Math 174 Final

December 7, 2016

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit**.

Print Name: ______________________________

Student ID: ______________________________

Signature and Date: ______________________________

Problem	Score
1	/25
2	/25
3	/25
4	/25
5	/25

Problem	Score
6	/25
7	/25
8	/25
Total	/200

1. (25 pts) Consider the ODE $x' = f(t,x)$, with $x(t_0) = x_0$. Suppose we have available for use a Matlab function with header

   function [value] = f(t,x)

   that evaluates the function $f(t,x)$. Now, given the following header for a Matlab function:

   function [xN] = predictorcorrector(t0,x0,h,N)

   complete this function so that it performs **predictor-corrector**, using **Midpoint method** for predictor and **Trapezoid method** for corrector, with inputs $t_0 = t0$, $x_0 = x0$, and stepsize h, to output the approximation x_N.

   Use only basic programming, such as for loops and if statements, and do **not** use any of Matlab's vector-vector or matrix-vector operations. Remember: Midpoint method

$$x_{i+1} = x_i + hf\left(t_i + \frac{h}{2}, x_i + \frac{h}{2}f(t_i, x_i)\right)$$

   and Trapezoid method

$$x_{i+1} = x_i + \frac{h}{2}(f(t_i, x_i) + f(t_{i+1}, x_{i+1})).$$

2. (25 pts) Let g be continuously differentiable and suppose there exists a $0 < \lambda < 1$ such that $|g'(x)| \leq \lambda$ for all x real numbers. Suppose p is a fixed point of g, and let $p_{i+1} = g(p_i)$ give fixed point iterations for an initial guess p_0. Show p_i **converges** to p.

   Do **not** use fixed point theorem on convergence, since you are being asked to prove it in a special case. Remember: Mean value theorem

$$f(x) - f(y) = f'(\xi)(x - y)$$

   for some ξ between x and y.

3. (25 pts) Use **Taylor series** to find p, k, and $C \neq 0$ such that:

$$f'(x) - \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} = Ch^p f^{(k)}(x) + \mathcal{O}(h^{p+1}),$$

for h small. Remember:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

4. (25 pts) Consider the linear system $Ax = b$, with $A = D - L - U$ nonsingular, where D is diagonal and nonsingular, L is strictly lower triangular, and U is strictly upper triangular. Consider the **iterative method** whose sequence of approximations satisfies:

$$x^{(k+1)} = (D - U)^{-1} L x^{(k)} + (D - U)^{-1} b.$$

When

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix},$$

write out the **iteration matrix** and use it to determine whether this iterative method's sequence of approximations will **converge** to the solution of $Ax = b$ for **any** initial guess. Remember:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

5. (25 pts) Solve the following short problems.

(a) Perform Gaussian elimination with **partial pivoting** with back substitution, as if in a 2-**digit rounding** machine, to **solve** the linear system

$$\begin{cases} x_1 + 2x_2 &= 2 \\ -3x_1 + 3x_2 &= 1. \end{cases}$$

(b) Let $x_i, i = 0, \ldots, 20$ be evenly spaced nodes from $x_0 = -1$ to $x_{20} = 1$. With $y_i = 1 + x_i^2$, let $p(x)$ be the **piecewise linear** interpolant for the data points $(x_i, y_i), i = 0, \ldots, 20$. Find $p(0.17)$.

6. (25 pts) Solve the following short problems.

(a) Find a, B, D so that the following is a **free** or **natural** ($S''(\text{endpoints}) = 0$) **cubic spline**

$$S(x) = \begin{cases} 2 - 5x - 3x^2 + ax^3, & \text{if } -1 \le x < 0, \\ D - 5x + Bx^2 + x^3, & \text{if } 0 \le x \le 1 \end{cases}$$

(b) Consider the ODE **system**

$$\begin{cases} y'(t) &= y(t) + z(t) + t \\ z'(t) &= ty(t) - z(t) \end{cases}$$

with initial conditions $y(3) = y_0, z(3) = z_0$. Use **Euler's method** with stepsize $h = 0.1$ and initial guesses $y_0 = 1,\ z_0 = 2$ to solve for y_1 and z_1. Remember Euler's method:

$$x_{i+1} = x_i + hf(t, x_i).$$

7. (25 pts) Let $(x_i, y_i), i = 0, \ldots, n$, be $n+1$ data points with distinct nodes. Let $p(x)$ and $q(x)$ be polynomials of degree $\leq n$ that interpolate the data points. Prove $p \equiv q$.

8. (25 pts) Given the function $f(x)$, consider the data points $(a-2h, f(a-2h)), (a-h, f(a-h)), (a, f(a))$. Write down the **Lagrange form** of the interpolating polynomial for these data points. Then use it to **approximate** $f'(a)$, writing your result in the **form**

$$\frac{Af(a-2h)+Bf(a-h)+Cf(a)}{h},$$

for constants A, B, C.