Homework #6

1. (a) The normal equations are

$$\begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix},$$

which in this case is

$$\begin{bmatrix} 5 & -0.5 \\ -0.5 & 3.25 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 6.2 \end{bmatrix}.$$

(b) Solving, we get $a_0 = 0.965625$ and $a_1 = 2.05625$, for the line $y = a_0 + a_1 x$.

(c) The normal equations are

$$\begin{bmatrix} n & \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i^3 \\ \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n x_i^2 y_i \end{bmatrix},$$

which in this case is

$$\begin{bmatrix} 5 & -0.5 & 3.25 \\ -0.5 & 3.25 & -0.875 \\ 3.25 & -0.875 & 3.0625 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 6.2 \\ 1.55 \end{bmatrix}.$$

4. The total error is

$$E(a_0, a_1, a_2) = \sum_{i=1}^n c_i(a_0 + a_1(1 + x_i) + a_2(1 + x_i + x_i^2) - y_i)^2,$$

Critical points satisfy

$$\begin{aligned} 0 = \frac{\partial E}{\partial a_0} &= \sum_{i=1}^n c_i(a_0 + a_1(1 + x_i) + a_2(1 + x_i + x_i^2) - y_i) \\ 0 = \frac{\partial E}{\partial a_1} &= \sum_{i=1}^n c_i(a_0 + a_1(1 + x_i) + a_2(1 + x_i + x_i^2) - y_i)(1 + x_i) \\ 0 = \frac{\partial E}{\partial a_1} &= \sum_{i=1}^n c_i(a_0 + a_1(1 + x_i) + a_2(1 + x_i + x_i^2) - y_i)(1 + x_i + x_i^2). \end{aligned}$$

Thus the normal equations are

$$\begin{bmatrix} \sum_{i=1}^n c_i & \sum_{i=1}^n c_i(1 + x_i) & \sum_{i=1}^n c_i(1 + x_i + x_i^2) \\ \sum_{i=1}^n c_i(1 + x_i) & \sum_{i=1}^n c_i(1 + x_i)^2 & \sum_{i=1}^n c_i(1 + x_i + x_i^2)(1 + x_i) \\ \sum_{i=1}^n c_i(1 + x_i + x_i^2) & \sum_{i=1}^n c_i(1 + x_i)(1 + x_i + x_i^2) & \sum_{i=1}^n c_i(1 + x_i + x_i^2)^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n (1 + x_i) y_i \\ \sum_{i=1}^n (1 + x_i + x_i^2) y_i \end{bmatrix}.$$

7.
$$\begin{aligned}&\tfrac{1}{h^2}(f(x+h)-2f(x)+f(x-h))-f''(x)\\&=\tfrac{1}{h^2}(f(x)+hf'(x)+\tfrac{h^2}{2}f''(x)+\tfrac{h^3}{6}f'''(x)+\tfrac{h^4}{24}f''''(x)+\ldots-2f(x)+\\&f(x)-hf'(x)+\tfrac{h^2}{2}f''(x)-\tfrac{h^3}{6}f'''(x)+\tfrac{h^4}{24}f''''(x)+\ldots)-f''(x)\\&=\tfrac{1}{h^2}\left(h^2f''(x)+\tfrac{h^4}{12}f''''(x)+\ldots\right)-f''(x)\\&=\tfrac{h^2}{12}f''''(x)+\ldots\end{aligned}$$

So the absolute error is $\mathcal{O}(h^2)$.

8. (Matlab)

   (a) See "hw6afn.m".

9. (Math 274) Taylor series says

$$\begin{aligned}Af(x)+Bf(x+h)+Cf(x-2h) &= Af(x)+B\left(f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{6}f'''(x)+\ldots\right)+\\&\quad C\left(f(x)-2hf'(x)+\frac{4h^2}{2}f''(x)-\frac{8h^3}{6}f'''(x)+\ldots\right)\\&= (A+B+C)f(x)+(B-2C)hf'(x)+\left(\frac{B}{2}+2C\right)h^2f''(x)\\&\quad+\left(\frac{B}{6}-\frac{4C}{3}\right)h^3f'''(x)+\ldots,\end{aligned}$$

so we want

$$\begin{aligned}A+B+C &= 0\\B-2C &= \frac{1}{h}\\\frac{B}{2}+2C &= 0.\end{aligned}$$

Solving, $B=\frac{2}{3h}$ and $C=-\frac{1}{6h}$ and $A=-\frac{1}{2h}$.