MATH 240C HOMEWORK 4

Important: Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

Problem # 1

Consider the metric space $\mathcal{C} = \prod_{n=1}^{\infty}\{0,2\}$ with the distance function $d(x,y) = \sum_n 2^{-n}|x_n - y_n|$ where $x = (x_1, x_2, \ldots)$ with $x_n \in \{0,2\}$ and similarly for y.

a) Show that $\mathcal{C}$ is compact, and in fact homeomorphic to the standard Cantor set under the mapping $\Phi(x) = \sum_n x_n 3^{-n}$.

b) On $\mathcal{C}$ consider the sequence of Borel probability measures:

$$\mu_n \ = \ 2^{-n}\sum_{k=1}^{2^n}\delta_{k,n} \ ,$$

where $\delta_{k,n}$ are point masses supported on (some enumeration of) each of the 2^n points of the form $x = (x_1, \ldots, x_n, 0, \ldots)$ (i.e. the points where every coordinate $x_k = 0$ for $k > n$). Prove that $\mu_n \rightharpoonup \mu$ to a Borel probability measure μ on $\mathcal{C}$.

c) Prove that the measure μ from the last part is exactly the Cantor Lebesgue measure (see page 39 of the text) when $\mathcal{C}$ is realized as its image in $[0,1]$ under the map Φ above.

d) Now consider the map $\Psi : \mathcal{C} \to [0,1]$ given by $\Psi(x) = \sum_n x_n 2^{-n-1}$. Prove that Ψ is a continuous surjection and that the standard Lebesgue measure m on $[0,1]$ is exactly the push forward of μ under Ψ. (Recall that this means $\int_0^1 f dm = \int_{\mathcal{C}} f \circ \Psi d\mu$ for every $f \in C([0,1])$).

Problem # 2

a) Construct a continuous Radon measure μ on $[0,1]$ such that $\mu \perp m$ where m is the standard Lebesgue measure and $\mu([a,b]) \neq 0$ for all $0 \leq a < b \leq 1$ (a continuous measure on an interval is one such that its CDF is continuous, or equivalently one such that the measure of any single point is zero).

b) Construct a continuous function $f : [0,1] \to \mathbb{R}$ which is strictly monotone, i.e. $f(x) < f(y)$ for all $0 \leq x < y \leq 1$, and such that its pointwise derivative f' is such that $f'(x) = 0$ for Lebesgue a.e. point $x \in (0,1)$. (Recall that a monotone function is automatically pointwise differentiable a.e. with respect to Lebesgue measure.)

Problem # 3

Let $F \in BV(\mathbb{R})$, and let $\mu = F'$ be is derivative in the sense of distributions.

a) Let $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(\epsilon^{-1}x)$ for $\varphi \in C^\infty_{comp}(\mathbb{R})$ be a standard mollifier (i.e. $\int \varphi dx = 1$). Show that the regularizations $F_\epsilon = \varphi_\epsilon * F$ are such that $\| F_\epsilon \|_{BV(\mathbb{R})} \leq \| F \|_{BV(\mathbb{R})}$ and $F_\epsilon \to F$ a.e. with respect to Lebesgue measure (assume $\varphi \geq 0$ here or get a constant C in the estimate).

b) Prove that the regularized measures $d\mu_\epsilon \ = \ F'_\epsilon dm$ converge weakly in the sense of measures to $d\mu$.

c) Prove that the regularizations F_ϵ converge to F strongly in the $BV(\mathbb{R})$ norm iff μ is absolutely continuous with respect to Lebesgue measure (for this assume F is its right continuous normalization).

d) Show that the set of functions $C^\infty_{comp}(\mathbb{R})$ is weakly dense in $\mathcal{M}(\mathbb{R})$ with the identification $\varphi \mapsto \varphi dm$.

e) Show that the set of finitely supported measures $\mathcal{F} = \{\mu \in \mathcal{M}(\mathbb{R}) \ \big| \ \mu = \sum_{k=1}^N c_k \delta_{x_k}\}$ is weakly dense in $\mathcal{M}(\mathbb{R})$. (Hint: Use the previous part and Riemann sums.)

f) Show that $\mathcal{M}(\mathbb{R}) = \mathcal{A} \oplus \mathcal{S} \oplus L^1(dm)$ where $\mathcal{A} \perp \mathcal{S} \perp L^1(dm)$, and where $\mathcal{A}$ is the norm closure of the set $\mathcal{F}$ defined above, $L^1(dm)$ is the norm closure of $C^\infty_{comp}(\mathbb{R})$ with the identification $\varphi \mapsto \varphi dm$, and where the remaining portion $\mathcal{S}$ is the set of "singular continuous measures" which is by definition the set of complex Borel measures with the property that $\mu \perp m$ and such that $F(x) = \mu((-\infty, x])$ is continuous (for example $\mathcal{S}$ contains the measures from Problems #2 and #3 above).

Problem # 4

Let $f_n \in L^p(\mathbb{R}^d)$ and let $\varphi_\epsilon(x) = \epsilon^{-d}\varphi(\epsilon^{-1}x)$ for $\varphi \in C^\infty_{comp}(\mathbb{R}^d)$ be a standard mollifier. Suppose that $f_n \rightharpoonup f$ weakly in $L^p(\mathbb{R}^d)$ (assume here $1 \leq p < \infty$). For each *fixed* $\epsilon > 0$ prove that the sequence of mollifications $\varphi_\epsilon * f_n \to \varphi_\epsilon * f$ strongly in $C^k(\Omega)$ for all $k \in \mathbb{N}$ and $\Omega \subset\subset \mathbb{R}^d$.

Problem # 5

Recall that for $0 < p < \infty$ a function f is said to be in $L^{p,\infty}(d\mu)$ if $\lambda_f(t) = \mu(\{|f| > t\})$ is such that:

$$\sup_{t>0} t^p \lambda_f(t) \ < \ \infty \ .$$

a) Assume that μ is σ-finite. Show that if $1 < p < \infty$ then one has:

$$\sup_{t>0} t\lambda_f^{\frac{1}{p}}(t) \ \leq \ \sup_{0<\mu(E)<\infty} \mu(E)^{\frac{1}{p}-1} \int_E |f| d\mu \ \leq \ \frac{p}{p-1} \sup_{t>0} t\lambda_f^{\frac{1}{p}}(t),$$

where the supremum is taken over all measurable sets of finite measure. In particular $L^{p,\infty}(d\mu)$ is a Banach space with norm $\| \cdot \|_{L^{p,\infty}(d\mu)}$ given by the middle term in the above chain of inequalities. (Hints: For a set E consider subsets $E_{>t} = E \cap \{|f| > t\}$ and $E_{\leq t} = E \setminus E_{>t}$. For the LHS choose $E_{\leq t} = \emptyset$, and for the RHS do a general calculation and then optimize the value of t. Don't forget to prove completeness!)

b) Prove that if $f \in L^1(\mathbb{R}^d)$ and $g \in L^{p,\infty}(\mathbb{R}^d)$ for $1 < p < \infty$, then $f * g \in L^{p,\infty}(\mathbb{R}^d)$ and in fact one has:

$$\| f * g \|_{L^{p,\infty}(\mathbb{R}^d)} \ \leq \ \| f \|_{L^1(\mathbb{R}^d)} \| g \|_{L^{p,\infty}(\mathbb{R}^d)} \ ,$$

where $\| \cdot \|_{L^{p,\infty}(\mathbb{R}^d)}$ the norm above.