I. Sobolev Spaces

A. Basic definitions and properties

Defn: Let $u \in S'(\mathbb{R}^n)$, then for any $\alpha \in \mathbb{N}^n$ we define $\partial^\alpha u$ via $\partial^\alpha u(f) = (-1)^{|\alpha|} u(\partial^\alpha f)$ for all $f \in S(\mathbb{R}^n)$.

Remark: Thus, if $u \in S'(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$ then we define $\partial^\alpha u$ via $(-1)^{|\alpha|} \int_{\mathbb{R}^n} u \cdot \partial^\alpha f \, dx$ all $f \in S(\mathbb{R}^n)$.
In genral for $u \in L^1_{loc}(\mathbb{R}^n)$ (any growth as $|x| \to \infty$) this formula makes sense for $f \in C_c^\infty(\mathbb{R}^n)$.

Defn: Let $K \in \mathbb{N}$. We say $u \in S'(\mathbb{R}^n)$ is in $H^K(\mathbb{R}^n)$ if $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq K$.
In this case we set $\|u\|_{H^K}^2 = \|u\|_{L^2}^2 + \sum_{|\alpha|=K} \|\partial^\alpha u\|_{L^2}^2$.
Note: we will replace this norm latr with an equivalent one.

Proposition: $u \in H^K$ iff $(1+|\xi|^2)^{K/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n)$.

pf: We have $\langle \widehat{\partial^\alpha u}, f \rangle = \langle \partial^\alpha u, \hat{f} \rangle = (-1)^{|\alpha|} \langle u, \partial_x^\alpha \hat{f} \rangle = \langle u, \widehat{(i\xi)^\alpha f} \rangle = \langle \hat{u}, (i\xi)^\alpha f \rangle$
$= \langle (i\xi)^\alpha \hat{u}, f \rangle$. Now $\partial^\alpha u \in L^2$ iff $|\langle \partial^\alpha u, g \rangle| \leq C \|g\|_{L^2}$ all $g \in S(\mathbb{R}^n)$, and
$\|\partial^\alpha u\|_{L^2}$ is the best constant in this inequality. Thus $\sup_{\substack{\|f\|_{L^2}=1 \\ f \in S}} |\langle \xi^\alpha \hat{u}, f \rangle| = \sup_{\substack{\|\hat{f}\|_{L^2} = (2\pi)^{n/2} \\ \hat{f} \in S}} |\langle \partial^\alpha u, \hat{f} \rangle| = (2\pi)^{n/2} \|\partial^\alpha u\|_2$
i.e. $\|\xi^\alpha \hat{u}\|_{L^2} = (2\pi)^{n/2} \|\partial^\alpha u\|_{L^2}$. So $u \in H^K$ implies
$\|\hat{u}\|_{L^2}^2 + \sum_{|\alpha|=K} \|\xi^\alpha \hat{u}\|_{L^2}^2 = \int (1 + \sum_{|\alpha|=K} |\xi^\alpha|^2) |\hat{u}(\xi)|^2 d\xi < \infty$. The proof follows by choosing $C_K > 0$
so that $C_K^{-1}(1+|\xi|^2)^K \leq 1 + \sum_{|\alpha|=K} |\xi^\alpha|^2 \leq C_K (1+|\xi|^2)^K$.

Defn: Let $s \in \mathbb{R}^n$, we define $H^s(\mathbb{R}^n) \subseteq S'(\mathbb{R}^n)$ to be all $u \in S'(\mathbb{R}^n)$ such that $\langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R}^n)$ where $\langle \xi \rangle^s = (1+|\xi|^2)^{s/2}$. We set $\|u\|_{H^s} = \|\langle \xi \rangle^s \hat{u}\|_{L^2}$.

Lemma: $H^s(\mathbb{R}^n)$ is a Hilbert space w/ inner product $\langle f, g \rangle_s = \langle \langle \xi \rangle^{2s} \hat{f}, \hat{g} \rangle_{L^2}$. The dual of H^s is H^{-s}.

In addition, for any $\alpha \in \mathbb{N}^n$ $\partial^\alpha : H^s \to H^{s-|\alpha|}$ is a continuous linear operator.

<u>pf</u>: The proofs are more or less immediate by doing things on the Fourier transform side.

B. Sobolev Embeddings

<u>Theorem</u>: Let $s > n/2$. Then every $u \in H^s(\mathbb{R}^n)$ can be changed on a set of measure zero so that $u \in C_0(\mathbb{R}^n)$, and one has the bound $\|u\|_{L^\infty} \leq C_s \|u\|_{H^s}$.

<u>pf</u>: We know $u \in H^s$ iff $\langle \xi \rangle^s \hat{u} \in L^2$. If $s > n/2$ then $\langle \xi \rangle^{-s} \in L^2$ so by Hölder $\hat{u} = \langle \xi \rangle^{-s} \langle \xi \rangle^s \hat{u} \in L^1(\mathbb{R}^n)$. Then $u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi \in C_0(\mathbb{R}^n)$ $(\hat{u} \in L^1)$, and we know this formula holds pointwise a.e. Finally $|u(x)| \leq \frac{1}{(2\pi)^n} \int |\hat{u}(\xi)|\, d\xi \leq \frac{1}{(2\pi)^n} \|\langle \xi \rangle^{-s}\|_{L^2} \|u\|_{H^s}$.

<u>Remark</u>: This proof also shows that for $u \in H^s$ with $s > n/2 + k$ we have $u \in C_0^k(\mathbb{R}^n)$, because by inversion we can compute the difference quotients $\Delta_{k,h} u = h^{-1}(u(x+he_k) - u(x))$ $= \frac{1}{(2\pi)^n} \int h^{-1}(e^{ihe_k\cdot\xi} - 1)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$ and take the limit (by DCT) to get $\partial^\alpha u = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} (i\xi)^\alpha \hat{u}(\xi)\, d\xi$ for $|\alpha| \leq k$, where $\xi^\alpha \hat{u} \in L^1(\mathbb{R}^n)$.

We now turn to Sobolev embeddings involving other L^p spaces. The most basic result is the following:

<u>Thm</u>: (Dyadic Sobolev Embedding) Let $u \in S'(\mathbb{R}^n)$ be such that $\operatorname{supp} \hat{u} \subseteq \{2^k \leq |\xi| \leq 2^{k+1}\}$, some $k \in \mathbb{Z}$. Then if $u \in L^2(\mathbb{R}^n)$ we have $u \in L^p(\mathbb{R}^n)$ for $p \geq 2$ and $\|u\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{H^{n(\frac{1}{2}-\frac{1}{p})}}$.

<u>pf</u>: This is just Bernstein's inequality, which gives $\|u\|_{L^p(\mathbb{R}^n)} \leq C\, (2^{nk})^{1/2-1/p} \|u\|_{L^2}$, and the fact that $(2^{nk})^{1/2-1/p} \|u\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \|(2^{nk})^{1/2-1/p} \mathbb{1}_{2^k \leq |\xi| \leq 2^{k+1}} \hat{u}\|_{L^2} \leq C \|\langle \xi \rangle^{n(1/2-1/p)} \hat{u}\|_{L^2} = C\|u\|_{H^s}$, $s = n(1/2 - 1/p)$.

$\star$ It would be nice to prove this result w/o the restriction on the Fourier support of u.

This would work if we could prove something like $\|u\|_{L^p} \lesssim C_p \left(\sum_k \|P_k u\|_{L^p}^2\right)^{1/2}$ where

$\widehat{P_k u} = P_k(\xi)\hat{u}(\xi)$ and $P_k(\xi)$ supported where $|\xi| \approx 2^k$, and $\sum P_k^2 \leq C$.

We'll get back to this issue in a bit. For now we give a sharp result using a different idea:

<u>Theorem</u>: Let $g(\xi) = |\xi|^{-s}$ for $0 < s < n$. Then $g^\vee(x) = C_{n,s}|x|^{s-n}$, where

$C_{n,s} = \pi^{-n/2} 2^{-s} \frac{\Gamma(\frac{n-s}{2})}{\Gamma(s/2)}$. In particular if $u \in H^s$ we can write $u = \frac{1}{|x|^{n-s}} * M_s u$, where

$\widehat{M_s u} = C_{n,s}|\xi|^s \hat{u}(\xi)$. Thus, from $\|M_s u\|_{L^2} \lesssim C_{n,s}\|u\|_{H^s}$ and the HLS inequality we have

$\|u\|_{L^p} \lesssim C_p \|u\|_{H^{n(1/2 - 1/p)}}$ for all $2 \leq p < \infty$. (Here $C_p \to \infty$ as $p \to \infty$).

<u>pf</u>: Recall that the definition of $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$. Rescaling this gives

$c_s|\xi|^{-s} = \int_0^\infty t^{s/2-1} e^{-t/2|\xi|^2} dt$ where $c_s = 2^{s/2}\Gamma(s/2)$. Recall that $(e^{-t/2|\xi|^2})^\vee = (2\pi)^{-n/2} t^{-n/2} e^{-\frac{1}{2t}|x|^2}$.

Thus $(c_s|\xi|^{-s})^\vee = (2\pi)^{-n/2}\int_0^\infty t^{\frac{s-n}{2}-1} e^{-\frac{1}{2t}|x|^2} dt = (2\pi)^{-n/2}|x|^{s-n} 2^{n-s/2}\int_0^\infty t^{\frac{n-s}{2}-1}e^{-t}dt = (2\pi)^{-n/2}C_{n-s}|x|^{s-n}$.

This shows $(|\xi|^{-s})^\vee = C_{n,s}|x|^{s-n}$. The formula $u = \frac{1}{|x|^{n-s}} * M_s u$, $\widehat{M_s u} = C_{n,s}|\xi|^s\hat{u}$ follows,

at least for the case $M_s u \in \mathcal{S}$, which is true if $u \in \mathcal{S}$ and $\hat{u}$ vanishes for $|\xi| << 1$ and $|\xi| >> 1$.

In general if $u \in H^s$ then $M_s u \in L^2$, so to make sense of the rest we only need to show

$\frac{1}{|x|^{n-s}} * : L^2 \to L^p$ for $0 < s = n(1/2 - 1/p) < n$. Recall that HLS says $|x|^{-\lambda} * : L^\delta \to L^p$ for

$\frac{\lambda}{n} + 1/\delta - 1/p = 1$. Applying this to $\delta = 2$ and $\lambda = n-s$ gives $s = n(1/2 - 1/p)$ as desired.