I. Schwartz Space

* We define $S(\mathbb{R}^d)$ as all functions $f \in C^\infty(\mathbb{R}^d)$ with $\|x^\alpha \partial_x^\beta \phi\|_{L^\infty} < \infty$ all $\alpha, \beta \in \mathbb{N}$.
* This is a Frechet space w/ seminorms $p_{\alpha\beta}(\phi) = \|x^\alpha \partial_x^\beta \phi\|_{L^\infty}$.
* Note that $C^\infty_{comp}(\mathbb{R}^d) \subseteq S(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$ & $C^\infty_{comp}(\mathbb{R}^d)$ is dense $(\chi(\epsilon x) f(x))$, while $S(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$.

__eg:__ $e^{-|x|^2} \in S(\mathbb{R}^d)$ but not $C^\infty_{comp}(\mathbb{R}^d)$.

__Lemma:__ $S(\mathbb{R}^d)$ is closed under products and $*$.

__pf:__ Note $|x^\alpha| \le C_\alpha \sum_{|\beta_1|+|\beta_2| = |\alpha|} |(x-y)^{\beta_1}| \cdot |y^{\beta_2}|$, thus $\|p_{\alpha\beta}(f*g)\| \le C_\alpha \sum_{|\beta_1| \le |\beta|} \|f\|_{p_{\alpha\beta_1}} \cdot \sum_{|\beta_2| \le |\beta|} \|g\|_{p_{\alpha\beta_2}}$.

__Lemma:__ the operators $x^\alpha, \partial_x^\alpha$ are continuous in the Frechet Topology.

__pf:__ In general if X, Y are TVS with $\mathcal{F}_{X,P}$, $\mathcal{F}_{Y,Q}$, then $T: X \to Y$ is continuous iff for each q $\exists$ $p_1, \dots, p_N$ with

$$|q(Tx)| \le C_q \sum_{i=1}^N |p_i(x)|.$$

II. The FT on $S(\mathbb{R}^d)$.

* Let $f \in S(\mathbb{R}^d)$, and define $\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$.

__Lemma:__ $|\hat{f}| \le \|f\|_{L^1(\mathbb{R}^d)}$.

__Prop:__ For $f \in S(\mathbb{R}^d)$ one has $\hat{f} \in S(\mathbb{R}^d)$, and in fact $\Phi(f) = \hat{f}$ is continuous. Furthermore:

i) $\widehat{D_x^\alpha f} = \xi^\alpha \hat{f}$, $D_j = \frac{1}{i}\partial_j$. Also $\tau_{x_0} f(x) = f(x - x_0)$, then $\widehat{\tau_{x_0} f} = e^{-ix_0\cdot\xi} \hat{f}$.

ii) $\widehat{x^\alpha f} = (-1)^{|\alpha|} D_\xi^\alpha \hat{f}$. Also $\widehat{e^{ix\cdot\xi^0} f} = \tau_{\xi^0} \hat{f}$

Pf: First show $\hat{f} \in C^\infty(\mathbb{R}^d)$. $\Delta_h \hat{f} = \int \Delta_h(e^{-ix\cdot\xi}) f(x)\,dx$, $|\Delta_h(e^{-ix\cdot\xi})| \leq |x|$.

By DCT $\partial_k \hat{f} = -i \int e^{-ix\cdot\xi} x_k f(x)\,dx$. By induction $\hat{f} \in C^\infty(\mathbb{R}^d)$

and ii) holds.

To get i) we use IBP $\Rightarrow$ $\int e^{-ix\cdot\xi} \partial_{x_k} f\,dx = i\xi_k \int e^{-ix\cdot\xi} f\,dx = i\xi_k \hat{f}(\xi)$.

By lemma $\|\hat{f}\|_{P_{\alpha\beta}} \leq \|\partial^\alpha(x^\beta f)\|_{L^1} \leq C_{\alpha\beta} \sum_{\substack{|\alpha'| \leq |\beta|+d+1 \\ |\beta'| \leq |\alpha|}} \|f\|_{P_{\alpha'\beta'}}$

Remark: If $f \in C^\infty_{comp}(\mathbb{R}^d)$ then $\hat{f}(\xi)$ is analytic in ξ, and extends to a holomorphic function on $\mathbb{C}^n$. Thus $\hat{f} \notin C^\infty_{comp}(\mathbb{R}^d)$.

Thm: $\Phi: S \to S$ is bijective and $f(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi$ all $f \in S(\mathbb{R}^d)$.

proof: Let $Rf(x) = f(-x)$, and consider the operator $T: S \to S$ given by

$T = R\Phi^2 = R \circ \Phi \circ \Phi$. Now $\Phi(D\cdot) = x\Phi(\cdot)$, $\Phi(x\cdot) = -\partial\Phi(\cdot)$, so $\Phi^2(x\cdot) = -x\Phi^2(\cdot)$

$\Phi^2(\partial\cdot) = -\partial\Phi^2(\cdot)$. Also $R(x\cdot) = -xR(\cdot)$, $R(\partial\cdot) = -\partial R(\cdot)$. Thus $[T,x] = [T,\partial] = 0$.

Now if $f \in S(\mathbb{R}^d)$ we have $f(x) = f(x_0) + \sum_{i=1}^d h_i(x;x_0)(x-x_0)_i$.

Letting $\chi(x) \in C^\infty_{comp}(\mathbb{R}^d)$ with $\chi \equiv 1$ for $|x| \leq 1$ we have $f(x) = f(x_0)\chi(x-x_0) + \sum_{i=1}^d \phi_i(x;x_0)(x-x_0)_i$

Thus $Tf(x) = f(x_0)g(x_0) + \sum_{i=1}^d (x-x_0)_i T\phi_i(\cdot;x_0)$, $g(x_0) = T\chi(\cdot - x_0)$.

Setting $x = x_0$ we get $Tf(x) = g(x)f(x)$ all x. Now $[\partial, T] = \partial g$, so $g \equiv g_0$.

Thus $f(x) = \frac{1}{g_0} \int e^{ix\cdot\xi} \hat{f}(\xi)\,d\xi$. To compute g_0 use a lemma.

Lemma: If $f(x) = e^{-\frac{1}{2}|x|^2}$ then $\hat{f}(\xi) = (2\pi)^{d/2} e^{-\frac{1}{2}|\xi|^2}$. In particular $g_0 = (2\pi)^d$.

pf: Taking a product reduce to $d=1$. Then completing the square $\frac{1}{2}x^2 + i\xi x = \frac{1}{2}(x+i\xi)^2 + \frac{1}{2}\xi^2$.

Thus $\hat{f}(\xi) = e^{-\frac{1}{2}\xi^2} \cdot \int e^{-\frac{1}{2}(x+i\xi)^2}\,dx = e^{-\frac{1}{2}\xi^2} \cdot \int e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}\, e^{-\frac{1}{2}\xi^2}$.

Thm: Let $f, g \in S$, then $\int f\bar{g}\,dx = \frac{1}{(2\pi)^d} \int \hat{f}\,\bar{\hat{g}}\,d\xi$. Thus $\|f\|_{L^2} = (2\pi)^{-d/2} \|\hat{f}\|_{L^2}$.

pf: Note that $\int \hat{h} k = \int h \hat{k}$. Also $\hat{\bar{g}} = R\bar{\hat{g}}$. So if $\hat{h} = f$, $k = \bar{g}$ we

get $h = \frac{1}{(2\pi)^d} R\hat{f}$, $\hat{k} = R\bar{\hat{g}}$, so $\int f\bar{g} = \frac{1}{(2\pi)^d}\int R\hat{f} \cdot R\bar{\hat{g}} = \frac{1}{(2\pi)^d}\int \hat{f}\cdot\bar{\hat{g}}$.

Note: this also gives $\int fg = \frac{1}{(2\pi)^d}\int \hat{f}\, R\hat{g}$ thanks to $\hat{\bar{g}} = R\bar{\hat{g}}$

Thm: If $f, g \in S(\mathbb{R}^d)$ then $\widehat{f \cdot g} = \frac{1}{(2\pi)^d}\hat{f} * \hat{g}$. $\widehat{f * g} = \hat{f}\cdot\hat{g}$

pf: $\int e^{-ix\cdot\xi} f(x) g(x)\,dx = \frac{1}{(2\pi)^d}\int \widehat{e^{-ix\cdot\xi} f(x)}(\eta)\, \hat{g}(-\eta)\,d\eta$. Also $\widehat{e^{-ix\cdot\xi} f(x)}(\eta) = \hat{f}(\xi+\eta)$.

After change of variables get first identity. The second is a direct

calculation using $e^{-ix\cdot\xi} = e^{-i(x-y)\cdot\xi} e^{-iy\cdot\xi}$ and change of variables.

Thm: The FT extends as a continuous operator $\Phi: L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$, $1 \le p \le 2$, with

$\|\hat{f}\|_{p'} \le (2\pi)^{d/2p}\|f\|_p$.

pf: For $p=1$ we have an extension w/ this bound. For $p=2$ let $f_n \to f$ in L^2

with $f_n \in S$. Then $\hat{f}_n$ is Cauchy in L^2, so define $\hat{f} = \lim \hat{f}_n$.

Then $\|\hat{f}\|_2 = (2\pi)^{d/2}\|f\|_2$. The general result follows by Riesz-Thorin.

Remark: If $f \in L^2$, then $\mathbb{1}_{B_R} f \in L^1$ so $\widehat{\mathbb{1}_{B_R} f}$ makes sense.

We see $\widehat{\mathbb{1}_{B_R} f} \to \hat{f}$ in L^2.