I. Convolutions

A. Translations and Reflections

<u>Defn:</u> For $f:\mathbb{R}^n \to \mathbb{C}$ and $y \in \mathbb{R}^n$ set $\tau_y f(x) = f(x-y)$. We set $Rf(x) = f(-x)$.

By translation invariance we get $\|\tau_y f\|_{L^p} = \|f\|_{L^p}$ all $y \in \mathbb{R}^n$ and $1 \le p \le \infty$.

In addition we can state the property of being uniformly continuous as

$\|\tau_y f - f\|_u \to 0$ as $|y| \to 0$.

<u>Proposition:</u> For $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$ then $y \mapsto \tau_y f$ is a uniformly continuous map from $\mathbb{R}^n \to L^p(\mathbb{R}^n)$.

<u>pf:</u> If $f \in C_c(\mathbb{R}^n)$ then the result follows from DCT. If $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ choose $\varphi \in C_c(\mathbb{R}^n)$ with $\|\varphi - f\|_{L^p} < \varepsilon$. Since $\|\tau_{x+y} f - \tau_x f\|_{L^p} = \|\tau_y f - f\|_{L^p}$, we can just prove $\|\tau_y f - f\|_{L^p} \to 0$ as $|y| \to 0$, and since $\overline{\lim_{|y|\to 0}} \|\tau_y f - f\|_{L^p} \le \overline{\lim_{|y|\to 0}} \|\tau_y (f - \varphi)\|_{L^p} + \overline{\lim_{|y|\to 0}} \|f - \varphi\|_{L^p} + \overline{\lim_{|y|\to 0}} \|\tau_y \varphi - f\|_{L^p}$ $< 2\varepsilon$, we may conclude by taking $\varepsilon \to 0$.

<u>Defn:</u> Let $f, g: \mathbb{R}^n \to \mathbb{C}$ be two measurable functions such that $f \cdot g \in L^1(\mathbb{R}^n)$.

Then we set $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) g(x)\, dx$, and $(f, g) = \langle f, \bar{g} \rangle$.

<u>Lemma:</u> If f, g and $y \in \mathbb{R}^n$.

1) If $\langle \tau_y f, g \rangle$ exists, then so does $\langle f, \tau_{-y} g \rangle$ and they are equal.

2) If $\langle Rf, g \rangle$ exists then so does $\langle f, Rg \rangle$ and they are equal.

<u>pf:</u> Recall that if $T: \mathbb{R}^n \to \mathbb{R}^n$ is a non-singular affine transformation $Tx = Ax - y$ then for $f \in L^1(\mathbb{R}^n)$, $f \circ T \in L^1(\mathbb{R}^n)$ and one has $\int_{\mathbb{R}^n} f(Ty) |\det(A)|\, dy = \int_{\mathbb{R}^n} f(y)\, dy$ (see thm 2.44). Thus for $Tx = x - y$ we have $f(x-y) g(x) = f(Tx) \tau_{-y} g(Tx)$ and the result follows. 2) is also direct.

B. Convolutions

<u>Defn</u>: Fix $x \in \mathbb{R}^n$, then if $f, g : \mathbb{R}^n \to \mathbb{C}$ are measurable and if $f(x-y)g(y) \in L^1(dy)$ we define $f * g(x) = \int f(x-y)g(y)dy$. Given two such measurable functions we say "$f*g$ exists" if $f*g(x)$ exists for a.e. $x \in \mathbb{R}^n$.

<u>Remark</u>: By Tonelli, $f*g$ exists and is measurable iff $\int |f(x-y)g(y)|dy < \infty$ a.e. $x \in \mathbb{R}^n$.

<u>Lemma</u>: Let $f, g, h \in \mathbb{R}^n \to \mathbb{C}$ be measurable functions.

1) If $f*g(x-y)$ exists then so does $\tau_y f * g$ and $f * \tau_y g$ and they are all equal.

2) If $f*g(-x)$ exists then so does $Rf * Rg(x)$ and they are all equal.

3) If $f*g(x)$ exists then so does $g*f(x)$ and they are equal

4) If $f*g$ and $g*h$ exists, then if $(|f|*|g|)*h(x)$ exists so does both $(f*g)*h(x)$ and $f*(g*h)(x)$ and they are equal.

5) $\mathrm{supp}(f*g) \subseteq \overline{\mathrm{supp}(f)+\mathrm{supp}(g)} = \{z \in \mathbb{R}^n \mid z = x+y,\ x \in \mathrm{supp}(f) \text{ and } y \in \mathrm{supp}(g)\}$ closure

<u>pf</u>: 1) we have $f*g(x) = \langle \tau_x Rf, g \rangle$, so because $\tau_{x-y}R = \tau_x \tau_{-y} R = \tau_x R \tau_y$ get $f*g(x-y) = \langle \tau_{x-y} Rf, g \rangle$
$= \langle \tau_x R \tau_y f, g \rangle = \langle \tau_x Rf, \tau_y g \rangle$ as needed.

2) $f*g(-x) = \langle \tau_{-x} Rf, g \rangle = \langle R\tau_x f, g \rangle = \langle \tau_x R Rf, Rg \rangle = Rf * Rg$.

3) From $\langle \tau_x Rf, g \rangle = \langle f, R\tau_{-x} g \rangle = \langle f, \tau_x Rg \rangle$

4) By definition and Fubini-Tonelli, if $\langle |f|*|g|, \phi \rangle$ exists then $\langle g, Rf*\phi \rangle$ exists and $\langle f*g, \phi \rangle = \langle g, Rf*\phi \rangle$.
In our case this gives $(f*g)*h(x) = \langle f*g, \tau_x Rh \rangle = \langle f, Rg * \tau_x Rh \rangle = \langle f, \tau_x R(g*h) \rangle = f*(g*h)(x)$.

5) Let $z \in \mathbb{R}^n$ such that $x+y = z \implies x \notin \mathrm{supp}(f)$ or $y \notin \mathrm{supp}(g)$. Then for any $y \in \mathbb{R}^n$ $f(z-y)g(y) = 0$ because either $f(z-y) = 0$ or $g(y) = 0$. Thus $f*g(z) = 0$.

<u>Remark</u>: Some of the above formulas become more clear if $|f|*|g| \in L^1_{loc}$ & $\phi \in C_c(\mathbb{R}^n)$, then by Fubini-Tonelli: $\langle f*g, \phi \rangle = \iint f(x)g(y)\,\phi(x+y)\,dxdy$.

<u>Proposition</u>: one has the formula $\partial^\alpha(f*g) = \partial^\alpha f * g$ in the following sense:

1) If $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then for all α, and $\|f*g\|_{C^K} \leq C(K)\|f\|_{\mathcal{S}}\|g\|_{L^p}$,

where $\|f\|_{C^K(\mathbb{R}^n)} = \sum_{|\alpha|\leq K} \|\partial^\alpha f\|_u$, for all $K \geq 0$.

2) If $f \in C^K(\mathbb{R}^n)$ and $\|f\|_{C^K} < \infty$ all $|\alpha| \leq K$ and $g \in L^1(\mathbb{R}^n)$, then $f*g \in C^K(\mathbb{R}^n)$

and $\|f*g\|_{C^K} \leq \|f\|_{C^K}\|g\|_{L^1}$.

3) If $f \in C_c^K(\mathbb{R}^n)$ and $f \in L^1_{loc}(\mathbb{R}^n)$, then $f*g \in C^K(\mathbb{R}^n)$. (Note: There is no uniform

estimate in this case.

4) If $f, g \in \mathcal{S}$ then $f*g \in \mathcal{S}$, and one can write $\partial^\alpha(f*g) = \partial^\alpha f * g = f * \partial^\alpha g$.

5) If $f \in C^{K_1}(\mathbb{R}^n)$ and $g \in C_c^{K_2}(\mathbb{R}^n)$, then $f*g \in C^{K_1+K_2}(\mathbb{R}^n)$ and $\partial^{\alpha+\beta}(f*g) = \partial^\alpha f * \partial^\beta g$

for all $|\alpha| \leq K_1$ and $|\beta| \leq K_2$.

<u>pf</u>: In all cases the proof is basically the same. To show $\int f(x-y)g(y)\,dy$

is differentiable at x_0 we just need $|\partial_{x_i} f(x-y)g(y)| \leq h(y) \in L^1(dy)$ all $|x-x_0| \ll 1$.

Higher derivatives can be treated by induction.

1) Here use $|\partial_{x_i} f(x-y)g(y)| \leq \|(1+|x|)^{n+1}\partial_{x_i} f(x)\|_u \cdot (1+|x-y|)^{-n-1}|g(y)|$, $h(y) = (1+|x-y|)^{-n-1}|g(y)| \in L^1(dy)$

and $\|\partial^\alpha f * g\|_u \leq \|\partial^\alpha f\|_{L^{p'}}\|g\|_{L^p} \leq C\|(1+|x|)^{n+1}\partial^\alpha f\|_u \cdot \|g\|_p \leq C_\alpha\|f\|_{\mathcal{S}}\|g\|_{L^p}$.

2) Similar argument, the continuity of $\partial^\alpha f * g$ for $|\alpha| = K$ follows from DCT. The bound

$\|\partial^\alpha f * g\| \leq \|\partial^\alpha f\|_u \|g\|_{L^1}$ follows from triangle inequality.

3) Here note the compact support of $f \in C_c^K(\mathbb{R}^n)$ guarantees $f*g$ converges. Then

$|\partial^\alpha f(x-y)g(y)| \leq \|f\|_{C^K} \cdot \chi_{B_R(x)}(y)|g(y)|$ where $\mathrm{supp}(f) \subseteq B_R(0)$.

4) If $f, g \in \mathcal{S}$ then $|x^\alpha \partial^\beta(f*g)(x)| \leq C_\alpha\left((|x|^{|\alpha|}|\partial^\beta f|) * |g|(x) + |\partial^\beta f| * (|x|^{|\alpha|}|g|)\right)$

because $|x^\alpha| \leq |x|^\alpha \leq 2^{|\alpha|}(|x-y|^\alpha + |y|^\alpha)$. Thus $\|f*g\|_{\alpha,\beta} \leq C_\alpha\left(\|f\|_{\alpha,\beta} \cdot \|(1+|x|)^{n+1}g\|_{L^\infty} + \|(1+|x|)^{n+1}\partial^\beta f\|_{L^\infty}\|g\|_{\alpha,0}\right)$

In fact this shows $T_g : \mathcal{S} \to \mathcal{S}$ where $T_g f = f*g$ is continuous.

5) Similar to 1)-4) above.

Remark: The previous result shows that convolution improves the local properties of functions. However, in general it worsens the global (i.e. decay) properties of functions.

Theorem (Young's ineq etc) Let $g \in L^p(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n) + L^{p'}(\mathbb{R}^n)$, then $f*g(x)$ exists for a.e. $x \in \mathbb{R}^n$. Moreover:

1) If $f \in L^1$ then $f*g \in L^p$ and $\|f*g\|_p \leq \|f\|_{L^1} \|g\|_{L^p}$.

2) If $f \in L^{p'}$ then $f*g$ is uniformly continuous and $\|f*g\|_u \leq \|f\|_{p'} \|g\|_p$. If $1<p<\infty$ then $f*g \in C_0(\mathbb{R}^n)$.

3) If $f \in L^q$ for $\frac{1}{p}+\frac{1}{q} = 1+1/r$ then $f*g \in L^r$ and $\|f*g\|_r \leq \|f\|_p \|g\|_q$ (note $1 \leq q \leq p'$, $r \geq q$).

4) If $g \in L^{p,\infty}$ then 3) holds for $1<p,q<\infty$, and $\|f*g\|_r \leq C(p,q) \|f\|_q \|g\|_{L^{p,\infty}}$.

pf: 1) Use Minkowski $\| \int g(x-y) f(y) dy \|_p \leq \int \|g(x-y)\|_{L^p_x} |f(y)| dy = \|g\|_p \|f\|_1$

2) Use Hölder $|\int f(x-y) g(y) dy| \leq \|f(x-y)\|_{L^{p'}_y} \|g\|_{L^p} = \|f\|_{p'} \|g\|_p$. If either $f \in C_c(\mathbb{R}^n)$ or $g \in C_c(\mathbb{R}^n)$ we get $\|\tau_y(f*g) - f*g\|_u \leq \|\tau_y f - f\|_{p'} \|g\|_p$ or $\leq \|f\|_{p'} \|\tau_y g - g\|_p$ which goes to zero as $|y| \to 0$. Then one can approximate by $C_c(\mathbb{R})$ because either $p \neq \infty$ or $p' \neq \infty$.

To get $f*g \in C_0(\mathbb{R}^n)$ we just need to approximate both f,g by $C_c(\mathbb{R}^n)$, which is possible when $1<p<\infty$.

3) We can use 1) & 2) above and Riesz-Thorin. Setting $T_g f = f*g$ we get $\|T_g\|_{L^{p'} \to L^\infty} \leq \|g\|_p$ and $\|T_g\|_{L^1 \to L^p} \leq \|g\|_p$. Thus, for $\frac{1}{q} = \frac{\theta}{p'} + \frac{1-\theta}{1} = 1 - \frac{\theta}{p}$ and $\frac{1}{r} = \frac{\theta}{\infty} + \frac{1-\theta}{p} = \frac{1}{p} - \frac{\theta}{p}$ we have $\|T_g\|_{L^q \to L^r} \leq \|g\|_p$. Then $\frac{1}{q} + \frac{1}{p} = 1 - \frac{\theta}{p} + \frac{1}{p} = 1 + \frac{1}{r}$, so $\|f*g\|_r \leq \|g\|_p \|f\|_q$.

4) We proved this using Marcinkiewicz interpolation.

C. Approximate Identities via convolution

The main use of convolutions is to smooth functions out in a uniform way. Thus, a major question is how regularizations $\varphi_\varepsilon * f$, $\varphi_\varepsilon = \varepsilon^{-n} \varphi(\varepsilon^{-1} x)$, $\varphi \in L^1(\mathbb{R}^n)$ converge to f as $\varepsilon \to 0$. We have already proved if $f \in L^p$ for $1 \leq p < \infty$ and $\int \phi = 1$

then $\varphi_\varepsilon * f \to f$ in L^p. Here we give a more general discussion.

<u>Definition</u>: Let $K_\delta(x) \in L^1(\mathbb{R}^n)$ be a sequence of functions defined for $0<\delta\leq\delta_0$, with:

1) $\int K_\delta(x)dx = 1$,

2) $\|K_\delta\|_{L^1} \leq C$

3) For each $\varepsilon>0$ $\int_{|x|>\varepsilon} |K_\delta(x)|\,dx \to 0$ as $\delta\to 0$.

We say $K_\delta(x)$ is a "good kernel" if 1)–3) hold.

<u>eg</u>: If $\varphi\in L^1(\mathbb{R}^n)$ and $\int\varphi=1$ then $K_\delta(x)=\delta^{-n}\varphi(\delta^{-1}x)$ has these properties.

<u>Theorem</u>: Let $K_\delta(x)$ be a good kernel. Then for $f\in L^p(\mathbb{R}^n)$ and $1\leq p<\infty$ $K_\delta * f \in L^p(\mathbb{R}^n)$ and $K_\delta * f \to f$ in L^p.

<u>Pf</u>: Write $(K_\delta * f - f)(x) = \int (\tau_y f(x) - f(x))\, K_\delta(y)\,dy$. Thus, for $\varepsilon>0$ by Minkowski we have $\|K_\delta * f - f\|_p \leq C \sup_{|y|\leq\varepsilon} \|\tau_y f - f\|_p + \sup_{|y|\geq\varepsilon} \|\tau_y f - f\|_p \cdot \|\chi_{|y|\geq\varepsilon} K_\delta\|_{L^1(y)}$. Thus $\overline{\lim_{\delta\to0}} \|K_\delta * f - f\|_p \leq C \sup_{|y|\leq\varepsilon}\|\tau_y f - f\|_p$ and since this works for $\varepsilon>0$ we get $\overline{\lim_{\delta\to 0}} \|K_\delta * f - f\|_p = 0$.

When studying the Fourier transform, it helps to know when $K_\delta * f\to f$ <u>pointwise</u>. By the previous result we have a sequence $\delta_n\to 0$ with $K_{\delta_n} * f \to f$ a.e. To get more information we put more conditions on K_δ.

<u>Defn</u>: Let $K_\delta(x)$ satisfy:

i) $\int K_\delta(x) = 1$

(ii') $|K_\delta(x)| \leq C\delta^{-n}$

(iii') $|K_\delta(x)| \leq C\, \delta/|x|^{n+1}$.

We call such K_δ an "approximate identity".

Remark: ii') & iii') $\Rightarrow$ $K_\delta \in L^1$ and i) - iii) above hold.

Thm: Let K_δ be an approximate identity. Then if $f \in L^p(\mathbb{R}^n)$ for any $1 \le p \le \infty$ we have $K_\delta * f(x) \to f(x)$ for all x in the Lebesgue set $\mathcal{L}_f = \{x \in \mathbb{R}^d \mid \lim_{r \to 0} \frac{1}{|B_r(x)|} \int |f(x) - f(y)| dy = 0\}$.

In particular $K_\delta * f \to f$ a.e.

pf: We have $|K_\delta * f(x) - f(x)| \le \int |f(x-y) - f(x)| \cdot |K_\delta(y)| dy$. Fix $\delta > 0$ and write

$$\int |f(x-y) - f(x)| \cdot |K_\delta(y)| dy \le C\delta^{-n} \int_{|y| \le \delta} |f(x-y) - f(x)| dy + \sum_{2^k \ge \delta} I_k(\delta) \quad \text{where}$$

$I_k(\delta) = \int_{2^k \le |y| \le 2^{k+1}} |f(x-y) - f(x)| \cdot |K_\delta(y)| dy$. Since the first integral $\to 0$ as $\delta \to 0$ when $x \in \mathcal{L}_f$, we need to show $\lim_{\delta \to 0} \sum_{2^k \ge \delta} I_k(\delta) \to 0$ as $\delta \to 0$. For a fixed k we have

$$I_k(\delta) \le \delta 2^{-k} \tilde{C} \frac{1}{|B_{2^k}(0)|} \int_{B_{2^k}(0)} |f(x-y) - f(x)| dy \le \delta \tilde{\tilde{C}} \left(2^{-k} |f(x)| + 2^{-nk/p} \|f\|_{L^p} \right).$$

Because $J_k = \frac{1}{|B_{2^k}(0)|} \int_{B_{2^k}(0)} |f(x-y) - f(x)| dy \to 0$ as $k \to -\infty$ choose $k_1 = k_1(\varepsilon)$ such that $J_k \le \varepsilon$ all $k \le k_1$. Then $\sum_{2^k \ge \delta} I_k(\delta) \le A\left(\varepsilon + \delta\left(2^{-k_1}|f(x)| + 2^{-nk_1/p}\|f\|_p\right)\right)$,

where $A > 0$ is universal (not dep on ε, δ, f). Choosing ε first we can make δ so small that RHS $\le 2A\varepsilon$, so we are finished.

Corollary: Let $\varphi \in S(\mathbb{R}^n)$ with $\int \varphi = 1$, and set $K_\delta(x) = \delta^{-n}\varphi(\delta^{-1}x)$. Then if $f \in L^p(\mathbb{R}^n)$ and $1 \le p < \infty$ we have $K_\delta * f \to f$ in L^p and pointwise for all x in the Lebesgue set of f.

If $f \in L^\infty(\mathbb{R}^n)$ then the pointwise result holds for all x in the Lebesgue set of f.