Homework #2

1. (a) $g'(x) = -1/x^2 + 1/2$, and so $g \in C^1[1.4, 1.45]$ and critical points of g are at $x = \pm\sqrt{2}$. Note, only $\sqrt{2}$ lies in the interval $[1.4, 1.45]$. Now, the max and min of g in the interval have to be at either the critical points of g or endpoints of the interval: $\sqrt{2}$, 1.4, or 1.45. Checking for the max and min, $g(\sqrt{2}) = \sqrt{2} = 1.41421356237310$, $g(1.4) = 1.41428571428571$, $g(1.45) = 1.41465517241379$. Thus the max is at $x = 1.45$, with a value of 1.41465517241379, and the min is at $x = \sqrt{2}$, with a value of 1.41421356237310. So

$$g : [1.4, 1.45] \to [1.41421356237310, 1.41465517241379] \subseteq [1.4, 1.45].$$

(b) $g''(x) = 2/x^3$, and so $g \in C^2[1.4, 1.45]$ and there are no critical points of g'. Now, the max of $|g'|$ has to be at either the critical points of g' or endpoints of the interval: 1.4 or 1.45. Checking for the max, $|g'(1.4)| = 0.0102040816326532$, $|g'(1.45)| = 0.0243757431629013$. Thus the max is at $x = 1.45$, with a value of 0.0243757431629013. This means $g'(x) \leq k < 1$ in $[1.4, 1.45]$, where $k = 0.0243757431629013$.

3. (a) Given starting guess p_0, let $p_{n+1} = g(p_n)$ for $n = 0, 1, \ldots$. Then

$$|p_n - p| = |g'(\xi_n)||p_{n-1} - p| \leq k|p_{n-1} - p|,$$

for some ξ_n between p_n and p. So

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \ldots \leq k^n|p_0 - p| \to 0,$$

as $n \to \infty$.

(b) Given starting guess p_0, let $p_{n+1} = g(p_n)$ for $n = 0, 1, \ldots$. Then

$$|p_n - p| = |g'(\xi_n)||p_{n-1} - p| \geq |p_{n-1} - p|,$$

for some ξ_n between p_n and p. So

$$|p_n - p| \geq |p_{n-1} - p| \geq |p_{n-2} - p| \leq \ldots \geq |p_0 - p|,$$

so if $p_0 \neq p$, then $|p_n - p|$ cannot converge to 0.

7. (a) Note, if $p_n = a > 0$, then

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = a - \frac{\sqrt{a}}{\frac{1}{2\sqrt{a}}} = a - 2a = -a.$$

Also, if $p_n = -a < 0$, then

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = -a - \frac{-\sqrt{a}}{\frac{1}{2\sqrt{a}}} = -a + 2a = a.$$

Thus $p_n = -a$ if n odd, and $p_n = a$ if n even, so $p_1 = a$, $p_2 = -a$, $p_3 = a$.

(c) The exact root is at $p = 0$, and $|p_n - p| = p_0$, whether n is even or odd. So if $p_0 \neq 0$, then $|p_n - p|$ cannot converge to 0. Checking the hypotheses of Newton's method's theorem on convergence, we notice that even though f is continuous,

$$f'(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0 \\ \frac{1}{2\sqrt{-x}}, & x < 0 \end{cases}$$

and so $f'(x) \to \infty$ as $x \to 0$, and f' is not continuous in any interval around the root. So one of the conditions in the hypotheses of the convergence theorem does not hold and the theorem cannot be applied.

8. (Matlab)

(a) See "hw2afn.m".

(b) See "hw2bfn.m".

(c) For $\epsilon = 10^{-2}$, the function returns $p_{11} = 0.735604740436347$. For $\epsilon = 10^{-5}$, the function returns $p_{29} = 0.739082298522402$. For $\epsilon = 10^{-10}$, the function returns $p_{58} = 0.739085133245110$.

9. (Math 274)

(a) We find $[c, d]$ in the form $[p - \delta, p + \delta]$, for some $\delta > 0$. First choose δ such that $[p - \delta, p + \delta] \subseteq [a, b]$, for example, $\delta = \min\{(p - a)/2, (b - p)/2\}$. Then, given $x \in [p - \delta, p + \delta]$,

$$|g(x) - p| = |g(x) - g(p)| \leq |g'(\xi)||x - p| \leq k|x - p| < \delta,$$

which implies $g(x) \in [p - \delta, p + \delta]$.