I. Regularity of Radon Measures

In what follows X will be a LCH space. We say X is σ-compact if $X=\bigcup_{n=1}^{\infty} K_n$, K_n compact.

<u>Proposition</u>: A Radon measure is regular on every σ-finite set. I.e. if $E=\bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n)<\infty$ then $\mu(E)=\sup\{\mu(K) \mid K\subseteq E \text{ compact}\}$.

<u>pf</u>: First start with the case where $\mu(E)<\infty$. Then $\exists$ $E\subseteq U$ open with $\mu(U)<\mu(E)+\epsilon$, and $F\subseteq U$ compact with $\mu(U)<\mu(F)+\epsilon$. Since $U\setminus E$ also has finite measure we have $U\setminus E\subseteq V$ open with $\mu(V)<\mu(U\setminus E)+\epsilon$.

Set $K=F\setminus V$ which is a closed subset of F and hence compact.

Since $V^c\subseteq(U\cap E^c)^c=U^c\cup E$, and since $F\subseteq U$ we get $K=F\cap V^c\subseteq E$. For it we compute $\mu(K)\geq\mu(F)-\mu(V)>\mu(U)-\epsilon-\mu(U\setminus E)-\epsilon=\mu(E)-2\epsilon$. Since this works for every $\epsilon>0$ we get $\mu(E)=\sup\{\mu(K) \mid K\subseteq E \text{ compact}\}$ as was to be shown.

Finally, in the case when $\mu(E)=\infty$ and $E=\bigcup_{n=1}^{\infty}E_n$, $\mu(E_n)<\infty$ we may WLOG assume $E_n\subseteq E_{n+1}$. Taking $K_n\subseteq E_n$ compact with $\mu(K_n)>\mu(E_n)-1$ we have $\lim\mu(K_n)\to\infty$ because $\mu(E_n)\to\mu(E)$ by continuity of measures.

<u>Corollary</u>: Every σ-finite Radon measure is regular. Every Radon measure on a σ-compact space is regular.

<u>Warning</u>: There exists a <u>compact</u> Hausdorff space, and a finite Borel measure μ on X such that μ is <u>not</u> regular (i.e. not a Radon measure). The problem is that in general even if X is compact, it might have "large" open sets that cannot be measured from within by compact sets. Taking complements its the same as having closed subsets which cannot be approximated from without by open sets. (On a compact Hausdorff space inner & outer regularity are the same thing.)

<u>Proposition</u>: Let μ be a σ-finite Radon measure on X. Then for every Borel subset $E \subseteq X$ we have:

a) For $\varepsilon > 0$ there exists U open and F closed with $F \subseteq E \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.

b) There exists an F_σ set A and G_δ set B with $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.

<u>pf</u>: For part a) we can write $E = \bigcup_{n=1}^\infty E_n$ with $E_n \cap E_m = \phi$, $n \neq m$, and $\mu(E_n) < \infty$.

For each n choose $E_n \subseteq U_n$ open with $\mu(U_n) < \mu(E_n) + 2^{-n-1}\varepsilon$. Then $E \subseteq U = \bigcup_{n=1}^\infty U_n$ and $U \setminus E \subseteq \bigcup_{n=1}^\infty U_n \setminus E_n$ and so $\mu(U \setminus E) < \frac{1}{2}\varepsilon$. Similarly, let $E^c \subseteq V$ with $\mu(V \setminus E^c) < \varepsilon/2$.

Setting $F = V^c$ we have F closed & $F \subseteq E$ with $\mu(E \setminus F) < \varepsilon/2$ and we are done by $U \setminus F = (U \setminus E) \cup (E \setminus F)$.

Part b) follows at once.

<u>Theorem</u>: Let X be a LCH for which every open set is σ-compact. Then if μ is a Borel measure which is <u>finite</u> on all compact subsets of X, then μ is a Radon measure (hence both inner and outer regular in this case).

<u>pf</u>: Since μ is finite on compact sets we must have $C_c(X) \subseteq L^1(d\mu)$. Thus, the map $f \mapsto \int_X f \, d\mu = I(f)$ is a positive linear functional on $C_c(X)$. Let ν be the Radon measure associated to I. Let $U = \bigcup_{n=1}^\infty K_n$ be an open union of compact sets. For each n choose f_n by induction so $(\bigcup_{i=1}^n K_i) \cup (\bigcup_{j=1}^{n-1} \operatorname{supp}(f_j)) \prec f_n \prec U$ (set $f_1 = 0$). Then $0 \leq f_n \nearrow \chi_U$ so by MCT $\mu(U) = \lim \int_X f_n \, d\mu = \lim_n \int_X f_n \, d\nu = \nu(U)$. Thus $\mu = \nu$ on open sets.

Next, if $E \subseteq X$ is any Borel set, since X is open ν is σ-finite so $\exists$ $F \subseteq E \subseteq U$ open & closed sets with $\nu(U \setminus F) < \varepsilon$, so $\mu(U \setminus F) < \varepsilon$ because $U \setminus F$ is open.

This implies $\mu(U) < \mu(E) + \varepsilon$ so μ is outer regular.

The inner regularity of μ on open sets follows at once from continuity of measures (take $U = \bigcup_{n=1}^\infty K_n$, $K_n \subseteq K_{n+1}$ compact).

Corollary: Let X be a 2^{nd} countable LCH space, then every Borel measure which is finite on compact sets is both inner & outer regular. E.g. each such Borel measure on $\mathbb{R}^n$ is regular.

Proposition: For all Radon measures on LCH X, $C_c(X)$ is dense in $L^p(d\mu)$ for all $1 \leq p < \infty$.

pf: Since simple functions are dense in $L^p(d\mu)$, it suffices to prove such theorems for characteristic functions of Borel sets E of finite measure. For such E one has $K \subseteq E \subseteq U$ with $\mu(U \setminus K) < \infty$. Let $K \prec f \prec U$, then $|f - \chi_E|^p \leq \chi_{U \setminus K}$, so $\|f - \chi_U\|_{L^p} \leq \varepsilon^{1/p}$.

Proposition (Lusin's Thm): Let μ be a Radon measure on an LCH X. Then if $f: X \to \mathbb{C}$ is a measurable function with $f \equiv 0$ outside a set of finite measure, for each $\varepsilon > 0$ $\exists$ $\varphi \in C_c(X)$ and a Borel set E with $\mu(E) < \varepsilon$ and $f = \varphi$ on E^c. If $\|f\|_u < \infty$ then we can also take $\|\varphi\|_u \leq \|f\|_u$.

pf: By the support condition on f & continuity of measures $\exists$ $E_1 \subseteq X$ with $\mu(E_1) < \varepsilon$ and $|\chi_{E_1^c} f| \leq C_\varepsilon < \infty$. Thus, it suffices to prove the same result for bounded f. Then $f \in L^1(d\mu)$ so take $\varphi_n \in C_c(X)$ with $\|f - \varphi_n\|_{L^1} \to 0$, and WLOG assume $\varphi_n \to f$ pointwise μ-a.e. Then by Egoroff and inner regularity $\varphi_n \to f$ uniformly on some compact set $K \subseteq \{x \in X \mid f(x) \neq 0\} = S$, with $K \subseteq S \subseteq U$ and $\mu(U \setminus K) < \varepsilon$. Then $f|_K$ is continuous, so we can find an extension $\varphi \in C_c(X)$ with $\varphi|_K = f$ and $\mathrm{supp}(\varphi) \subseteq U$. In addition $\|\varphi\|_u \leq \|f|_K\|_u$. Since $\{x \in X \mid |f(x) - \varphi(x)| > 0\} \subseteq U \setminus K$ we are done. Note that we have used:

Thm (LCH Tietze Extension) Let X be a LCH and $K \subseteq U$ compact & open subsets. Let $f: K \to \mathbb{C}$ be continuous. Then $\exists$ $\varphi \in C_c(X)$ with $\varphi|_K = f$ and $\mathrm{supp}(\varphi) \subseteq U$.