I. Dual of $C_0(X)$.

A. Linear functionals on $C_0(X)$.

Defn: Let X be a LCH space. Set $C_0(X) \subseteq B(X)$ to be the closure of $C_c(X)$ with respect to the uniform norm $\|f\|_u = \sup_{x \in X} |f(x)|$. This is the set of all continuous functions on X which "vanish at infinity".

Lemma: $f \in C_0(X)$ iff f extends to $f \in C(X^*)$ via $f(x_*) = 0$. $C_0(X)$ with norm $\|\cdot\|_u$ is a Banach space.

Lemma: A PLF on X extends to a CLF on $C_0(X)$ iff its Radon measure is finite, and this happens iff $\exists\, C > 0$ s.t. $|I(f)| \leq C\|f\|_u$ all $f \in C_c(X)$.

pf: Since $\mu(X) = \sup\{I(f) \mid f \in C_c(X),\ 0 \leq f \leq 1\} = \sup\{|I(f)| \mid f \in C_c(X), \|f\|_u \leq 1\}$ we get $|I(f)| \leq C\|f\|_u$ iff μ finite. If $I \mapsto \mu$ is finite, then clearly it extends to $C_0(X)$. On the other hand if $I: C_0(X) \to \mathbb{C}$ is a CLF iff $|I(f)| \leq C\|f\|_u$ all $f \in C_c(X)$ by density.

Corollary: Let $L \in (C_0(X))^*$ be positive in the sense that $L(f) \geq 0$ all $f \geq 0$. Then $L(f) = \int_X f\, d\mu$ for a unique finite Radon measure μ.

Now we extend this result to all of $(C_0(X))^*$.

B. Real Linear Functionals

Proposition (Jordan decomp for CLF): Let $L \in C_0(X, \mathbb{R})^*$ be a real linear functional. Then there exists positive linear functionals $L^{\pm} \in C_0(X, \mathbb{R})^*$ such that $L = L^+ - L^-$.

pf: First, for $f \in C_0(X)$ and $f \geq 0$ we define $L^+(f) = \sup\{L(g) \mid g \in C_0(X) \text{ and } 0 \leq g \leq f\}$.
Let $C = \|L\|_{C_0(X)^*}$ then $L(g) \leq C\|g\|_u \leq C\|f\|_u$ all $0 \leq g \leq f$ in $C_0(X)$. Thus $0 \leq L^+(f) \leq C\|f\|_u$.
Next we wish to show L^+ is linear on $C_0(X, [0,\infty))$. If $c > 0$ then $0 \leq g \leq cf \Leftrightarrow 0 \leq c^{-1}g \leq f$

so $L^+(cf) = \sup\{L(g) \mid 0 \leq g \leq cf\} = \sup\{cL(c^{-1}g) \mid 0 \leq c^{-1}g \leq f\} = cL^+(f)$.

Also if $f_1, f_2 \geq 0$ then $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2 \Longrightarrow 0 \leq g_1 + g_2 \leq f_1 + f_2$

So $L^+(f_1+f_2) = \sup\{L(g) \mid 0 \leq g \leq f_1+f_2\} \geq \sup\{L(g_1+g_2) \mid 0 \leq g_1 \leq f_1 \text{ and } 0 \leq g_2 \leq f_2\} = L^+(f_1) + L^+(f_2)$

Also, if $0 \leq g \leq f_1+f_2$ then $g = g_1+g_2$ where $g_1 = \min\{f_1, g\}$ and $g_2 = g - g_1$ and $0 \leq g_1 \leq f_1$, $g_1 \leq g$, and $g \leq g_1 + f_2$, so $0 \leq g_2 \leq f_2$. Thus $\sup\{L(g) \mid 0 \leq g \leq f_1+f_2\} \leq \sup\{L(g_1)+L(g_2) \mid 0 \leq g_1 \leq f_1 \text{ and } 0 \leq g_2 \leq f_2\}$ $= L^+(f_1) + L^+(f_2)$ so we are done.

Next, if $f = f_1 - f_2$ where $f_1, f_2 \geq 0$ we set $L^+(f) = L^+(f_1) - L^+(f_2)$. This is well defined because if $f = \tilde{f}_1 - \tilde{f}_2$ with $\tilde{f}_1, \tilde{f}_2 \geq 0$ then $\tilde{f}_1 + f_2 = f_1 + \tilde{f}_2 \geq 0$ so $L^+(\tilde{f}_1) + L^+(f_2) = L^+(f_1) + L^+(\tilde{f}_2)$. This also shows L^+ (defined on all of $C_0(X,\mathbb{R})$ now) is linear, e.g. if $f = f_1 - f_2$ & $g = g_1 - g_2$ with $f_i, g_i \geq 0$ then $f+g = (f_1+g_1) - (f_2+g_2)$ so $L^+(f+g) = L^+(f_1+g_1) - L^+(f_2+g_2) = L^+(f) + L^+(g)$.

Finally, we have $f = f^+ - f^-$ with $\|f\|_u = \max\{\|f^+\|_u, \|f^-\|_u\}$ so $|L^+(f)| \leq \max\{L^+(f^+), L^+(f^-)\} \leq C\|f\|_u$ ($C = \|L\|_{C_0(X)^*}$). Setting $L^- = L^+ - L$ we have $L^- \in C_0(X,\mathbb{R})^*$ is real and in fact for $f \geq 0$ in $C_0(X)$ $L^-(f) = \sup\{L(g) \mid 0 \leq g \leq f\} - L(f) \geq 0$ (choosing $g=f$).

In addition its worth pointing out $\sup\{L(g) \mid 0 \leq g \leq f\} - L(f) = \sup\{-L(f-g) \mid 0 \leq f-g \leq f\} = (-L)^+(f)$, so $\|L^-\|_{C_0(X)^*} \leq \|L\|_{C_0(X)^*}$ as well.

<u>Corollary</u>: Let $L \in C_0(X,\mathbb{R})^*$ be a real linear functional, then $\exists$ finite (positive) Radon measures $\mu^\pm$ such that $L(f) = \int_X f\,d\mu$ all $f \in C_0(X,\mathbb{R})$, and $\mu = \mu^+ - \mu^-$.

C. Complex Radon Measures

<u>Defn</u>: A finite real "signed Radon measure" on a LCH space X is a signed Borel measure $\mu = \mu^+ - \mu^-$ where $\mu^\pm$ are finite Radon measures.

A "complex Radon measure" is a Borel measure $\mu = \mu_{re} + i\mu_{im}$ where

μ_{re} & μ_{im} are finite signed Radon measures.

Recall that given a complex measure space $(X, \mathcal{B}, \mu)$ (say X LCH, $\mathcal{B}$ is Borel sets, μ Radon) there is a unique positive finite measure $|\mu|$ such that $d\mu = e^{i\theta} d|\mu|$ where $\theta : X \to \mathbb{R}$ is some measurable (Borel) function ($|\mu|$ is the total variation of μ). The measure $|\mu|$ can be specified by the criteria that $|\mu|(E) = \sup\{|\int_E f\, d\mu| \mid 0 \leq |f| \leq 1 \text{ meas}\}$. (see problem #21 in Ch. 3). One has the triangle inequality $|\mu + \nu| \leq |\mu| + |\nu|$, and $|\lambda \mu| = |\lambda| \cdot |\mu|$ for any two such measures μ, ν and $\lambda \in \mathbb{C}$.

<u>Defn</u>: Let X be a LCH. We denote by $M(X)$ the set of all complex Radon measures on X. We set $\|\mu\| = |\mu|(X)$ for $\mu \in M(X)$.

<u>Lemma</u>: The pair $(M(X), \|\cdot\|)$ is a normed vector space which is closed under $\mu \mapsto |\mu|$.

<u>pf</u>: NVS part clear from $\mu \in M(X) \Rightarrow \mu = \mu_{re}^+ - \mu_{re}^- + i(\mu_{im}^+ - \mu_{im}^-)$ $\mu_{re}^{\pm}, \mu_{im}^{\pm}$ all Radon, and the triangle inequality $|\mu+\nu| \leq |\mu| + |\nu|$, once we can prove that $\mu_1, \mu_2 \in M(X)$ and positive $\Rightarrow \mu_1 + \mu_2 \in M(X)$. For the second part if $\mu \in M(X)$ then $|\mu| \leq \mu_{re}^+ + \mu_{re}^- + \mu_{im}^+ + \mu_{im}^-$. Thus, the main thing we need to prove is the following:

<u>Lemma (Regularity Criterion)</u>: Let μ be a complex Borel measure on a LCH space. Then μ is a Radon measure iff for each Borel set $E \subseteq X$ and $\varepsilon > 0$ there exists $K \subseteq E \subseteq U$, K compact & U open, with $|\mu|(U \setminus K) < \varepsilon$.

<u>pf</u>: If μ is Radon then $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where μ_j are σ-finite Radon so $\exists$ $K_j \subseteq E_j \subseteq U_j$, F_j compact and U_i open with $\mu_i(U_i \setminus K_i) < \varepsilon/4$. Setting $K = \bigcup_{i=1}^{4} K_i$ and $U = \bigcap_{i=1}^{4} U_i$ we have $U \setminus K \subseteq U_i \setminus K_i$ any i so $|\mu|(U \setminus K) \leq \sum_{i=1}^{4} \mu_i(U \setminus K) \leq \sum_{i=1}^{4} \mu_i(U_i \setminus K_i) < \varepsilon$.

For the other direction suppose $|\mu|(U\setminus K) < \varepsilon$, then let $\mu = \mu_{re}^+ - \mu_{re}^- + i(\mu_{im}^+ - \mu_{im}^-)$, where $\mu_{re}^\pm$ is the Jordan decomp of μ_{re}, and similarly for $\mu_{im}^\pm$. Then

$\mu_{re}^\pm \leq |\mu_{re}| \leq |\mu|$ and $\mu_{im}^\pm \leq |\mu|$ as well. Thus $\mu_{re}^\pm(U\setminus K) < \varepsilon$ etc so $\mu_{re}^\pm$ & $\mu_{im}^\pm$ are inner & outer regular.

D. Complex Linear Functionals on $C_0(X)$

By taking real and imaginary parts we have $C_0(X,\mathbb{C}) = C_0(X,\mathbb{R}) \oplus i C_0(X,\mathbb{R})$, and any $L \in C_0(X,\mathbb{C})^*$ is determined by $L|_{C_0(X,\mathbb{R})}$. Also, $L^{re} = \mathrm{Re}(L(f))$, $f \in C_0(X,\mathbb{R})$ and $L^{im} = \mathrm{Im}(L(f))$ are real linear functionals. In addition $\|L^{re}\|_{C_0(X,\mathbb{R})^*} \leq \|L\|_{C_0(X,\mathbb{C})^*}$ and similarly for L^{im}, and of course $L = L^{re} + iL^{im}$. Thus, using the material above we have:

Theorem (Riesz Representation for $C_0(X,\mathbb{C})^*$): For each $L \in C_0(X,\mathbb{C})^*$ there is a unique complex Radon measure μ so that $L(f) = \int_X f\, d\mu$ all $f \in C_0(X,\mathbb{C})$. In addition $\|L\|_{C_0(X,\mathbb{C})^*} = \|\mu\|_{M(X)}$. Thus $C_0(X,\mathbb{C})^* = M(X)$ isometrically.

pf: We have already shown the existence of μ. If we can show the isometry identity uniqueness follows directly as well. $|L(f)| = |\int_X f\, d\mu| \leq \int_X |f|\, d|\mu| \leq \|f\|_u \cdot |\mu|(X) = \|\mu\|_{M(X)} \|f\|_u$. Thus $\|L\|_{C_0(X)^*} \leq \|\mu\|_{M(X)}$. On the other hand $\|\mu\|_{M(X)} = \sup\{|\int_X f\, d\mu| \mid 0 \leq |f| \leq 1 \text{ measurable}\}$. If $|\int_X f\, d\mu| > \|\mu\|_{M(X)} - \epsilon/2$, $0 \leq |f| \leq 1$, by Lusin's Thm $\exists\, \varphi \in C_0(X)$ with $0 \leq |\varphi| \leq 1$ and $|\mu|(\{x \in X \mid \varphi(x) \neq f(x)\}) < \varepsilon/4$. Thus $|\int_X f\, d\mu| < |L(\varphi)| + \varepsilon/2 \leq \|L\|_{C_0(X)^*} + \epsilon/2$. This gives $\|L\|_{C_0(X)^*} > \|\mu\|_{M(X)} - \epsilon$ all $\epsilon > 0$ so we are done.

E. Weak Compactness, L^1, etc.

Here is one of the main uses for the material thus far:

Defn: Recall that the weak* topology on $M(X)$ is given by $\mu_\alpha \rightharpoonup \mu$ iff

$\int_X f\,d\mu_\alpha \to \int_X f\,d\mu$ all $f \in C_0(X)$. We also call $\mu_\alpha \rightharpoonup \mu$ "vague convergence".

Recall from the Banach-Alaoglu Thms we have:

1) If $\mu_\alpha \in M(X)$ is a <u>net</u> with $\|\mu_\alpha\|_{M(X)} \leq C$ then $\mu_{\alpha_k} \rightharpoonup \mu$ for some subnet μ_{α_k}.

2) If $C_0(X)$ is separable, then any bounded <u>sequence</u> $\|\mu_n\|_{M(X)} \leq C$ has a convergent subsequence $\mu_{n_k} \rightharpoonup \mu$. For example this happens on $\mathbb{R}^n$.

Recall that $L^1(d\mu)$ is rarely reflexive. This makes it difficult to use weak convergence arguments. However, one does have:

<u>Lemma:</u> Let X be a LCH space and μ a positive (not nec finite) Radon measure. Then if $f \in L^1(d\mu)$ the measure $d\nu = f\,d\mu$ is in $M(X)$. Moreover the map $L^1(d\mu) \hookrightarrow M(X)$ is an isometric embedding.

<u>pf:</u> The fact that ν is Radon follows from density of $C_c(X)$ in $L^1(d\mu)$. This implies we can find a compact subset $K \subseteq X$ with $\int_{K^c} |f|\,d\mu < \varepsilon/2$. Then if $E \subseteq X$ Borel we can find $F \subseteq E \cap K \subseteq V$ with F compact V open & $\mu(V \setminus F) < \delta$. By the a.c. condition we can find δ so small that $\int_{V \setminus F} |f|\,d\mu < \varepsilon/2$ as well. WLOG $\delta < \varepsilon/2$, so if $U = K^c \cup V$ we get $|\nu|(U \setminus F) < \varepsilon$.

Finally, we compute that $\int_X |f|\,d\mu = \int_X d|\nu| = \|\nu\|_{M(X)}$.

<u>Corollary:</u> Let X be a LCH space with $C_0(X)$ separable. Then if μ is any positive (not nec finite) Radon measure, and $f_n \in L^1(d\mu)$ with $\|f_n\|_{L^1} \leq C$, $\exists\, \mu \in M(X)$ and $f_{n_k} \rightharpoonup \mu$ in the sense of measures.