Homework #5

3. (a) The divided difference table is

$$
\begin{array}{ccccc}
-1 & 0 & & & \\
 & & 1 & & \\
-1 & 0 & & 0 & \\
 & & 1 & & -\frac{1}{2} \\
1 & 2 & & -1 & \\
 & & -1 & & \\
1 & 2 & & &
\end{array}
$$

Then $p(x) = (x+1) - \frac{1}{2}(x+1)^2(x-1)$.

(b) The divided difference table is

$$
\begin{array}{ccccccc}
-1 & 0 & & & & & \\
 & & 1 & & & & \\
-1 & 0 & & 0 & & & \\
 & & 1 & & -\frac{1}{2} & & \\
1 & 2 & & -1 & & -\frac{1}{2} & \\
 & & -1 & & -1 & & \frac{1}{2} \\
1 & 2 & & -2 & & 0 & \\
 & & 1 & & -1 & & \\
0 & 1 & & -1 & & & \\
 & & 2 & & & & \\
0 & 1 & & & & &
\end{array}
$$

Then $p(x) = (x+1) - \frac{1}{2}(x+1)^2(x-1) - \frac{1}{2}(x+1)^2(x-1)^2 + \frac{1}{2}(x+1)^2(x-1)^2 x$.

4. Let p be the polynomial interpolating the data. Then $p(x_i) = f(x_i)$, for $i = 0, 1, 2, 3$, gives 4 equations; and $p'(x_i) = f'(x_i)$, for $i = 0, 1, 2$ gives 3 more equations; and $p''(x_i) = f''(x_i)$, for $i = 0, 1$ gives 2 more equations; and $p'''(x_0) = f'''(x_0)$ gives 1 more equation. This totals to 10 equations. Writing

$$p(x) = a_0 + a_1 x + \ldots + a_d x^d,$$

we see p has the $d+1$ unknowns: $a_0, \ldots, a_d$. Matching equations with unknowns, we take $d + 1 = 10$, or $d = 9$.

7. $S'(0) = S_0'(0) = -1$ implies $b = -1$; and $S_0(1) = S_1(1) = 1$ implies $a+b+c+d = 1$; and $S_0'(1) = S_1'(1) = 1$ implies $b + 2c + 3d = 1$; and $S_0''(1) = S_1''(1) = 2$ implies $2c + 6d = 2$. Thus, $2c + 3d = 2$, so $d = 0$; so $c = 1$; so $a = 1$.

8. (Matlab)

   (a) See “hw5afn.m”.

   (b) The function gives a result of 0.649630271332711.

9. (Math 274) Let x be a fixed point. If $x = x_k$, then $f(x) = p(x)$ and the formula holds. If $x \neq x_k$, for any k, then we define the function

$$g(t) = f(t) - p(t) - (f(x) - p(x)) \frac{\prod_{i=0}^{n}(t - x_i)^2}{\prod_{i=0}^{n}(x - x_i)^2},$$

so that g has roots at all the nodes x_i and at point x. Let

$$y_0 < y_1 < \ldots < y_{n+1},$$

such that each y_i is either a node or x. Then Rolle's Theorem says g' has roots in (y_i, y_{i+1}), for each $i = 0, \ldots, n$. Additionally, g' has roots at each of the nodes x_i. Thus we count at least $(n+1) + (n+1) = 2n+2$ roots for g'. Note $f \in C^{2n+2}[a,b]$ implies $g \in C^{2n+2}[a,b]$ so, by Generalized Rolle's Theorem, there exists $\xi(x) \in [a,b]$ such that $(g')^{(2n+1)}(\xi(x)) = 0$. Thus, noting $p^{(2n+2)} \equiv 0$, since p is degree $\leq 2n+1$,

$$\begin{aligned} 0 &= g^{(2n+2)}(\xi(x)) \\ &= f^{(2n+2)}(\xi(x)) - (f(x) - p(x)) \frac{(2n+2)!}{\prod_{i=0}^{n}(x - x_i)^2}. \end{aligned}$$

Solving, we get

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!} \prod_{i=0}^{n}(x - x_i)^2.$$