f(x)|x-y|^{-\lambda}g(y)dxdy\big| \ \leqslant\ C(n,p,\lambda)\| f\|_{L^p}\| g\|_{L^q} \ . \tag{7}$$

To prove this one might be tempted to view it as a consequence of Young's inequality. Indeed, setting $h(x)=|x|^{-\lambda}$ the integral becomes $\iint f(x)h(x-y)g(y)dxdy$, so if $h\in L^r$ for $r=\frac{n}{\lambda}$ we would be done. However we see $\| h\|_{L^r}^r=\int |x|^{-n}dx$ which fails to be integrable (by a log) both as $|x|\to 0$ and $|x|\to\infty$. Thus, using Young's inequality directly to prove (7) is out. In spite of this, in a moment we will show that (7) in fact *does* follow from a version of Young's inequality once matters have been localized correctly to dyadic scales. In the process of doing this we will also show that there is nothing special about the exact form of the integral (7) and in fact a more general result holds. The key observation in this regard is the fact that $\| |x|^{-\lambda}\|_{L^{\frac{n}{\lambda},\infty}}=|B_1(n)|^{\frac{\lambda}{n}}$ where $|B_1(n)|$ is the volume of the unit ball in $\mathbb{R}^n$ and $\| f\|_{L^{r,\infty}}^r=\sup_{t>0}t^r\lambda_f(t)$. Thus, to prove (7) it suffices to show:

Theorem 2.2 (Weak Young's Inequality)**.** *Let $\frac{1}{p}+\frac{1}{r}+\frac{1}{q}=2$ be such that $1<p,q,r<\infty$ and let $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$, and $h\in L^{r,\infty}(\mathbb{R}^n)$. Then $f(x)h(x-y)g(y)\in L^1(\mathbb{R}^{2n})$ and there exists a constant $C=C(p,q)$ such that:*

$$|\iint_{\mathbb{R}^{2n}} f(x)h(x-y)g(y)dxdy\big| \ \leqslant\ C(p,q)\| h\|_{L^{r,\infty}}\| f\|_{L^p}\| g\|_{L^q} \ . \tag{8}$$

To show this theorem we will use a version of the usual Young's inequality adapted to $\ell^p=L^p(\mathbb{Z})$.

Lemma 2.3. *Let $(a_n)\in\ell^p$, $(b_n)\in\ell^q$, $(c_n)\in\ell^r$ where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\geqslant 2$. Then one has the estimate:*

$$|\sum_{n,m}a_nb_mc_{n-m}\big| \ \leqslant\ \|(a_n)\|_{\ell^p}\|(b_n)\|_{\ell^q}\|(c_n)\|_{\ell^r} \ . \tag{9}$$

Proof. Choose $\widetilde{p}\geqslant p$, $\widetilde{q}\geqslant q$, $\widetilde{r}\geqslant r$ with $\frac{1}{\widetilde{p}}+\frac{1}{\widetilde{q}}+\frac{1}{\widetilde{r}}=2$. By the same calculation used to produce Young's inequality on $\mathbb{R}^n$ we have LHS(9)$\leqslant\|(a_n)\|_{\ell^{\widetilde{p}}}\|(b_n)\|_{\ell^{\widetilde{q}}}\|(c_n)\|_{\ell^{\widetilde{r}}}$. Then (9) follows from the discrete L^p bound $\|(a_n)\|_{\ell^{\widetilde{p}}}\leqslant\|(a_n)\|_{\ell^p}$, and similarly for (b_n) and (c_n). □

Proof of (8). The key observation that makes this estimate tick is that for indicator functions of Lebesgue measureable sets F,G,H the usual version of Young's inequality is highly inefficient if the sizes of these sets are widely separated. More specifically by trading one of χ_F, χ_G, or χ_H, for 1 in the convolution we have:

$$\iint_{\mathbb{R}^{2n}}\chi_F(x)\chi_G(x-y)\chi_H(y)dxdy \ \leqslant\ \frac{|F|\,|G|\,|H|}{\max\{|F|,|G|,|H|\}} \ =\ M(F,G,H)\|\chi_F\|_{L^p}\|\chi_G\|_{L^q}\|\chi_H\|_{L^r} \ ,$$

where:

$$M(F,G,H) \ \leqslant\ \Big(\frac{\min\{|F|,|G|,|H|\}\cdot\text{med}\{|F|,|G|,|H|\}}{\max\{|F|,|G|,|H|\}^2}\Big)^{\epsilon} \ , \qquad 0<\epsilon=\min\{1-\frac{1}{p},1-\frac{1}{q},1-\frac{1}{r}\} \ .$$

Applying this estimate to the functions $|f|\leqslant\sum_i a_i\chi_{F_i}$, $|g|\leqslant\sum_j b_j\chi_{G_j}$, $|h|\leqslant\sum_k c_k\chi_{H_k}$ as defined in Theorem 1.5 we have the estimate:

$$\iint_{\mathbb{R}^{2n}}|f(x)|\,|h(x-y)|\,|g(y)|dxdy \ \leqslant\ \sum_{i,j,k}2^{\epsilon(i+j+k-3\max\{i,j,k\})}a_i2^{\frac{1}{p}i}b_j2^{\frac{1}{q}j}c_k2^{\frac{1}{r}k} \ . \tag{10}$$

Using the inequality $\sum_k 2^{\epsilon(i+j+k-3\max\{i,j,k\})}\leqslant C\epsilon^{-1}2^{-\epsilon|i-j|}$ we directly have:

$$RHS(10)\leqslant C\epsilon^{-1}\| c_k2^{\frac{1}{r}k}\|_{\ell^\infty}\sum_{i,j}2^{-\epsilon|i-j|}a_i2^{\frac{1}{p}i}b_j2^{\frac{1}{q}j} \ . \tag{11}$$

Notice that by the condition $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$ we automatically have $\frac{1}{p}+\frac{1}{q}+1\geqslant 2$, so we can apply (9) with $r=1$ and $c_n=2^{-\epsilon|n|}$ to the RHS sum above to yield (for a new C uniform in p,q,r):

$$RHS(11) \ \leqslant \ C\epsilon^{-2}\| a_i 2^{\frac{1}{p}i} \|_{\ell^p}\| b_j 2^{\frac{1}{q}j} \|_{\ell^q}\| c_k 2^{\frac{1}{r}k} \|_{\ell^\infty} \ .$$

The final estimate (8) follows by applying (6) which gives $\| a_i 2^{\frac{1}{p}i} \|_{\ell^p} \leqslant C\| f \|_{L^p}$, $\| b_j 2^{\frac{1}{q}j} \|_{\ell^q} \leqslant C\| g \|_{L^q}$, and $\| c_k 2^{\frac{1}{r}k} \|_{\ell^\infty} \leqslant C\| h \|_{L^{r,\infty}}$. □

References

[1] Folland, Gerald B. Real analysis. **Modern techniques and their applications.** Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[2] Grafakos, Loukas **Classical Fourier analysis.** Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.

[3] Lieb, Elliott H.; Loss, Michael **Analysis.** Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[4] Stein, Elias M.; Weiss, Guido **Introduction to Fourier analysis on Euclidean spaces.** Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971.

Department of Mathematics, University of California, San Diego (UCSD) 9500 Gilman Drive, Dept 0112 La Jolla, CA 92093-0112 USA

Email address: `jsterben@math.ucsd.edu`