Homework #5

1. (a) Draw a graph of the piecewise linear interpolating polynomial for the data given in the following table:

x	0	1	3	4	6
$f(x)$	1	0	1	2	1

   (b) Write down the equation for the linear piece in the interval $[4, 6]$.

   (c) Suppose we create the piecewise quadratic interpolating polynomial by using the parabola passing through the first three points in $x \in [0, 3]$ and the parabola passing through the last three points in $x \in [3, 6]$. Find the values of this piecewise polynomial at $x = 2$ and also at $x = 5$.

2. Let $x_0, \ldots, x_n$ be distinct nodes in $[a, b]$ and let $f \in C^{n+1}[a, b]$. If p is the Lagrange interpolating polynomial for values $f(x_i)$ at the nodes, prove for $x \in [a, b]$,

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

for some $\xi(x) \in (a, b)$ (Hint: Look in the book).

3. Given $x_0 = -1, x_1 = 1$ and $f(x_0) = 0, f(x_1) = 2$ and $f'(x_0) = 1, f'(x_1) = -1$:

   (a) Find the Hermite interpolating polynomial using divided differences.

   (b) Add the information $x_2 = 0$ and $f(x_2) = 1$ and $f'(x_2) = 2$ and find the resulting Hermite interpolating polynomial.

   (c) Find instead the form for the piecewise cubic Hermite interpolating polynomial interpolating the data of the previous part.

4. Suppose we have nodes $x_i, i = 0, 1, 2, 3$ and are given values of $f(x_i), i = 0, 1, 2, 3$, first derivative values $f'(x_i), i = 0, 1, 2$, second derivative values $f''(x_i), i = 0, 1$ and third derivative value $f'''(x_0)$. The theory says there is a unique degree $\leq d$ polynomial that interpolates this data. Guess d by counting unknowns and equations (degrees of freedom and conditions).

5. Consider knots and nodes at $t_0 < t_1 < \ldots < t_n$.

   (a) Count the number of degrees of freedom afforded by using pieces of degree $\leq k$ polynomials in $[t_i, t_{i+1})$, $i = 0, \ldots, n-1$.

   (b) Count the number of conditions imposed if the piecewise polynomial is forced to be continuous in $[t_0, t_n]$ while interpolating given values at t_i, $i = 0, \ldots, n$.

   (c) Count the number of conditions there are from additionally forcing the first to kth derivatives to be continuous in $[t_0, t_n]$.

(d) If we choose pieces of degree ≤ 5 polynomials and require interpolation of given values at t_i, $i = 0, \ldots, n$ in addition to continuity up to the 4th derivative in $[t_0, t_n]$, How many more degrees of freedom are there than conditions?

6. Find b, c, d so that the following is a natural cubic spline

$$S(x) = \begin{cases} 1 + 2x - x^3, & \text{if } 0 \leq x < 1, \\ 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \leq x < 2. \end{cases}$$

7. Let

$$S(x) = \begin{cases} S_0(x) = a + bx + cx^2 + dx^3, & \text{if } 0 \leq x \leq 1, \\ S_1(x) = 1 + (x-1) + (x-1)^2 + (x-1)^3, & \text{if } 1 \leq x \leq 2, \end{cases}$$

be a cubic spline on $[0, 2]$ satisfying the boundary conditions $S'(0) = -1$ and $S'(2) = 6$ and with knots at $0, 1, 2$. Find a, b, c, d.

8. (Matlab)

(a) Write a Matlab function that inputs x, an array of x-coordinates of data points, with $x_1 < \ldots < x_n$; y, an array of y-coordinates of data points; n, the total number of data points; and z, a location on the x-axis. Have it output an approximation of the value at z using piecewise linear interpolation. Print out or write out your program.

(b) Apply your function to the case where $x = -1 : 0.01 : 1$ and $y = \sin(x)$, with $z = \sqrt{2}/2$, and print out or write out your result.

9. (Math 274) Let $x_0, \ldots, x_n$ be distinct nodes in $[a, b]$ and let $f \in C^{2n+2}[a, b]$. If p is the Hermite interpolating polynomial for values $f(x_i)$ and first derivatives $f'(x_i)$ at the nodes, prove for $x \in [a, b]$,

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2,$$

for some $\xi(x) \in (a, b)$.