Homework #1

1. Suppose we have a computer that performs 3-digit rounding. Find the absolute and relative errors for the following numbers computed in the computer:
   (a) 0.000224729
   (b) 42381900
   (c) $2012.34 + 3.114$
   (d) $2.728 \cdot 0.0036$
   (e) $0.3856121 - 0.3840256$

2. Prove k-digit rounding of any real number x leads to a floating point number $fl(x)$ with relative error $\leq 5 \cdot 10^{-k}$.

3. In a computer that performs 4-digit rounding, find examples of the following:
   (a) Positive real numbers a, b such that $fl(fl(a) - fl(b))$ has absolute error $\leq 10^{-5}$ but relative error $\geq 90\%$.
   (b) Positive real numbers a, b such that $fl(fl(a) - fl(b))$ has absolute error $\geq 10^5$ but relative error $\leq 10\%$.

4. Let $u = 5 \cdot 10^{-k}$, so the relative error of k-digit rounding of any real number is $\leq u$. We of course take $k \geq 1$, so note $u < 1$. Given positive, real numbers x, y, show $x + y$, performed in a k-digit rounding machine, has relative error $\leq 2u + \mathcal{O}(u^2)$.

5. (a) Prove, using the Intermediate Value Theorem, that there is a root of $f(x) = x^2 - 3$ in the interval $[1, 2]$.
   (b) Starting with this interval, draw a description of the bisection method on $f(x)$, labeling the approximations p_1, p_2, p_3.
   (c) Compute the values of p_1, p_2, p_3.
   (d) Determine a bound on the absolute error of p_4 without computing p_4. Then compute p_4 and determine the actual absolute error (using the actual exact solution computed by calculator or computer). Does it satisfy the bound?

6. Suppose f is continuous and $f(x) < 0$ in $[-1, 2]$, and $f(x) > 0$ in $[2.5, 4]$. Using the bisection method on $f(x)$ with starting interval $[-1, 4]$, compute the values of p_1, p_2, p_3.

7. Suppose f is continuos in $[-3, 2]$ and $f(-3) < 0$ and $f(2) > 0$.
   (a) Let p_{15} be bisection method's approximation, applied to $f(x)$ with starting interval $[-3, 2]$, after 15 iterations. Bound the absolute error of this approximation without computing p_{15}.
   (b) Use error bounds to determine n such that the absolute error of p_n is guaranteed to be $\leq 10^{-12}$.

8. (Matlab)

   (a) Using the "cos" command in Matlab, write a Matlab function that inputs a number x and outputs the value $\cos x - x$. Print out or write out the function.

   (b) Write a Matlab function that inputs the endpoints of a starting interval a, b and the number N and outputs the bisection method's p_N approximation of the root for the function in part (a). Print out or write out the function.

   (c) Run your program for the starting interval $[0, \pi/2]$ (use "pi" for π) and write down the values of p_1, p_5, p_{10}, p_{20}.

9. (Math 274) Suppose we have a k-digit rounding machine. Let $u = 5 \cdot 10^{-k}$, so the relative error of k-digit rounding of any real number is $\leq u$. We of course take $k \geq 1$, so note $u < 1$.

   (a) If x_i are real numbers, show $\prod_{x=1}^{n} x_i$, performed in the machine, has relative error bounded by $(2n-1)u + \mathcal{O}(u^2)$.

   (b) If $x_i > 0$ are positive floating point numbers, show $((x_1+x_2)+x_3)+x_4$, performed in the machine, has relative error bounded by $3u + \mathcal{O}(u^2)$. Guess an analogous result for

$$((((((x_1 + x_2) + x_3) + x_4) + x_5) + x_6) + x_7) + x_8.$$

   (c) If $x_i > 0$ are positive floating point numbers, show $(x_1+x_2)+(x_3+x_4)$, performed in the machine, has relative error bounded by $2u + \mathcal{O}(u^2)$. Guess an analogous result for

$$((x_1 + x_2) + (x_3 + x_4)) + ((x_5 + x_6) + (x_7 + x_8)).$$