Homework #0

1. Let f be continuous everywhere and suppose $f(0) = 1$. Does there exist an interval $(-a, a)$, where $a > 0$, such that $f(x) < 1.1$ for $x \in (-a, a)$? What about $f(x) > 0.9$?

2. Let f be infinitely differentiable and suppose it has at least 10 distinct roots.

   (a) Use Rolle's Theorem to show f' has at least 9 distinct roots.

   (b) Now that f' has at least 9 distinct roots, conclude f'' has at least 8 distinct roots.

   (c) Continuing, what can you say about the number of roots of $f^{(3)}, f^{(4)}, \ldots$?

3. Let $f(x) = \frac{x}{2} - \frac{1}{x}$.

   (a) Find the maximum of $|f(x)|$ in the interval $[1, 2]$.

   (b) Find the maximum of $|f'(x)|$ in the interval $[1, 2]$.

4. Let $a = x_0 < x_1 < \ldots < x_9 = b$ and let f be such that $f(x_k) = (-1)^k$ and $|f(x)| \leq 1$. Let g be a function such that $|g(x)|$ has maximum value < 1 in $[a, b]$. Use the Intermediate Value Theorem to show $f - g$ has at least 9 roots.

5. Let f be an infinitely differentiable function and let x_0 be a point. For $h > 0$ sufficiently small, use Taylor series to write the expression $f(x_0 + h) + 2f(x_0) + 3f(x_0 - h)$ in the form
$$C_0 f(x_0) + C_1 f'(x_0)h + C_2 f''(x_0)h^2 + \mathcal{O}(h^3).$$

6. Suppose p is a polynomial of degree ≤ 7. Suppose also that p has at least 10 distinct roots. Use the Fundamental Theorem of Algebra (or its corollaries) to determine the actual expression for p?

7. (Matlab)

   (a) Write a Matlab function using basic programming (for loops, while loops, and if statements) that inputs a square matrix and outputs the square root of the sum of squares of its entries:
$$\left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2},$$
   where $A = (a_{ij})$ is the matrix. Print out or write out your function.

   (b) Apply your function to the matrix $[1, -2; 3, -4]$ and print out or write out the results.

   (c) Apply your function to a matrix generated by $rand(5)$ and print out or write out the matrix and results.

8. (Math 274) Let f be an infinitely differentiable function and let x_0 be a point.

(a) Use integration by parts to derive the Taylor series expression:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{1}{2!}\int_{x_0}^{x} (x_0 + h - t)^2 f'''(t)\ dt.$$

(b) Use Weighted Mean Value Theorem for Integrals to then derive the Taylor series expression:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(\xi)}{3!}h^3,$$

for some ξ between x_0 and $x_0 + h$.