I. Rellich Compactness

A. Failure of compactness in $H^s(\mathbb{R}^d)$.

★ Note that just like in $L^2(\mathbb{R}^d)$, weakly convergent $H^s(\mathbb{R}^d)$ sequences can fail to be strongly convergent for <u>two</u> reasons:

a) Convergence to infinity in x, $u_n(x) = u_0(x-y_n)$ where $|y_n| \to \infty$, $u_0 \in H^s$.

Then $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^d)$.

b) Convergence to infinity in ξ, $u_n(x) = \langle D \rangle^{-s}\left(e^{i\xi_n \cdot x} u_0(x)\right)$, $u_0 \in L^2(\mathbb{R}^d)$.

Then $\hat{u}_n(\xi) = \langle \xi \rangle^{-s} \hat{u}_0(\xi - \xi_n)$, $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^d)$ (test on $f \in H^{-s}(\mathbb{R}^d)$, so $\langle f, u_n \rangle = \int \langle D \rangle^{-s} f(x)\, e^{i\xi_n x} u_0(x)\, dx \to 0$).

B. Rellich's Thm

★ Now prove the above examples are the only obstruction:

<u>Thm</u>: Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^d)$. Then for all $\sigma < s$ and $\chi \in S(\mathbb{R}^d)$ one has $\chi u_n \to \chi u$ in $H^\sigma(\mathbb{R}^d)$.

<u>eg</u>: Let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^d)$, and $\Omega \subset\subset \mathbb{R}^d$. Then $u_n \to u$ in $L^2(\Omega)$, because we can take $\chi \equiv 1$ on Ω.

<u>pf</u>: First show that for $u \in H^s$ and $\chi \in S(\mathbb{R}^d)$ that $\|\chi u\|_{H^s} \le C_\chi \|u\|_{H^s}$. This follows by computing $\widehat{\chi u} = \hat{\chi} * \hat{u}$, where $\hat{u} \in L^2_{loc}(\mathbb{R}^d)$ with $\langle \xi \rangle^{-s} \hat{u}(\xi) \in L^2$.

Thus $\widehat{\chi u}(\xi) = \int \hat{\chi}(\xi - \eta)\hat{u}(\eta)\, d\eta$. Then use

$\langle \xi \rangle^s \le C_s \langle \xi - \eta \rangle^s \langle \eta \rangle^s$ when $s \ge 0$, and $\langle \xi \rangle^s \le C_s\left(\langle \xi - \eta \rangle^{-s} \langle \eta \rangle^s\right)$ when $s < 0$.

In either case we conclude via Young's inequality and $\langle \xi \rangle^{|s|} \hat{\chi}(\xi) \in L^2$.

In light of the previous estimate, the principle of uniform boundedness and $\| \langle \xi \rangle^t \hat{u} \|_{L^2(|\xi| \geq R)} \leq C R^{t-s} \| u \|_{H^s}$ for $t < s$ and $R \to \infty$, we only need to prove $u_n \rightharpoonup 0$ in H^s implies $\| \hat{\chi} * \hat{u}_n \|_{L^2(|\xi| \leq R)} \to 0$, for $\chi \in S(\mathbb{R}^d)$. In fact we claim $\hat{\chi} * \hat{u}_n \to 0$ in $C^k(|\xi| < R)$ all k. This is because $\hat{\chi} * \hat{u}_n \to 0$ <u>pointwise</u>, and in addition $\partial_\xi^\alpha (\hat{\chi} * u_n) = \partial_\xi^\alpha \hat{\chi} * u_n$, and $\left| \int \partial_\xi^\alpha \hat{\chi}(\xi - \eta) u_n(\eta) d\eta \right| \leq \| \langle \eta \rangle^{-s} \partial_\xi^\alpha \hat{\chi}(\xi - \eta) \|_{L^2(d\eta)} \cdot \| u_n \|_{H^s(\mathbb{R}^d)} \leq C(\xi, \alpha, \chi) \| u_n \|_{H^s(\mathbb{R}^d)}$ where C locally bounded in ξ.