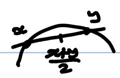
I. Uniform Convexity

✱ **Defn:** A normed vector space X is "uniformly convex" if for each $\epsilon > 0$

$\exists$ a $\delta = \delta(\epsilon) > 0$ such that for $\|x\| = \|y\| = 1$, $\|x-y\| \geq \epsilon \Rightarrow \|\frac{x+y}{2}\| \leq 1-\delta$.



eg: $L^p(d\mu)$ is uniformly convex when $1 < p < \infty$. For example when $p \geq 2$ we have

that $|\frac{a-b}{2}|^p + |\frac{a+b}{2}|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p$ which follows by WLOG assuming

$a, b \geq 0$, then rescaling to $a = 1 \leq b$, and then setting $b = 1+s$, $s \geq 0$.

The inequality is now $2^{-p}s^p + (1+\frac{1}{2}s)^p \leq \frac{1}{2} + \frac{1}{2}(1+s)^p$, which

follows from the differential version $2^{-r}s^r + (1+\frac{1}{2}s)^r \leq (1+s)^r$, $r = p-1$.

Integration gives $(1+s)^r - (1+\frac{1}{2}s)^r \geq r\int_{\frac{1}{2}s}^{s}(1+t)^{r-1}dt \geq \frac{r}{2}s(1+\frac{1}{2}s)^{r-1} \geq \frac{r}{2^r}s^r \geq 2^{-r}s^r$.

Finally, integration of (✱) gives $\|\frac{f-g}{2}\|_p^p + \|\frac{f+g}{2}\|_p^p \leq \frac{1}{2}\|f\|_p^p + \|g\|_p^p$,

so when $\|f\|_p = \|g\|_p = 1 \implies \|\frac{f+g}{2}\|_p \leq \left(1 - 2^{-p}\|f-g\|_p^p\right)^{1/p}$.

set $\delta = 1 - (1 - 2^{-p}\epsilon)^{1/p}$.

Thm: Let X be a uniformly convex Banach space. Then if $x_n \rightharpoonup x$

and $\|x_n\| \to \|x\|$, one must have $x_n \to x$ strongly.

pf: If $x = 0$ we are done. Otherwise WLOG assume $\|x_n\|, \|x\| \neq 0$,

and set $\hat{x}_n = x_n/\|x_n\|$, $\hat{x} = x/\|x\|$. Then $\hat{x}_n \rightharpoonup \hat{x}$, so WLOG

assume $\|x_n\| \equiv \|x\| = 1$ to begin with. Now assume $x_n \not\to x$, and WLOG

assume $\|x_n - x\| \geq \epsilon > 0$. Then $\|\frac{x_n+x}{2}\| \leq 1-\delta$ some $\delta > 0$. But then $\frac{x_n+x}{2} \rightharpoonup x$,

so $\|x\| \leq \underline{\lim}\|\frac{x_n+x}{2}\| \leq 1-\delta$, a contradiction.

II. Convolutions.

* Let $f, g \in L^1(\mathbb{R}^d)$, then set $f*g \in L^1(\mathbb{R}^d)$ by $f*g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$.
* If f, g are Lebesgue measurable, we define $f*g(x)$ at all x for which $f(x-y)g(y) \in L^1(dy)$.

Thm: Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, and if $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$ then $f*g \in L^r(\mathbb{R}^d)$, and one has $\|f*g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$.

pf: Let $h \in L^{r'}(\mathbb{R}^d)$, and consider $\iint |f(x)|\cdot|h(x-y)|\cdot|g(y)|\,dx\,dy = I$

The condition on the indices means $\frac{1}{p'}+\frac{1}{q'}+\frac{1}{r} = 1$, so we set:

$F(x,y) = |f(x)|^{p/q'}\cdot|h(x-y)|^{r'/q'}$, $G(x,y) = |g(y)|^{q/p'}\cdot|h(x-y)|^{r'/p'}$, $H(x,y) = |f(x)|^{p/r}|g(y)|^{q/r}$.

Then $I = \iint FGH\,dx\,dy \leq \|F\|_{L^{q'}}\|G\|_{L^{p'}}\|H\|_{L^r}$.

Now compute $\|F\|_{L^{q'}} = \left(\iint |f(x)|^p|h(x-y)|^{r'}\,dx\,dy\right)^{1/q'} = \|f\|_{L^p}^{p/q'}\|h\|_{L^{r'}}^{r'/q'}$, and similarly $\|G\|_{L^{p'}} = \|g\|_{L^q}^{q/p'}\|h\|_{L^{r'}}^{r'/p'}$, $\|H\|_{L^r} = \|f\|_{L^p}^{p/r}\|g\|_{L^q}^{q/r}$. Taking the product gives $I \leq \|h\|_{L^{r'}}\|f\|_{L^p}\|g\|_{L^q}$. Taking $\sup_{\|h\|_{L^{r'}}} I \Rightarrow \int |f(x-y)g(y)|\,dy \in L^r(\mathbb{R}^d)$, so in particular $f(x-y)g(y) \in L^1(dy)$ a.e. $x \in \mathbb{R}^d$, and $f*g(x) \in L^r(\mathbb{R}^d)$.

Thm: Let $f \in L^1(\mathbb{R}^d)$ with $\int f\,dx = a$, and set $f_\epsilon(x) = \epsilon^{-d}f(\epsilon^{-1}x)$.

Then for $g \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$ one has $f_\epsilon * g \to ag$ in $L^p(\mathbb{R}^d)$.

pf: Let $\phi, \psi \in C_{comp}(\mathbb{R}^d)$, $\int \psi\,dx = a$, then writing $f_\epsilon * g - ag = \text{same} \pm a\phi \pm \psi_\epsilon * \phi \pm f_\epsilon * \phi$

we get $\|f_\epsilon * g - ag\|_p \leq a\|\phi - g\|_p + \|\psi_\epsilon * \phi - a\phi\|_p + \|(\psi - f)_\epsilon * \phi\|_p + \|f_\epsilon * (g-\phi)\|_p$

So the proof reduces to showing $\psi_\epsilon * \phi - a\phi \xrightarrow{L^p} 0$. Now $|\psi_\epsilon * \phi| \leq \int|\psi|\,dx \cdot \|\phi\|_{L^\infty}$ and $\operatorname{supp}(\psi_\epsilon * \phi) \subseteq \epsilon\cdot\operatorname{supp}(\psi) + \operatorname{supp}(\phi)$. So by DCT we just need $\psi_\epsilon * \phi(x) \to a\phi(x)$ pointwise. For that $\psi_\epsilon * \phi(x) - a\phi(x) = \int(\phi(x-y) - \phi(x))\psi_\epsilon(y)\,dy = \int(\phi(x-\epsilon y) - \phi(x))\psi(y)\,dy$.

Since $(\phi(x-\epsilon y) - \phi(x))\psi(y) \to 0$, DCT again gives it. Now use density.

Thm: Let $f \in L^1_{loc}(\mathbb{R}^d)$, and $\psi \in C^k_{comp}(\mathbb{R}^d)$. Then $f * \psi \in C^k(\mathbb{R}^d)$, and for every $\Omega \subset\subset \mathbb{R}^d$ one has $\| f * \psi \|_{C^k(\Omega)} \leq \| \psi \|_{C^k(\mathbb{R}^d)} \sup_{x \in \Omega} \| f \|_{L^1(x - \mathrm{supp}(\psi))}$.

pf: For $\psi \in C^0_{comp}(\mathbb{R}^d)$, $\psi * f \in C^0(\mathbb{R}^d)$ by a DCT argument.

Also $\Delta_h(\psi * f) = \Delta_h \psi * f$. Also, as $h \to 0$ $\Delta_h \psi$ has uniformly compact support and $|\Delta_h \psi| \leq \| \partial \psi \|_{L^\infty} \leq \| \psi \|_{C^1(\mathbb{R}^d)}$. So by DCT $\Delta_h(\psi * f) \to (\partial \psi) * f$, which is continuous. By induction $\partial^k(\psi * f) = (\partial^k \psi) * f$.

Finally $|\partial^k \psi * f(x)| \leq \| \partial^k \psi \|_{L^\infty(\mathbb{R}^d)} \cdot \int_{\mathrm{supp}(\psi)} |f(x-y)|\, dy = \| \partial^k \psi \|_{L^\infty(\mathbb{R}^d)} \cdot \| f \|_{L^1(x - \mathrm{supp}(\psi))}$.

Thm: The space $C^\infty_{comp}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$.

pf: Let $\psi \in C^\infty_{comp}(\mathbb{R}^d)$, and $\int \psi\, dx = 1$, and $\chi \in C^\infty_{comp}(\mathbb{R}^d)$ with $\chi(0) = 1$, $0 \leq \chi \leq 1$.

Let $f_\varepsilon(x) = \chi(\varepsilon x)\, \psi_\varepsilon * f(x)$. Then $f_\varepsilon \in C^\infty_{comp}(\mathbb{R}^d)$, and $\psi_\varepsilon * f \to f$ in L^p, so we just need $(1 - \chi(\varepsilon x)) f(x) \to 0$ in L^p, or $|1 - \chi(\varepsilon x)| \cdot |f(x)|^p \to 0$ in L^1. But this follows from DCT and $|1 - \chi(\varepsilon x)| \to 0$ all x.