**Homework #2**

1. Let $g(x) = 1/x + x/2$ and consider the interval $[1.4, 1.45]$.
   (a) Show $g(x) \in [1.4, 1.45]$ for $x \in [1.4, 1.45]$ by finding the maximum and minimum values of $g$ in the interval.
   (b) Find $0 \leq k < 1$ such that $|g'(x)| \leq k$ for all $x \in [1.4, 1.45]$ by finding the maximum and minimum values of $g'(x)$ in the interval.
   (c) Use this $k$, along with error bounds, to estimate $n$ such that $p_n$ of fixed point iteratiions will have absolute error $\leq 10^{-4}$, when $p_0 = 1.425$. Do the same for absolute error $\leq 10^{-10}$.
   (d) Perform fixed point iterations with initial guess $p_0 = 1.425$ until $|p_k - p_{k-1}| \leq 10^{-4}$ is satisfied.
2. Suppose $g \in C^1[a, b]$ and there exists $0 \leq k < 1$ such that $g'(x) \leq k$ for all $x \in [a, b]$. Prove $g$ has at most one fixed point in $[a, b]$.
3. Suppose $g$ and $g'$ are continuous functions.
   (a) Prove if there is a $0 < k < 1$ such that $|g'(x)| \leq k$ for all $x$, and if $g$ has a fixed point, then fixed point iterations will converge for any starting guess.
   (b) Prove if $|g'(x)| \geq 1$ everywhere, then fixed point iterations will not converge to any fixed point when the starting guess is not itself a fixed point.
4. Consider the root-finding problem $x^2 - 3 = 0$.
   (a) Consider $x^2 + x - 3 = x$, obtained by adding $x$ on both sides. Study the value of $|g'(\sqrt{3})|$ and comment on the convergence of fixed point iterations.
   (b) Find a different way of turning $x^2 - 3 = 0$ into a fixed point problem that gives a fixed point function $g(x)$ that satisfies $|g'(\sqrt{3})| < 1$. Comment on the convergence of fixed point iterations.
5. (a) Give a graphical description showing how fixed point iterations converge for the fixed point function $g(x) = x/2$.
   (b) Give a graphical description showing how fixed point iterations do not converge for the fixed point function $g(x) = 2x$.
6. (a) Starting with initial guess $p_0 = 1$, find approximations $p_1, p_2, p_3$ to the root of $f(x) = x^2 - 3$ using Newton's method.
   (b) Give a graphical description of how Newton's method arrives at these approximations.

7. (a) Consider

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0. \end{cases}$$

Starting with initial guess $p_0 = a > 0$, find 3 additional approximations to the root of $f(x)$ using Newton's method.

(b) Give a graphical description of how Newton's method arrives at these approximations.

(c) Will Newton's method converge to the exact root at 0 for any $p_0 \neq 0$? Why does this not violate the theorem on convergence of Newton's method?

8. (Matlab)

(a) Using the "cos" command in Matlab, write a Matlab function that inputs a number $x$ and outputs the value $\cos x$. Print out or write out the function.

(b) Write a Matlab function that inputs a starting guess $p_0$ and tolerance $\epsilon$, performs fixed point iterations on the function of part (a), and outputs the number of iterations $N$ and the final fixed point approximation $p_N$ satisfying $|p_N - p_{N-1}| \leq \epsilon$. Print out or write out the function.

(c) Run your function using $p_0 = 1$ and $\epsilon = 10^{-2}, 10^{-5}, 10^{-10}$ and print out or write out the results.

9. (Math 274) Let $g \in C^1[a, b]$.

(a) Suppose there exists $k < 1$ such that $|g'(x)| \leq k$ for all $x \in [a, b]$. If there is a fixed point $p$ in $(a, b)$, prove there exists an interval $[c, d] \subseteq [a, b]$ such that $g(x) \in [c, d]$ for all $x \in [c, d]$.

(b) If there is a fixed point $p$ in $(a, b)$, and if $|g'(p)| < 1$, explain why there exists an interval $[c, d] \subseteq [a, b]$ such that $g(x) \in [c, d]$ for all $x \in [c, d]$.