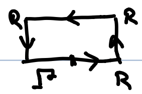
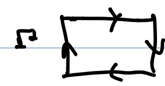
I. FT of $\frac{1}{x \pm i\epsilon}$.

* Now compute $\hat{f}(\xi)$ where $f(x) = \frac{1}{x+i\epsilon}$.

$\hat{f}(\xi) = \lim_{R\to\infty} \int_{-R}^{R} e^{-ix\xi} \frac{dx}{x+i\epsilon}$, $\xi \neq 0$. Two cases: $\xi > 0$, $\xi < 0$.

* $\xi < 0$  $\int_{\Gamma} e^{-i\xi z} \frac{dz}{z+i\epsilon} = 0$. One also checks side and top contours go to zero as $R\to\infty$ $z = \pm R + it$, $0 \le t \le R$ and $z = x + iR$, $-R \le x \le R$

Thus $\hat{f}(\xi) \equiv 0$ all $\xi < 0$.

* Now let $\xi > 0$, then Γ  $\int_{\Gamma} e^{-i\xi z} \frac{dz}{z+i\epsilon} = \frac{2\pi}{i}\left(\frac{1}{2\pi i} \int_{\square} \frac{e^{-i\xi z} dz}{z-(-i\epsilon)}\right)$ $= -2\pi i e^{-\epsilon\xi}$. Thus

$$\hat{f}(\xi) = -2\pi i \, H(\xi) e^{-\epsilon\xi}, \quad H(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi > 0 \end{cases}.$$

* As an exercise let's invert: $c \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\cdot\xi} d\xi = -2\pi i c \int_0^{\infty} e^{(-\epsilon + ix)\xi} d\xi$
$= 2\pi i c \frac{1}{-\epsilon + ix} = \frac{2\pi c}{x + i\epsilon}$.

Thus $c = \frac{1}{2\pi}$ as before.

* A similar calculation using $\hat{\bar{f}} = R\bar{\hat{f}}$ shows $g(x) = \frac{1}{x - i\epsilon} = \bar{f}(x)$, $H(-\xi) = 1 - H(\xi)$

$\hat{g}(\xi) = 2\pi i (1 - H(\xi)) e^{\epsilon\xi}$.

II. Limits as $\epsilon \to 0$.

* These formulas also show $\frac{1}{x \pm i\epsilon}$ have good weak limits as $\epsilon \to 0$, call them $\frac{1}{x \pm i0}$. Then $\frac{1}{2\pi i} \cdot \widehat{\frac{1}{x - i0}} = (1 - H(\xi))$, $\frac{-1}{2\pi i} \widehat{\frac{1}{x + i0}} = H(\xi)$.

* Because of this we get the formulas $\frac{1}{2\pi i}\left(\frac{1}{x-i0}+\frac{1}{x+i0}\right)=\delta_0$

* Also $\frac{1}{2\pi}\left(\frac{1}{x+i0}+\frac{1}{x-i0}\right)=\frac{1}{i}\left(2H(\xi)-1\right)$

In x this distribution is given by $\lim\limits_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{x+i\epsilon}+\frac{1}{x-i\epsilon}\right)f(x)\,dx$

$$= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+\epsilon^2}f(x)\,dx = \lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_{|x|>\epsilon}\frac{f(x)}{x}\,dx\,, \quad f\in S(\mathbb{R}).$$

$$= \text{P.V. } \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{x}f(x)\,dx\,.$$