I. LCH spaces and Radon Measures

A. Basics of LCH spaces

<u>Lemma</u>: Let X be a Hausdorff space and set $X^* = \{X, x_*\}$ $(x_* \notin X)$, and $\mathcal{T}_{X^*} = \{U \in \mathcal{T}_X,\ X^*\setminus C \text{ where } C \subseteq X \text{ is compact}\}$. Then $\mathcal{T}_{X^*}$ is a topology on X^* making it into a compact space. Finally, $i: X \hookrightarrow X^*$ is an embedding.

<u>pf</u>: First note that if $U \subseteq X$ open and $C \subseteq X$ compact then $U \cup X^*\setminus C = U \cup C^c \cup \{x_*\} = (U^c \cap C)^c \cup \{x_*\}$ (c in X) $= X^* \setminus (U^c \cap C)$, and $U^c \cap C$ is compact in X. In addition $U \cap (X^*\setminus C) = U \cap C^c$ which is open in X.

Thus, we only need to show that $U_1, U_2, U_\alpha \in \mathcal{T}_X$ & C_1, C_2, C_α compact in X implies $U_1 \cap U_2 \in \mathcal{T}_X$, $\bigcup_\alpha U_\alpha \in \mathcal{T}_X$, $(X^*\setminus C_1) \cap (X^*\setminus C_2) = X^* \setminus (C_1 \cup C_2)$ & $C_1 \cup C_2$ compact in X, and $\bigcup_\alpha (X^*\setminus C_\alpha) = X^* \setminus (\bigcap_\alpha C_\alpha)$ where $\bigcap_\alpha C_\alpha$ compact in X. These are all obvious.

Finally, let $X^* = \bigcup_\alpha V_\alpha$, $V_\alpha \in \mathcal{T}_{X^*}$. Then some $V_{\alpha_0} = X^* \setminus C_{\alpha_0}$ because $x_* \in V_{\alpha_0}$. Then $C_{\alpha_0} \subseteq \bigcup_\alpha (V_\alpha \cap X)$, and $V_\alpha \cap X$ is always open. Thus $C_{\alpha_0} \subseteq \bigcup_{i=1}^{N} V_{\alpha_i}$ & $V_{\alpha_0}, \dots, V_{\alpha_N}$ covers X^*.

Finally, $i^{-1}(V) = V \cap X$ is open all $V \in \mathcal{T}_{X^*}$. Also $i(U) \subseteq X^*$ is open all $U \in \mathcal{T}_X$.

<u>Defn</u>: A Hausdorff space is called "locally compact" (LCH) if X^* is Hausdorff. We call X^* the "one point compactification" of X.

<u>Lemma</u>: If X is a LCH and $C \subseteq U \subseteq X$ with C compact and U open. Then $\exists$ $f: C(X, [0,1])$ with $f|_C \equiv 1$ and $f \equiv 0$ outside some compact subset of U.

<u>pf</u>: Note that $C \subseteq U \subseteq X^*$ are compact resp open. Thus $X^* \setminus U$ is closed and there exists $C \subseteq \tilde{U}$ and $X^* \setminus U \subseteq V$ with $\tilde{U}, V \in \mathcal{T}_{X^*}$ and $\tilde{U} \cap V = \emptyset$.

Thus $C \subseteq \tilde{U} \subseteq \overline{\tilde{U}} \subseteq V^c \subseteq U$, and $\overline{\tilde{U}}$ is compact in X. By Urysohn's Lemma there exists $f: C(X^*, [0,1])$ with $f|_C = 1$ and $f|_{\tilde{U}^c} = 0$. Thus $\mathrm{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq \overline{\tilde{U}}$ is compact.

Restricting f to X yields the desired function.

Remark: Note that one of the steps in the proof shows if $C \subseteq U \subseteq X$ w/ C compact and U open then $\exists$ open V with $C \subseteq V \subseteq \bar{V} \subseteq U$ with $\bar{V}$ compact.

Defn: Let X be an topological space. If $U \subseteq X$ is an open set then we for $f \in C(X)$ we write $f \prec U$ if $0 \leq f \leq 1$ and $\mathrm{supp}(f)$ is a compact subset of U. If $C \subseteq X$ is a compact subset we write $C \prec f$ if $f|_C = 1$, $0 \leq f \leq 1$, and $\mathrm{supp}(f)$ is compact. Thus, the conclusion of the previous result can be written as $C \prec f \prec U$.

Lemma: Let X be a LCH space and $C \subseteq \bigcup_{i=1}^{n} U_i$ a compact subset with open covering U_i. Then there exists a collection of $\varphi_i \in C(X)$ with $\varphi_i \prec U_i$ and $C \prec \sum_{i=1}^{n} \varphi_i$.

pf: As shown in the Remark above, for each $x \in C$ with $x \in U_i$ $\exists$ an open set V_x with $x \in V_x \subseteq \overline{V_x} \subseteq U_i$ and $\overline{V_x}$ compact. Let V_{x_j}, $j = 1, \ldots, m$, cover C. For each i set $F_i = \bigcup_{\substack{j: \\ x_j \in U_i}} \overline{V_{x_j}}$. Then $F_i \subseteq U_i$ are compact and $C \subseteq \bigcup_{i=1}^{n} F_i$. Now let $f_i \in C(X, [0,1])$ with $F_i \prec f_i \prec U_i$. Then $\sum_{i=1}^{n} f_i \geq 1$ on C and $\sum_{i=1}^{n} f_i \prec U = \bigcup_{i=1}^{n} U_i$. Let $\psi \in C(X, [0,1])$ be given so that $C \prec \psi \prec \{x \in X \mid \sum_{i=1}^{n} f_i(x) > 0\}$. Setting $\varphi_i = f_i \big/ \left(1 - \psi + \sum_{j=1}^{n} f_j\right)$ gives the desired result.

B. Radon Measures

Defn: Let X be a LCH space. Then a "Radon measure" on $(X, \underbrace{B_X}_{\text{Borel sets}})$ is a Borel measure such that $\mu(C) < \infty$ for a compact $C \subseteq X$, and such that:

i) (Outer Regularity) $\mu(E) = \inf\{\mu(U) \mid U \supseteq E \text{ and } U \text{ open}\}$ all $E \in B_X$,

ii) (Inner Regularity on Open Sets) $\mu(U) = \sup\{\mu(C) \mid C \subseteq U, C \text{ compact}\}$ all U open.

eg: If $F:\mathbb{R}\to\mathbb{R}$ is monotone increasing and right continuous there is a unique Radon measure on $\mathbb{R}$ such that $\mu((a,b])=F(b)-F(a)$. And all Radon measures on $\mathbb{R}$ are given in this way. In fact one only needs to assume $\mu(e)<\infty$ all $e\subseteq\mathbb{R}$ compact and we even get μ is inner regular on all Borel sets (get back to this in general later).

Defn: Let X be a LCH space. We set $C_c(X)$ the set of all $f\in C(X)$ such that $\mathrm{supp}(f)=\overline{\{x\in X \mid f(x)\neq 0\}}$ is compact. A "positive linear functional" on $C_c(X)$ is a linear map $I: C_c(X)\to\mathbb{C}$ such that $I(f)\geq 0$ all $f\geq 0$.

Proposition: Let X be a LCH space and I a positive linear functional on $C_c(X)$. Define the function $\mu:\mathcal{T}_X\to\mathbb{R}$ by $\mu(U)=\sup\{I(f) \mid f\prec U\}$, and $\mu(\phi)=\phi$.
Then the function $\mu^*:P(X)\to\mathbb{R}$ defined by $\mu^*(A)=\inf\{\mu(U) \mid A\subseteq U \text{ open}\}$ is an outer measure.

pf: By a 240A result we just need to show if $U=\bigcup_{i=1}^{\infty}U_i$ then $\mu(U)\leq\sum_{i=1}^{\infty}\mu(U_i)$ because then $\mu^*(A)=\inf\{\sum_{i=1}^{\infty}\mu(U_i) \mid A\subseteq\bigcup_{i=1}^{\infty}U_i \text{ is an open cover}\}$.
To show the inequality let $f\prec U$ and set $K=\mathrm{supp}(f)$. Then $K\subseteq\bigcup_{i=1}^{n}U_i$, so let $\psi_i\in C(X)$ be given with $\psi_i\prec U_i$ and $K\prec\sum_{i=1}^{n}\psi_i\prec U$. Then $f=\sum_{i=1}^{n}\psi_i f$ and since $\psi_i f\prec U_i$ we get $I(f)=\sum_{i=1}^{n}I(\psi_i f)\leq\sum_{i=1}^{n}\mu(U_i)\leq\sum_{i=1}^{\infty}\mu(U_i)$. Taking the sup over all $f\prec U$ gives the result.

Proposition: Let X, I, μ^* be as above. Then all open sets are μ^*-measurable.
In particular μ^* induces a Borel measure μ on X with $\mu(U)=\sup\{I(f) \mid f\prec U\}$ and $\mu(E)=\inf\{\mu(U) \mid E\subseteq U \text{ open}\}$.

pf: We need to show that for all sets $A\subseteq X$ with $\mu^*(A)<\infty$ that $\mu^*(A)\geq \mu^*(A\cap U)+\mu^*(A\setminus U)$ all $U\subseteq X$ open. Given $\varepsilon>0$ choose $V\supseteq A$ open with $\mu(V)<\mu^*(A)+\epsilon$. Then if $\mu^*(V\cap U)+\mu^*(V\setminus U)\leq \mu^*(V)$ we get $\mu^*(A\cap U)+\mu^*(A\setminus U)\leq \mu^*(A)+\epsilon$ for all $\varepsilon>0$. So it suffices to prove the inequality for open sets of finite outer measure. Let V be such a set.

For $\varepsilon>0$ choose $f\prec V\cap U$ with $I(f)>\mu(V\cap U)-\epsilon$. Since $V\setminus \mathrm{supp}(f)$ is open and finite measure we can also find $g\prec V\setminus \mathrm{supp}(f)$ with $I(g)>\mu(V\setminus \mathrm{supp}(f))-\epsilon$. Then $f+g\prec V$ so $\mu^*(V)\geq I(f+g)>\mu(V\cap U)+\mu(V\setminus \mathrm{supp}(f))-2\epsilon \geq \mu^*(V\cap U)+\mu^*(V\setminus U)-2\epsilon$. Take $\epsilon\to 0$.

Proposition: Let X, I, μ be as above. Then μ is a Radon measure, and in fact one has

(★) $\mu(C)=\inf\{I(f)\mid C\prec f\}$ for all $C\subseteq X$ compact.

pf: Fix C compact and let $C\subseteq U$ open. Then $\exists$ $C\prec f\prec U$. Thus $\inf\{I(f)\mid C\prec f\}\leq \inf\{I(f)\mid C\prec f\prec U\}\leq \mu(U)$. Since this works for all $C\subseteq U$ we certainly have $\inf\{I(f)\mid C\prec f\}\leq \mu(C)$. On the other hand let $C\prec f$ and set $U_\epsilon=\{x\in X\mid f(x)>1-\epsilon\}$.
Then $C\subseteq U_\epsilon$ and for all $g\prec U_\epsilon$ we have $f/_{1-\epsilon}-g\geq 0$ so $I(g)\leq \frac{1}{1-\epsilon}I(f)$.
Supping over such g gives $\mu(C)\leq \mu(U_\epsilon)\leq \frac{1}{1-\epsilon}I(f)$, and taking $\varepsilon\to 0$ gives $\mu(C)\leq I(f)$ all $C\prec f$.
Thus $\mu(C)\leq \inf\{I(f)\mid C\prec f\}$ and we are done.
This main identity (★) also shows $\mu(C)<\infty$ all compact $C\subseteq X$.
Finally, if $U\subseteq X$ open choose $\alpha<\mu(U)$ and $f\prec U$ with $I(f)>\alpha$. Let $C=\mathrm{supp}(f)$, then for any $C\prec g$ we have $g-f\geq 0$ so $I(g)\geq I(f)>\alpha$, and by the main identity of the proposition we get $\mu(C)>\alpha$ as well. Thus $\sup\{\mu(C)\mid C\subseteq U \text{ compact}\}\geq \mu(U)$, and the other direction is immediate from monotonicity.

<u>Theorem</u>: Let X be a LCH space and $I: C_c(X) \to \mathbb{C}$ a positive linear functional. Then there exists a unique Radon measure μ such that $I(f) = \int_X f \, d\mu$. Moreover $\mu(U) = \sup\{I(f) \mid f \prec U\}$ for all U open while $\mu(C) = \inf\{I(f) \mid C \prec f\}$ all C compact.

<u>pf</u>: We have already given the construction $I \mapsto \mu$ where μ is Radon with the correct formulas for $\mu(U)$ and $\mu(C)$.

To get the integral formula, by splitting into real and imaginary parts, then positive and negative parts, and finally rescaling it suffices to show $\int_X f \, d\mu = I(f)$ all $f \in C_c(X, [0,1])$.

Given such an f choose $N \in \mathbb{N}$ and for $j = 1, \dots, N$ and define $K_j = \{x \in X \mid f(x) \geq j N^{-1}\}$.

Since K_j are closed and $\subseteq supp(f) = K_0$ they are compact. For each $1 \leq j \leq N$ set

$$f_j(x) = \begin{cases} 0, & x \notin K_{j-1}, \\ f(x) - (j-1)N^{-1}, & x \in K_{j-1} \setminus K_j, \\ N^{-1}, & x \in K_j. \end{cases} = \min\left\{\max\left\{f - \tfrac{j-1}{N}, 0\right\}, \tfrac{1}{N}\right\}.$$

Thus $N^{-1}\chi_{K_j} \leq f_j \leq N^{-1}\chi_{K_{j-1}}$. In addition we have $\sum_{j=1}^{N} f_j = f$. Thus $N^{-1}\mu(K_j) \leq \int_X f_j \, d\mu \leq N^{-1}\mu(K_j)$ and we get

$$\frac{1}{N}\sum_{j=1}^{N} \mu(K_j) \leq \int_X f \, d\mu \leq \frac{1}{N}\sum_{j=0}^{N-1} \mu(K_j).$$

On the other hand if $K_{j-1} \subseteq U$ open we get $N f_j \prec U$ so $I(f_j) \leq N^{-1}\mu(U)$, and taking inf over $K_{j-1} \subseteq U$ we get $I(f_j) \leq N^{-1}\mu(K_{j-1})$. But since $K_j \prec N f_j$ we also have $N^{-1}\mu(K_j) \leq I(f_j)$ by the previous proposition. Thus $\frac{1}{N}\sum_{j=1}^{N} \leq I(f) \leq \frac{1}{N}\sum_{j=0}^{N-1} \mu(K_j)$ as well.

Taking the difference yields $\left|I(f) - \int_X f \, d\mu\right| \leq N^{-1}(\mu(K_0) - \mu(K_N)) < \frac{\mu(supp(f))}{N}$.

Taking $N \to \infty$ gives the desired result.

Finally we prove the uniqueness result. If $\tilde{\mu}$ is any Radon measure such that $I(f) = \int_X f \, d\tilde{\mu}$ all $f \in C_c(X)$. If $U \subseteq X$ is open and $f \prec U$ then $0 \leq f \leq \chi_U$ so integration gives $I(f) \leq \tilde{\mu}(U)$. On the other taking $K_n \subseteq U$ compact with $\tilde{\mu}(K_n) \to \tilde{\mu}(U)$, we can find $K_n \prec f_n \prec U$ so integration gives $\int_X f_n \, d\tilde{\mu} \to \tilde{\mu}(U)$. Thus $\tilde{\mu}(U) = \sup\{I(f) \mid f \prec U\} = \mu(U)$, where μ is the Radon measure constructed above.

Since $\tilde{\mu}(E) = \inf\{\tilde{\mu}(U) \mid E \subseteq U \text{ open}\} = \inf\{\mu(U) \mid E \subseteq U \text{ open}\} = \mu(E)$ all Borel E we are done.