Homework #7

1. Consider the table of values

x	x_0	x_1	x_2	x_3	x_4
y	y_0	y_1	y_2	y_3	y_4

with $x_i = i/4$ and $y_i = 4/(1+x_i^2)$.

   (a) Use composite trapezoidal rule to approximate the integral. What is the exact absolute error given that the exact value is π?

   (b) Use instead Simpson's rule. What is the exact absolute error?

   (c) Use Simpson's rule on the table with more values:

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

   with $x_i = i/8$ and $y_i = 4/(1+x_i^2)$. How many times smaller is the exact absolute error compared to the results in part (b)?

2. Use Taylor series to show

$$\int_0^h f(x)\ dx - hf(h/2) = \mathcal{O}(h^3).$$

3. Let

$$I = \int_a^b f(x)\ dx$$

and let $T(h)$ denote composite trapezoidal rule approximating I using stepsize h. Assume error formula

$$I = T(h) + C_2h^2 + C_4h^4 + C_6h^6 + \dots$$

   (a) Use Richardson extrapolation with stepsizes h and $2h$ to derive Simpson's rule.

   (b) (not due) Use Richardson extrapolation with stepsizes $h, 2h, 4h$ to write out the $\mathcal{O}(h^6)$ approximation formula.

4. Consider the ODE

$$y' = -2ty$$

with $y(0) = 2$. The exact solution is $y(t) = 2e^{-t^2}$.

   (a) Use Euler's method with stepsize $h = 0.5$ to approximate $y(1)$ and find the absolute error $E(0.5)$ of this approximation.

   (b) Use Euler's method with stepsize $h = 0.25$ to approximate $y(1)$ and find the absolute error $E(0.25)$ of this approximation.

   (c) Compute $E(0.5)/E(0.25)$.

5. (not due) Use Trapezoid Method with stepsize $h = 0.5$ to solve the ODE

$$y' = t/y$$

for $y(2)$ given $y(1) = 2$.

6. (not due) Consider the ODE

$$y' = -2ty$$

with $y(0) = 2$. The exact solution is $y(t) = 2e^{-t^2}$.

(a) Use Midpoint method with stepsize $h = 0.5$ to approximate $y(1)$ and find the absolute error $E(0.5)$ of this approximation.

(b) Use Midpoint method with stepsize $h = 0.25$ to approximate $y(1)$ and find the absolute error $E(0.25)$ of this approximation.

(c) Compute $E(0.5)/E(0.25)$.

7. (Matlab)

(a) For a given $f(t, y)$, write a Matlab function that inputs t, y, and outputs the vaue of $f(t, y)$. Then write a Matlab function that inputs:

- t_0 and w_0;
- stepsize h;
- number of iterations N;

and uses the Midpoint method, with stepsie h, to solve the ODE

$$y' = f(t, y),$$

with initial value $y(t_0) = w_0$, and outputs approximation w_N. Write out or print out this latter function.

(b) Apply your function to the case where $f(t, y) = \sin t + y$, $t_0 = 0$, $w_0 = 1$, and $(h, N) = (0.5, 2)$ and $(h, N) = (0.05, 20)$ and $(h, N) = (0.01, 100)$. Write out or print out your results in each case.

8. (Math 274) Consider, for $\lambda > 0$,

$$y' = -\lambda y,$$

with $y(0) = w_0 \neq 0$. The exact solution is $y(t) = w_0 e^{-\lambda t}$, and $\lim_{t\to\infty} y(t) = 0$.

(a) Apply Euler's method to the ODE and write down the approximation w_n in terms of w_0.

(b) For what h does $\lim_{n\to\infty} w_n = 0$? What happens to $\lim_{n\to\infty} w_n$ for other $h > 0$?

(c) Similarly analyze the case $\lambda < 0$: what is $\lim_{t\to\infty} y(t)$ and when does $\lim_{n\to\infty} w_n$ satisfy similar results?