* Gen defn of weak topology, case of L^p & $M(X)$.

I. Weak topologies on X and X^*.

A. Topologies generated by seminorms

* $p: X \to \mathbb{R}$ is a seminorm if $p(0)=0$, $p \geq 0$, $p(x+y) \leq p(x)+p(y)$.

* Let P_α, $\alpha \in I$ be a collection of seminorms on X. We define $\mathcal{T}_{X,P_\alpha}$ by letting $\mathcal{U} = \bigcap_{i=1}^{k} \mathcal{U}_{x_i \epsilon_i \alpha_i}$ be a base with $\mathcal{U}_{x\epsilon\alpha} = \{y \in X \mid P_\alpha(x-y) < \epsilon\}$.

Lemma: Sets of the form $\bigcap_{i=1}^{k} \mathcal{U}_{x \epsilon_i \alpha_i}$ form a nbd base at x (i.e. any $x \in \mathcal{U}$ open $\exists$ $\bigcap_{i=1}^{n} \mathcal{U}_{x\epsilon_i \alpha_i} \subseteq \mathcal{U}$).

pf: Since $\bigcap_{i=1}^{k} \mathcal{U}_{x_i \epsilon_i \alpha_i}$ form a base for $\mathcal{T}_{X,P_\alpha}$ sps $x \in \mathcal{U}_{x_i \epsilon_i \alpha_i}$ $i=1,\dots,k$. Set $\delta_i = \epsilon_i - p_{\alpha_i}(x_i - x)$.

Then $\mathcal{U}_{x\delta_i\alpha_i} \subseteq \mathcal{U}_{x_i \epsilon_i \alpha_i}$ because if $p_{\alpha_i}(x-y) < \delta_i = \epsilon_i - p_{\alpha_i}(x_i - x) \Rightarrow p_{\alpha_i}(y - x_i) < \epsilon_i$.

Thm: In the topology $\mathcal{T}_{X,P_\alpha}$ one has $x_i \to x$ iff $p_\alpha(x_i - x) \to 0$ for each α. (Here x_i is a net).

pf: $x_i \to x$ in $\mathcal{T}_{X,P_\alpha}$ iff for each $\epsilon > 0$, α $x_i \in \mathcal{U}_{x\epsilon\alpha}$ for $i_0 \leq i$ some $i_0 = i_0(\epsilon,\alpha)$. This is equivalent to $p_\alpha(x_i - x) \to 0$.

eg: Let X be a Banach space & $p_\alpha(x) = \|\alpha(x)\|_X$ for each $\alpha \in X^*$. Then $\mathcal{T}_{X,P_\alpha} = \mathcal{T}_{weak}$.

Then $x_i \to x$ iff $\alpha(x_i) \to \alpha(x)$ for each $\alpha \in X^*$.

Note that this is exactly the weak (Tychonoff) topology on X generated by $\alpha \in X^* \subseteq C(X;\mathbb{C})$.

(Recall for $\mathcal{S} \subseteq C(X;\mathbb{C})$, define $\mathcal{T}_{X,\mathcal{S}}$ on $\mathcal{S}$ by sets $f^{-1}(\mathcal{O})$ for $\mathcal{O} \in \mathbb{C}$, $f \in \mathcal{S}$. Exc #34 in Ch#6)

eg: Let X be a Banach space (or NVS). Then $X \subseteq (X^*)^*$ via $x(\alpha) = \alpha(x)$, $x \in X$, $\alpha \in X^*$.

Also $\|x\|_{(X^*)^*} = \sup_{\|\alpha\|_{X^*}=1} |\alpha(x)| = \|x\|$ (by Hahn-Banach define $\|\alpha\|_{X^*} = 1$ from $\alpha_0(x) = \|x\|_X$ on $\mathbb{C}x$, then extend).

On X^* we can define the weak-* topology via $p_x(\alpha) = |\alpha(x)|$ all $\alpha \in X^*$ ranging over $x \in X$.

Thus $\alpha_i \rightharpoonup \alpha$ iff $\alpha_i(x) \to \alpha(x)$ for all $x \in X$. In other words this is exactly the topology of pointwise convergence of linear functionals.

* Note, if $X^{**} = X$ then weak-* topology on X^* is the weak topology generated by $(X^*)^*$.

In general $X \subsetneq (X^*)^*$, so the weak-* topology is weaker than the weak topology.

eg: Let $1 < p < \infty$ and (X, M, μ) a (positive) measure space. Then for each $\alpha \in L^p(d\mu)^*$ $\exists$ $g \in L^{p'}(d\mu)$ with $\alpha(f) = \int_X fg \, d\mu$ all $f \in L^p(d\mu)$. Thus $L^p(d\mu)^{**} = L^p(d\mu)$ because $(p')' = p$.

eg: Let $X^* = M(X)$, $X = C_0(X)$ when X is LCH. Then every bounded Borel measurable f gives $\langle f, \mu \rangle = \int_X f \, d\mu$, $|\langle f, \mu \rangle| \leq \sup_{x \in X} |f(x)| \cdot |\mu|(X)$. Thus $X^{**} \supsetneq C_0(X)$.

II. Tychonoff Topologies & Compactness

* Let D_x, $x \in X$, where $D_x \subseteq D$. Then $Q = \prod_{x \in X} D_x = \{ f : X \to D \mid f(x) \in D_x \}$.

The weak (Tychonoff) topology on Q is the weak topology generated by $\pi_x : Q \to D_x$ ($\pi_x(f) = f(x)$).

Thus $f_i \rightharpoonup f$ iff $f_i(x) \to f(x)$ each $x \in X$.

Thm: If each D_x is compact then Q is compact.

eg: Let $X = [0,1]$ and $D_x = [-1,1]$. Then $Q = \prod_{x \in X} D_x = \{ f : [0,1] \to [-1,1] \}$, and $f_n \rightharpoonup f$ iff $f_n(x) \to f(x)$.

Consider $f_n(x) = \sin(2\pi n x)$. Then $\exists$ a subnet $f_{n_k}(x) \to f(x)$ some $|f(x)| \leq 1$.

If f_{n_k} was a sequence, then $f_{n_k} \to f$ in L^2. But that's impossible.

III. General Alaoglu's Theorem

Thm: Let X any NVS, then $B_1^* \subseteq X^*$ is compact in the weak-* topology.

pf: For each $x \in X$ set $D_x = \{ z \in \mathbb{C} \mid |z| \leq \|x\|_X \}$, and $Q = \prod_{x \in X} D_x$. Then $Q = \{ \phi : X \to \mathbb{C} \mid |\phi(x)| \leq \|x\| \}$.

So $D_1^* \subseteq Q$ is a subset. We claim its weak-* closed in Q. Let $\alpha_n \in D_1^*$ with $\alpha_n \to \phi$

Thus $\alpha_n(x) \to \phi(x)$ each x. Now $\alpha_n(x+y) \to \lim_n \alpha_n(x) + \lim_n \alpha_n(y) = \phi(x) + \phi(y)$, and similarly $\alpha_n(\lambda x) \to \lambda \phi(x)$.

eg: Let X be a LCH. Let $\mu_n \in M(X) = C_0(X)^*$ with $\sup_n |\mu_n|(X) < \infty$. Then $\exists$ μ_{n_k} and $\mu \in M(X)$

such that $\int_X f \, d\mu_{n_k} \to \int_X f \, d\mu$ for each $f \in C_0(X)$.

eg: If X is 2^{nd} countable and μ_n are Borel probability measures.

IV. The case of separable spaces

* Let P_n be a countable family of semi-norms. Then we can metrize $\mathcal{F}_{X,P_n}$ w/ a

translation invariant distance via $d(x,y) = \sum_{n=1}^{\infty} 2^{-n} P_n(x-y) / 1 + P_n(x-y)$. Here we assume $P_n(x) = 0$ all $n \Rightarrow x = 0$.

Note that $\frac{p}{1+p} = 1 - \frac{1}{1+p}$ is monotone increasing. Thus $\frac{P_n(x-y)}{1+P_n(x-y)} \leq \frac{P_n(x-z) + P_n(y-z)}{1+P_n(x-z)+P_n(y-z)} \leq \frac{P_n(x-z)}{1+P_n(x-z)} + \frac{P_n(y-z)}{1+P_n(y-z)}$.

Thus $d(x,y) \leq d(x,z) + d(y,z)$.

* Now $d(x_i, x) \to 0$ iff $P_n(x - x_i) \to 0$ each n.

* If $\mathcal{F}_{X,P_n}$ is complete we call it a "Frechet space".

Thm: If X is separable, then B_R^* is a compact metric space.

pf: Let $x_n \in X$ be a countable dense subset. Now $\alpha_i \to \alpha$ in B_R^* iff $\alpha_i(x) \to \alpha(x)$

for each $x \in X$. Suppose instead we just know $\alpha_i(x_n) \to \alpha(x_n)$ for each n.

Let $\epsilon > 0$ and choose x_n with $\|x - x_n\|_X < \frac{\epsilon}{3R}$, and then $i_0 \in I$ s.t.

$|\alpha_i(x_n) - \alpha(x_n)| < \epsilon/3$ all $i_0 \leq i$. Then $|\alpha_i(x) - \alpha(x)| \leq |\alpha_i(x - x_n)| + |\alpha(x - x_n)| + |\alpha_i(x_n) - \alpha(x_n)| \leq$

$\leq \|\alpha_i\|_{X^*} \|x - x_n\|_X + \|\alpha\|_{X^*} \cdot \|x - x_n\|_X + \epsilon/3 < \epsilon$. Thus $\alpha_i(x) \to \alpha(x)$ all $x \in X$ iff $\alpha_i(x_n) \to \alpha_i(x_n)$

all x_n. So $d(\alpha, \beta) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\alpha(x_n) - \beta(x_n)|}{1 + |\alpha(x_n) - \beta(x_n)|}$ gives the weak-* topology on B_R^*.

__eg__: Let X be a 2^{nd} countable LCH space. Then $C_0(X)$ is separable, so $B_R^* \subseteq M(X)$ is a compact metric space. In particular all Borel probability measures on $[0,1]$, or measures $|\mu|(\mathbb{R}^d) \le 1$ on $\mathbb{R}^d$ form a compact metric space in the weak-* topology.

* Warning: X^* is __not__ a Fréchet space in the weak-* topology. Reason - B_R^* is weak-* closed and __nowhere__ __dense__ in $\mathcal{T}_{X^*}$. (every weakly open set is norm unbounded).

V. A note on semicontinuity.

* Let $x_n \rightharpoonup x$ weakly in X. Then $\alpha(x_n) \to \alpha(x)$ all $\alpha \in B_1^*$. Choose α so $|\alpha(x)| = \|x\|_X$

Then $\|x\|_X = \lim_n |\alpha(x_n)| \le \varliminf \|\alpha\|_{X^*} \|x_n\|_X = \varliminf \|x_n\|_X$.

* Also, if $x_n \rightharpoonup x$ then $\varlimsup \|x_n\|_X < \infty$, because $\sup_n |x_n(\alpha)| < \infty$ each $\alpha \in X^*$, thus $\sup_n \|x_n\|_{X^{**}} < \infty$ by PUB.

* If $\alpha_n \in X^*$ and $\alpha_n \rightharpoonup \alpha$, then $\|\alpha\|_{X^*} = \sup_{\|x\|_X = 1} |\alpha(x)| = \sup_{\|x\|_X = 1} \lim_n |\alpha_n(x)| \le \varliminf \sup_{\|x\|_X = 1} |\alpha_n(x)| = \varliminf \|\alpha_n\|_{X^*}$.

__eg__: If $\mu_n \rightharpoonup \mu$ then $|\mu|(X) \le \varliminf |\mu_n|(X)$. Can get strict inequality if $\mu_n(E) = m(E \cap [n,n+1])$.

Then $\|\mu_n\| \equiv 1$ but $\mu_n \rightharpoonup 0$.

__eg__: $\mu_n(E) = \int_E e^{inx}\,dm$, $E \subseteq [0,2\pi]$. $\mu_n = e^{inx} \mathbb{1}_{[0,2\pi]} \rightharpoonup 0$. $\|\mu_n\| \equiv 2\pi$.

__eg__: Now let $\mu_n \rightharpoonup \mu$ in $M(X)$, where μ_n, μ are positive measures.

* If $\mu_n \rightharpoonup \mu$, let K compact. Then for $K \prec f$ we have $\int_X f\,d\mu_n \to \int_X f\,d\mu$.

Thus $\varlimsup \mu(K_n) \le \int_X f\,d\mu$ all $K \prec f$. Taking inf in f get $\varlimsup \mu_n(K) \le \mu(K)$ all compact K.

* If $\mu_n \rightharpoonup \mu$ & U is open, taking $f \prec U$ we get $\int f\,d\mu_n \le \mu_n(U) \Rightarrow \int f\,d\mu \le \varliminf \mu(U_n)$.

Taking the sup in f gives $\mu(U) \le \varliminf \mu_n(U)$.