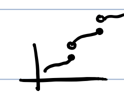
★ Overview: semicontinuity, weak derivatives - BV, AC, etc.

I. Semicontinuity

A. Defn & examples

<u>Defn</u>: A function $f:X\to\mathbb{R}$ is "lower semicontinuous" if $f^{-1}((a,\infty))$ is open for all $a\in\mathbb{R}$. A function $g:X\to\mathbb{R}$ is "upper semicontinuous" if $f=-g$ is LSC (i.e. $g^{-1}((-\infty,a))$ open all $a\in\mathbb{R}$).

<u>eg</u>: If $f:I\to\mathbb{R}$ is monotone increasing & <u>left</u> continuous its LSC 

<u>eg</u>: If $f(x)=\chi_U(x)$, U open then f is LSC. If $g(x)=\chi_C(x)$ then g is USC.

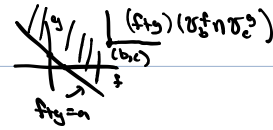
B. Basic Properties

<u>Thm</u>: If $f:X\to\mathbb{R}$ is LSC then for each $x_n\to x$ one has (*) $f(x)\le\underline{\lim}\, f(x_n)$.
If X is 1st countable (e.g. a metric space), then if (*) holds for every $x_n\to x$ f is LSC.

<u>pf</u>: If f is LSC let $x_n\to x$, and choose some $a<f(x)$. Since $U_a=f^{-1}((a,\infty))\ni x$ is open $f(x_n)\in U_a$ for $n>N$. Thus $a\le\underline{\lim}\, f(x_n)$. Taking sup $a\le f(x)$ get the result.
Now suppose $f(x)\le\underline{\lim}\, f(x_n)$ all $x_n\to x$. If f not USC then $\exists\ a\in\mathbb{R}$ such that U_a not open, so $\exists\ x\in U_a\cap\partial U_a$. Thus for $x\in V$ open $\exists\ x_V\in U_a^c$ with $x_V\in V$. Doing this for a countable nbd base at x gives a sequence $x_n\notin U_a$ and $x_n\to x\in U_a$. Then $f(x_n)<a$ all n, so $\underline{\lim}\, f(x_n)\le a<f(x)$ #.

<u>Thm</u>: If $f,g:X\to\mathbb{R}$ are LSC, $\lambda\ge 0$, then so are λf, $f+g$.

pf: $(f+g)^{-1}((a,\infty)) = \bigcup_{b+c>a} \mathcal{U}_b^f \cap \mathcal{U}_c^g$.



Thm: If $f: X \to \mathbb{R}$ is LSC and $K \subseteq X$ is compact, then $\exists\, x \in K$ s.t. $f(x) \leq f(y)$ all $y \in K$.

pf: $K \subseteq \bigcup_{a \in \mathbb{R}} \mathcal{U}_a$, and there is a finite subcover.

Thm: Let $ELSC(X) = \{ f: X \to (-\infty, \infty] \mid \mathcal{U}_a \text{ open all} \}$, then if $f_\alpha \in ELSC(X)$, $\alpha \in I$,

$f = \sup_{\alpha \in I} f_\alpha$ is in $ELSC(X)$.

pf: $f^{-1}((a,\infty)) = \bigcup_{\alpha \in I} f_\alpha^{-1}((a,\infty))$.

C. LSC functions & Integration

★ Recall that for Radon Measures $\int_X \chi_{\mathcal{U}}\, d\mu = \sup\{ \int_X f\, d\mu \mid f \prec \mathcal{U} \}$, $\mathcal{U}$ open,

and $\int_X \chi_E\, d\mu = \inf\{ \int_X \chi_{\mathcal{U}}\, d\mu \mid E \subseteq \mathcal{U}, \text{open} \}$, where E is Borel.

★ Since open sets are like LSC functions, and Borel sets are like arbitrary non-negative measurable functions we would expect similar results.

Thm: Let $f: X \to [0,\infty]$ be an ELSC function. Then if μ is a positive Radon measure
$\int_X f\, d\mu = \sup\{ \int_X \varphi\, d\mu \mid 0 \leq \varphi \leq f,\ \varphi \in C_c(X) \}$.

pf: Let $\int_X f\, d\mu = \int_0^\infty \mu(\{x \mid f(x) > t\})\, dt$ be the layer cake representation of f. Since $\lambda_f(t) = \mu(\{x \mid f(x) > t\})$ is monotone its Riemann integrable on $[0,N^2]$, so $\int_X f\, d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N^2} \lambda_f(kN^{-1}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N^2} \mu(\mathcal{U}_k^N)$ where $\mathcal{U}_k^N = \{ f > kN^{-1} \}$ is open. Now $f(x) \geq \frac{1}{N} \sum_{k=1}^{N^2} \chi_{\mathcal{U}_k^N}(x)$ each $x \in X$, and for each number $a_k^N < \mu(\mathcal{U}_k^N)$ which is >0 we can fin $\varphi_k^N \prec \mathcal{U}_k^N$ with $\int_X \varphi_k^N\, d\mu > a_k^N$. Thus $\frac{1}{N} \sum_{k=1}^{N^2} \lambda_f(kN^{-1}) \leq \sup\{ \int_X \varphi\, d\mu \mid 0 \leq \varphi \leq f \}$.

This shows $\int_X f\,d\mu \leq \sup\{\int_X \varphi\,d\mu \mid 0\leq\varphi\leq f,\ \varphi\in C_c(X)\}$ and the other direction is trivial.

<u>Thm:</u> Let μ be a (positive) Radon Measure on LCH X. Then for each $f:X\to[0,\infty]$ Borel measurable one has $\int_X f\,d\mu = \inf\{\int_X g\,d\mu \mid g\geq f \text{ and } g\in ELSC\}$.

If $\{x \mid f(x)>0\}$ is σ-compact then $\int_X f\,d\mu = \sup\{\int_X g\,d\mu \mid 0\leq g\leq f,\ g\in EUSC\}$.

In particular if $f:X\to\mathbb{R}$ & $f\in L^1(d\mu)$ $\exists$ $g_U\leq f\leq g_L$ with $\int_X (g_L-g_U)\,d\mu<\epsilon$.

<u>pf:</u> Let $\phi_n\nearrow f$ be a series of simple functions, and let g_n LSC with

$g_n\geq \phi_n-\phi_{n-1}$, $\phi_{-1}\equiv 0$, and such that $\int_X g_n\,d\mu \leq \int_X(\phi_n-\phi_{n-1})\,d\mu + \epsilon 2^{-n}$ (including $\int g_n=\infty$).

Note that $\phi_n-\phi_{n-1}=\sum_k a_{k,n}\chi_{E_{k,n}}$, $a_{k,n}>0$, and $\exists$ $E_{k,n}\subseteq \mathcal{U}_{k,n}$ with $\mu(\mathcal{U}_{k,n})\leq\mu(E_{k,n})+\frac{\epsilon 2^{-n}}{a_{k,n}}$.

Then $f=\sum_{n=1}^{\infty}(\phi_n-\phi_{n-1})\leq g=\sum_{n=1}^{\infty} g_n$. And g is ELSC with $\int_X g\,d\mu\leq\int_X f\,d\mu+\epsilon$.

* The proof of the USC part is similar and in the text.

II. CDF, BV, AC

* Let $\mu\in M(\mathbb{R})$ be a finite (complex) Borel measure. Define $F(x)=\mu((-\infty,x])$ is the (left continuous) CDF of μ.

* We know that $F\in BV(\mathbb{R})$, and $\|F\|_{BV}=\sup_{x_0<\dots<x_n}\sum_{k=1}^{n}|F(x_k)-F(x_{k-1})| = \|\mu\|_{M(X)}$.

* If $F\in BV(\mathbb{R})$ then $\lim_{x\to\pm\infty}F(x)=F(\pm\infty)$ exists.

* Let $BV_0=\{f\in BV(\mathbb{R}) \mid F(-\infty)=0\}$. For $F\in BV_0(\mathbb{R})$ set $L: C^1_{comp}(\mathbb{R})\to\mathbb{C}$ by $L(\varphi)=-\int_{\mathbb{R}} F\varphi'\,dm$.

* Let $h\neq 0$ and define $\Delta_h(f)(x)=h^{-1}(f(x+h)-f(x))$. Then $L(\varphi)=\lim_{h\to 0}-\int_{\mathbb{R}} F\,\Delta_h\varphi\,dm$

$=\lim_{h\to 0}\int_{\mathbb{R}}\Delta_h F\cdot\varphi\,dm$. We estimate it as $\int_{\mathbb{R}}\Delta_h F(x)\varphi(x)\,dx=\sum_{n\in\mathbb{Z}}\frac{1}{h}\int_{nh}^{(n+1)h}(F(x+h)-F(x))\varphi(x)\,dm$

Now $\sup_{x\in[nh,(n+1)h]}|F(x+h)-F(x)|\leq\|F\|_{BV([nh,(n+2)h])}$, so $|L(\varphi)|\leq 2\|F\|_{BV(\mathbb{R})}\cdot\|\varphi\|_{L^\infty}$.

$\star$ This shows L extends to $C_0(\mathbb{R})$, so $\exists$ $\mu \in M(\mathbb{R})$ such that

$(\star)$ $- \int_{\mathbb{R}} F \varphi' \, dm = L(\varphi) = \int_{\mathbb{R}} \varphi \, d\mu$ all $\varphi \in C'_{comp}(\mathbb{R})$.

<u>Defn</u>: Let $F \in BV(\mathbb{R})$. A point $x_0 \in \mathbb{R}$ is called a "jump" if $F(x_0^-) \neq F(x_0^+)$.

(recall that $\lim_{x \to x_0^\pm} F(x) = F(x_0^\pm)$ always exists for $F \in BV$). We call x_0 a "continuous point" if its not a jump. Its also clear $\text{jump}(F)$ is countable and $\sum_{x \in \mathbb{R}} |F(x^+) - F(x^-)| \le \|F\|_{BV(\mathbb{R})}$.

<u>Thm</u>: Let $F \in BV_0(\mathbb{R})$, and define $d\mu = dF$ by $(\star)$. Let $G(x) = \mu((-\infty, x])$ be the right continuous CDF of μ. Then $\text{jump}(F) = \text{jump}(G) = \{x \mid G \text{ is discontinuous at } x\} = \text{atoms}(\mu)$

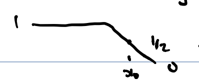
<u>pf</u>: Its the same to say that the "continuous points" of F are exactly the continuous points of $G(x)$. These are exactly the points when $\mu(\{x\}) = 0$. Now Fix any $x_0 \in \mathbb{R}$.

Let $\varphi_n' = \begin{cases} \chi(x+n) \\ -n\chi(n(x-x_0)) \\ 0, \text{ otherwise} \end{cases}$, then $\int \varphi_n \, d\mu \to \frac{1}{2}(G(x_0^-) + G(x_0^+))$. Here $\int \chi = 1$ is a smooth even bump.

On the other hand $\lim -\int F \varphi_n' \, dm = \lim_n n \int F(x) \chi(n(x-x_0)) dx$

For the integral we compute $= \int F(x_0 + \frac{1}{n}x) \chi(x) dx = \int_{-\infty}^0 (F(x_0 + \frac{1}{n}x) - F(x_0^-)) \chi(x) dx + \int_0^\infty (F(x_0 + \frac{1}{n}x) - F(x_0^+)) \chi(x) dx$

$+ \frac{1}{2}(F(x_0^+) + F(x_0^-)))$. The First two terms go to zero so $F(x) = G(x)$ at all points of continuity.

( $\to$ $\mathbb{1}_{(-\infty, x_0)} + \frac{1}{2}\mathbb{1}_{\{x_0\}}$ $\quad \mu((-\infty, x_0)) + \frac{1}{2}\mu(\{x_0\}) = G(x_0) - \frac{1}{2}\mu(\{x_0\}) = \frac{1}{2}G(x_0^+) + \frac{1}{2}G(x_0^-)$.)

<u>Thm</u>: Let $\mu_n \to \mu$ in $\overset{+}{M}(X)$ and be "tight" in the sense that given $\epsilon > 0$ $\exists$ $R > 0$ so that $\mu_n((-\infty, -R]) < \epsilon$ all n. Then if $G_n(x), G(x)$ are the CDF of μ_n, μ, we have $G_n(x) \to G(x)$ all points of continuity of G.

<u>pf</u>: We also have tightness for G because $\mu((-\infty, R)) \le \underline{\lim} \, \mu_n((-\infty, R))$. Choose a sequence $R_k \to -\infty$ with $\mu(\{R_k\}) = 0$, and let x be a continuous point of μ. Then

$\mu((R_k, x)) = \mu((R_k, x]) = \mu([R_k, x])$. Thus $\overline{\lim} \, \mu_n([R_k, x]) = \mu((R_k, x]) \le \underline{\lim} \, \mu_n((R_k, x))$.

Since $\mu_n((R_k, x)) \le \mu_n([R_k, x])$ this implies <u>both</u> $\mu_n((R_k, x))$ & $\mu_n([R_k, x]) \to \mu((R_k, x])$

Thus $G_n(x) - G(x) - (G_n(R_k) - G(R_k)) \to 0$, so $\overline{\lim} |G_n(x) - G(x)| \le \overline{\lim} |G_n(R_k) - G(R_k)| \le o_k(1)$.