Homework #7

1. (a) Composite trapezoidal rule gives an approximation of

$$\frac{0.25}{2}\left[\frac{4}{1+0^2}+\frac{8}{1+0.25^2}+\frac{8}{1+0.5^2}+\frac{8}{1+0.75^2}+\frac{4}{1+1^2}\right]=3.13117647058824.$$

The absolute error is $|\pi - 3.13117647058824| = 0.0104161830015577$.

(b) Composite Simpson's rule gives an approximation of

$$\frac{0.25}{3}\left[\frac{4}{1+0^2}+\frac{16}{1+0.25^2}+\frac{8}{1+0.5^2}+\frac{16}{1+0.75^2}+\frac{4}{1+1^2}\right]=3.14156862745098.$$

The absolute error is $|\pi - 3.14156862745098| = 2.40261388126939 \cdot 10^{-5}$.

(c) Composite Simpson's rule gives an approximation of

$$\frac{0.125}{3}\left[\frac{4}{1+0^2}+\frac{16}{1+0.125^2}+\frac{8}{1+0.25^2}+\frac{16}{1+0.375^2}+\frac{8}{1+0.5^2}+\frac{16}{1+0.625^2}+\frac{8}{1+0.75^2}+\frac{16}{1+0.875^2}+\frac{4}{1+1^2}\right]=3.14159250245871.$$

The absolute error is $|\pi - 3.14159250245871| = 1.51131086312262 \cdot 10^{-7}$, which is 158.975492064233 times smaller.

2. Let $F(x) = \int_0^h f(x)\,dx$. Then

$$\begin{aligned}F(h) &= F(0) + hF'(0) + \frac{h^2}{2}F''(0) + \frac{h^3}{6}F'''(0) + \dots\\ &= 0 + hf(0) + \frac{h^2}{2}f'(0) + \frac{h^3}{6}f''(0) + \dots\end{aligned}$$

Also,

$$\begin{aligned}f\left(\frac{h}{2}\right) &= f(0) + \frac{h}{2}f'(0) + \frac{\left(\frac{h}{2}\right)^2}{2}f''(0) + \dots\\ &= f(0) + \frac{h}{2}f'(0) + \frac{h^2}{8}f''(0) + \dots\end{aligned}$$

So

$$\begin{aligned}\int_0^h f(x)\,dx - hf(h/2) &= 0 + hf(0) + \frac{h^2}{2}f'(0) + \frac{h^3}{6}f''(0) + \dots -\\ &\quad h\left(f(0) + \frac{h}{2}f'(0) + \frac{h^2}{8}f''(0) + \dots\right)\\ &= \left(\frac{h^3}{6} - \frac{h^3}{8}\right)f''(0) + \dots\\ &= \frac{h^3}{24}f''(0) + \dots\\ &= \mathcal{O}(h^3).\end{aligned}$$

4. (a) Note, $f(t, y) = -2ty$, and, with $h = 0.5$, $t_i = 0.5i$. The initial condition says $w_0 = 2$. So

$$\begin{aligned} w_1 &= w_0 + hf(t_0, w_0) = 2 + 0.5(-2 \cdot 0 \cdot 2) = 2 \\ w_2 &= w_1 + hf(t_1, w_1) = 2 + 0.5(-2 \cdot 0.5 \cdot 2) = 1, \end{aligned}$$

and $y(1) \approx w_2 = 1$. Thus, $E(0.5) = |2e^{-1} - 1| = 0.264241117657115$.

(b) Note, $f(t, y) = -2ty$, and, with $h = 0.25$, $t_i = 0.25i$. The initial condition says $w_0 = 2$. So

$$\begin{aligned} w_1 &= w_0 + hf(t_0, w_0) = 2 + 0.25(-2 \cdot 0 \cdot 2) = 2 \\ w_2 &= w_1 + hf(t_1, w_1) = 2 + 0.25(-2 \cdot 0.25 \cdot 2) = 1.75 \\ w_3 &= w_2 + hf(t_2, w_2) = 1.75 + 0.25(-2 \cdot 0.5 \cdot 1.75) = 1.3125 \\ w_4 &= w_3 + hf(t_3, w_3) = 1.3125 + 0.25(-2 \cdot 0.75 \cdot 1.3125) = 0.8203125 \end{aligned}$$

and $y(1) \approx w_4 = 0.8203125$. Thus, $E(0.25) = |2e^{-1} - 0.8203125| = 0.0845536176571153$.

(c) $E(0.5)/E(0.25) = 3.12513083388903$.

7. (Matlab)

(a) See "hw7afn.m".

(b) For $(h, N) = (0.5, 2)$, we get $w_2 = 3.24238828923149$; for $(h, N) = (0.05, 20)$, we get $w_{20} = 3.38452755318898$; and for $(h, N) = (0.01, 100)$, we get $w_{100} = 3.38645337737018$.

8. (Math 274)

(a) Note, $f(t, y) = -\lambda y$. So

$$\begin{aligned} w_n &= w_{n-1} + hf(t_{n-1}, w_{n-1}) = w_{n-1} - \lambda h w_{n-1} = (1 - \lambda h) w_{n-1} \\ &= (1 - \lambda h)^2 w_{n-2} \\ &= \dots \\ &= (1 - \lambda h)^n w_0. \end{aligned}$$

(b)

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} (1 - \lambda h)^n w_0,$$

and, since $w_0 \neq 0$, this $= 0$ if and only if $|1 - \lambda h| < 1$. Now $|1 - \lambda h| < 1$ if and only if $-1 < 1 - \lambda h < 1$. So $\lim_{n\to\infty} w_n = 0$ if and only if $0 < h < 2/\lambda$. Also, $\lim_{n\to\infty} w_n$ does not converge for $h = 2/\lambda$, and $\lim_{n\to\infty} w_n = \infty$, if $h > 2/\lambda$.