I. The FT on $S(\mathbb{R}^n)$ class Functions.

<u>Defn</u>: For $f \in S(\mathbb{R}^n)$ we set $F(f)(\xi) = \hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx$.

<u>Proposition</u>: $F : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ and is continuous. Also $F D_j = \xi_j F$ where $D_j = \frac{1}{i}\partial_{x^j}$,

and $F x^j = -D_j F$ where $D_j = \frac{1}{i}\partial_{\xi^j}$.

<u>pf</u>: The Formula $\widehat{\partial_{x^j} f}(\xi) = i\xi_j \hat{f}(\xi)$ Follows at once via IBP. $D_j \hat{f}(\xi) = -\widehat{(x^j f)}(\xi)$ is also direct.

By induction $\xi^\alpha \partial_\xi^\beta \hat{f}$ exists and $|\xi^\alpha \partial_\xi^\beta \hat{f}(\xi)| = |\widehat{D_x^\alpha x^\beta f}(\xi)| \leq \| D_x^\alpha x^\beta f\|_{L^1} \leq \|(1+|x|)^{n+1} D_x^\alpha x^\beta f\|_u$

$\leq C_{\alpha,\beta} \sum_{\substack{|\alpha'| \leq |\beta| + n+1 \\ |\beta'| \leq |\alpha|}} \|f\|_{\alpha,\beta}$, which also Shows $F : S \to S$ is continuous.

<u>Lemma</u>: Let $T : S \to S$ be a linear transformation such that $[T, x^j] = [T, \partial_{x^j}] = 0$.

Then $Tf = cf$ For some fixed $c \in \mathbb{C}$.

<u>pf</u>: First let $f \in S$ be such that $f(x_0) = 0$. Then using Taylor's Formula with remainder

we have $f(x) = \sum_{j=1}^n \int_0^1 \partial_j f(\lambda x + (1-\lambda)x_0)\, d\lambda\, (x - x_0)^j$, so choosing $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ on $|x| \leq 1$

we have $f(x) = \sum_1^n (x - x_0)^j \phi_j(x)$, where $\phi_j(x) = \chi(x - x_0) \int_0^1 \partial_j f(\lambda x + (1-\lambda)x_0)\, d\lambda + (1 - \chi(x - x_0)) f(x) \frac{(x - x_0)^j}{|x - x_0|^2} \in S(\mathbb{R}^n)$.

Since $[T, (x - x_0)^j] = 0$ we get $Tf(x) = \sum_1^n (x - x_0)^j T\phi_j(x)$ so $Tf(x_0) = 0$.

Now let $f \in S(\mathbb{R}^n)$ be general, then $f_{x_0}(x) = f(x) - \chi(x - x_0) f(x_0)$ is such that $f_{x_0}(x_0) = 0$,

So $0 = Tf_{x_0}(x_0) = Tf(x_0) - f(x_0)\psi(x_0)$, where $\psi(x_0) = T\chi(\cdot - x_0)$. Since $Tf \in S$, setting

$f \equiv 1$ in the nbd of fixed points shows $\psi \in C^\infty(\mathbb{R}^n)$. On the other hand

since $[T, \partial_j] = 0$ we get $\partial_j \psi \cdot f = 0$ all $f \in S$ & $j = 1, \ldots, n$. Hence $\psi = c$ some constant.

<u>Theorem</u> (Fourier inversion) One has $RF^2 = c\,\mathrm{Id}$ for some $c \in \mathbb{C}$.

<u>pf</u>: We have $F^2 D_j = F \xi_j F = -D_j F^2$ and $F^2 x^j = -F D_j F = -x^j F^2$.

Hence $[RF^2, \partial_j] = [RF^2, x^j] = 0$ and the result Follows.

Thm (Fourier inversion cont.) Let $f = e^{-\frac{1}{2}|x|^2}$. Then $\hat{f}(\xi) = (2\pi)^{n/2} e^{-\frac{1}{2}|\xi|^2}$. Thus for every $f \in \mathcal{S}$ one has $f(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$.

pf: Note that $(iD_j + x^j) f = 0$, thus $(i\xi_j - D_j)\hat{f} = 0$, or $(iD_j + \xi_j)\hat{f} = 0$ as well.

The only functions which satisfy these ODE in each variable are $h = c e^{-\frac{1}{2}|x|^2}$.

Thus $\hat{f}(\xi) = c e^{-\frac{1}{2}|\xi|^2}$ for some $c \in \mathbb{C}$, and $f(x) = \frac{1}{c^2} \int e^{ix\cdot\xi} \hat{f}(\xi) d\xi$ in general by the previous result. To find c note that $\hat{f}(0) = \int f(x)\,dx$,

so $c = \int e^{-\frac{1}{2}|x|^2} dx = (2\pi)^{n/2}$.

We now recap and extend the results thus far:

Theorem: Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f}, \hat{g} \in \mathcal{S}(\mathbb{R}^n)$ and

1) $f(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$

2) $\widehat{D_j f} = \xi_j \hat{f}$ and $\widehat{x^j f} = -D_j \hat{f}$. Schematically $(x, \xi) \xrightarrow{F} (\xi, -x)$.

3) $\widehat{\tau_y f} = e^{-iy\cdot\xi}\hat{f}$, $\widehat{e^{ix\cdot\eta} f} = \tau_\eta \hat{f}$, and $\widehat{f\circ A} = \frac{1}{|A|} \hat{f} \circ (A^{-1})^t$ all $A \in Gl_n$.

4) $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$

5) $\widehat{f * g} = \hat{f}\hat{g}$

6) $\widehat{fg} = \frac{1}{(2\pi)^n} \hat{f} * \hat{g}$

7) $(f,g) = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})$. Thus $\|f\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \|\hat{f}\|_{L^2}$.

pf: We have already shown 1) & 2), and 3) is a direct calculation.

4) Both integrals are $= \iint e^{-ix\cdot\xi} f(x) g(\xi)\, dx\, d\xi$ by Fubini.

5) For $f, g \in \mathcal{S}$ and $\phi \in L^\infty$ we have $\int \phi(x)\, f*g(x)\, dx = \iint f(x) g(y) \phi(x+y)$. Setting $\phi(x) = e^{-ix\cdot\xi}$ it follows.

6) By 1), 3), 5) $\frac{1}{(2\pi)^n} RF(\hat{f} * \hat{g}) = \frac{1}{(2\pi)^n} R(\hat{\hat{f}} \cdot \hat{\hat{g}}) = \frac{1}{(2\pi)^{2n}} RF^2 f \cdot RF^2 g = f \cdot g$.

Applying F to both sides and using $\frac{1}{(2\pi)^n} FRF = \frac{1}{(2\pi)^n} RF^2 = Id$ gives the identity.

7) Note that $\hat{\bar{g}} = R\overline{\hat{g}}$, by 1), 3), 4) we have $(f,g) = \frac{1}{(2\pi)^n} \langle RF\hat{f}, \bar{g}\rangle = \frac{1}{(2\pi)^n} \langle \hat{f}, RF\bar{g}\rangle = \frac{1}{(2\pi)^n} \langle \hat{f}, \bar{\hat{g}}\rangle$.

II. The Fourier transform on $S'(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$.

Defn: Let X be LCTVS whose topology is given by seminorms $\|\cdot\|_j$, $j \in J$ (possibly uncountable). We set $X^* = \{u : X \to \mathbb{C} \mid |u(x)| \leq C_u \sum_{k=1}^{N(u)} \|x\|_{j_k}$, where j_k is some finite collection of indices depending on u.

Defn: We call $S'(\mathbb{R}^n) = S^*(\mathbb{R}^n)$ the space of "tempered distributions". We give S' the weak-* topology $u_n \rightharpoonup u$ iff $u_n(f) \to u(f)$ all $f \in S$.

If $u \in S'(\mathbb{R}^n)$ we define $\hat{u} \in S'(\mathbb{R}^n)$ via the formula $\hat{u}(f) = u(\hat{f})$ all $f \in S(\mathbb{R}^n)$.

We say a $u \in S'$ "is a function" if $u(f) = \langle g, f \rangle$ for some $g \in L^1_{loc}(\mathbb{R}^n)$ with $(1+|x|)^{-N} g \in L^1$ for N large.

We say $u \in S'$ "is in L^p" if $u(f) = \langle g, f \rangle$ some $g \in L^p(\mathbb{R}^n)$.

In general we'll write $u(f) = \langle u, f \rangle$ for the pairing $S' \times S \to \mathbb{C}$.

Lemma: If $u \in S'(\mathbb{R}^n)$ is a function then $\exists$ a unique $g \in L^1_{loc}(\mathbb{R}^n)$ (up to a.e. equivalence) such that $\langle u, f \rangle = \int g f \, dx$ all $f \in S(\mathbb{R}^n)$. In addition, $u \in L^p(\mathbb{R})$ for $1 < p \leq \infty$ iff $\exists$ $C > 0$ such that $|\langle u, f \rangle| \leq C \|f\|_{L^{p'}(\mathbb{R}^n)}$ all $f \in S(\mathbb{R}^n)$.

Remark: If $|\langle u, f \rangle| \leq C \|f\|_{L^\infty(\mathbb{R}^n)}$ we cannot conclude that $u \in L^1(\mathbb{R}^n)$, but we do know there exists a unique complex Radon measure μ such that $\langle u, f \rangle = \int_{\mathbb{R}^n} f \, d\mu$ all $f \in S(\mathbb{R}^n)$.

pf: Let $\int (g - \tilde{g}) f = 0$ all $f \in C_c^\infty(\mathbb{R}^n)$ with $g - \tilde{g} \in L^1_{loc}$. Then one can take a sequence $f_n \to \chi_{B_r(x_0)}$ pointwise and $0 \leq f_n \leq B_{2r}(x_0)$ by the C_c^∞ version of Urysohn's Lemma. Thus $\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (g - \tilde{g}) = 0$ all $x_0 \in \mathbb{R}^n$ and $r > 0$. Thus $g = \tilde{g}$ at every Lebesgue point of $g - \tilde{g}$, hence a.e.

that, if $|u(f)| \leq C\|f\|_{L^{p'}}$ some $1 \leq p' < \infty$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, then by density u extends to an element of $(L^{p'})^*$. Thus $u(f) = \int_{\mathbb{R}^n} g f \, dx$ some $g \in L^p(\mathbb{R}^n)$ all $f \in \mathcal{S} \subseteq L^{p'}$. Note that the $g \in L^p$ representing u is unique (even from old results).

We'll develop the theory of $\mathcal{S}'(\mathbb{R}^n)$ later. For now we use it to give the result:

<u>Thm</u> (Hausdorff-Young): Let $u \in \mathcal{S}'(\mathbb{R}^n)$, then if $u \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$ $\hat{u} \in L^{p'}(\mathbb{R}^n)$ and one has the estimate $\|\hat{u}\|_{L^{p'}(\mathbb{R}^n)} \leq (2\pi)^{n/p'} \|u\|_{L^p(\mathbb{R}^n)}$. More specifically one has:

i) If $u \in L^1(\mathbb{R}^n)$ the $\hat{u} \in C_0(\mathbb{R}^n)$ and $\|\hat{u}\|_{L^\infty} \leq \|u\|_{L^1}$.

ii) If $u \in L^2(\mathbb{R}^n)$ then $\hat{u} \in L^2(\mathbb{R}^n)$ and $\|\hat{u}\|_{L^2} = (2\pi)^{n/2} \|u\|_{L^2}$.

<u>pf:</u> First prove i) and ii). If $u \in L^1$ and $f \in \mathcal{S}$ then we can define $\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx$ directly, and we still have $\langle \hat{u}, f \rangle = \langle u, \hat{f} \rangle = u(\hat{f}) = \hat{u}(f)$. Thus $\hat{u}$ as a distribution is given by integration against $\hat{u}(\xi)$ as a function. The result then follows from the uniqueness lemma above. That $\hat{u} \in C_0(\mathbb{R}^n)$ follows from density of $\mathcal{S} \subseteq L^1$ and $\mathcal{F}: \mathcal{S} \to \mathcal{S}$.

Now let $u \in L^2$. Then for all $f \in \mathcal{S}(\mathbb{R}^n)$ we have the estimate:

$|\hat{u}(f)| = |u(\hat{f})| = |\int u \cdot \hat{f}| \leq \|u\|_{L^2} \|\hat{f}\|_{L^2} = (2\pi)^{n/2} \|u\|_{L^2} \|f\|_{L^2}$. Thus, by the above lemma $\exists$ unique $g \in L^2$ with $\hat{u}(f) = \int g(\xi) f(\xi) d\xi$ all $f \in \mathcal{S}$. We call $\hat{u}(\xi) = g(\xi)$ (by "abuse of notation"). By the above inequality we actually have $\|\hat{u}\|_{L^2} = \sup_{\substack{\|f\|_{L^2}=1 \\ f \in \mathcal{S}}} |\langle \hat{u}, f \rangle| = \sup_{\substack{\|f\|_{L^2}=1 \\ f \in \mathcal{S}}} |\langle u, \hat{f} \rangle|$.

But $\|f\|_{L^2} = (2\pi)^{-n/2} \|\hat{f}\|_{L^2}$ all $f \in \mathcal{S}$, and $\mathcal{F}$ is a bijection on $\mathcal{S}$

So the RHS above becomes $\sup_{\substack{\|\hat{f}\|_{L^2} = (2\pi)^{n/2} \\ f \in \mathcal{S}}} |\langle u, \hat{f} \rangle| = (2\pi)^{n/2} \sup_{\substack{\|f\|_{L^2}=1 \\ f \in \mathcal{S}}} |\langle u, f \rangle| = (2\pi)^{n/2} \|u\|_{L^2}$.

iii) We now know $\mathcal{F}: L^1 + L^2 \to L^\infty + L^2$ with bounds $\|\mathcal{F}\|_{L^1 \to L^\infty} \leq 1$ and $\|\mathcal{F}\|_{L^2 \to L^2} = (2\pi)^{n/2}$

So the result follows from Riesz-Thorin interpolation. Notice that for $u \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, the function $\hat{u} \in L^{p'}(\mathbb{R}^n)$ given by the Riesz-Thorin thm must agree with $\hat{u}$ as a distribution because if $u_k \in L^1(\mathbb{R}^n)$ are such that $u_k \to u$ in L^p then $\hat{u}_k \to \hat{u}$ in $L^{p'}$ where $\hat{u}_k$ & $\hat{u}$ are given by Riesz-Thorin, but $\hat{u}_k(\xi) = \int e^{-ix\cdot\xi} u_k(x)\,dx$ is also the distributional FT as explained in 1) above, and thus $\hat{u}_k \rightharpoonup \hat{u}$ as distributions so the RT & distributional FT of u agree because the distributional FT is continuous under weak limits.