Homework #8

1. Use Euler's method as predictor and Trapezoid Method as corrector with $h = 0.5$ to solve the ODE
$$y' = \sin y$$
for $y(1)$ given $y(0) = 1$.

2. Use Euler's method with stepsize $h = 0.25$ to solve the system of ODE's
$$\begin{aligned} y' &= -z \\ z' &= y \end{aligned}$$
for $y(1), z(1)$ given $y(0) = 1, z(0) = 0$.

3. (a) Write out each step of Gaussian elimination (without pivoting) on the augmented matrix for the linear system
$$\begin{cases} 2x - 3y + z &= 1 \\ x + y - z &= 2 \\ -4x + 4z &= -1. \end{cases}$$
(b) Now use back substitution to solve for x, y, z.

4. (a) Count the number of additions/subtractions needed to perform back substitution on $Ux = b$, where U is upper triangular, from the formula
$$x_i = \left[b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right] / u_{ii}.$$
(b) Also count the number of multiplications/divisions.

5. Prove one step of Gaussian elimination (without pivoting) on an $n \times n$ symmetric matrix A with $a_{11} \neq 0$ produces a matrix B such that $B(2:n, 2:n)$ is symmetric.

6. (a) Give an example of a 2×2 matrix that doesn't have an LU factorization.
(b) Use Gaussian elimination (without pivoting) to find the LU factorization of the matrix
$$A = \begin{bmatrix} -2 & 0 & 1 & -1 \\ -1 & 2 & 0 & 1 \\ 4 & -1 & -2 & -4 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$
(c) Use the LU factorization and forward and back substitution to solve the linear system $Ax = b$, where $b = [0, 1, 2, 3]^T$.

7. (a) Count the number of additions/subtractions and multiplications/divisions it takes to get the LU factorization (assuming it exists) of a tridiagonal $n \times n$ matrix (taking advantage of all the zeros). Thus, how many times longer does it take to compute the LU factorization of a $2n \times 2n$ tridiagonal matrix compared to a $n \times n$ tridiagonal matrix?

   (b) Count the number of additions/subtractions and multiplications/divisions it takes to perform forward and back substitution to solve $LUx = b$, using the LU factorization a tridiagonal $n \times n$ matrix (assuming it exists and taking advantage of all the zeros).

8. (Matlab)

   (a) Write a Matlab function that inputs the dimension n and a $n \times n$ matrix A, performs finds the LU factorization of A (stored in a single matrix), and outputs the number of flops used. Print out or write out your function.

   (b) Apply your function to 10×10, 20×20, 100×100, and 200×200 matrices and print out your results in each case.

9. (Math 274) An $n \times n$ matrix A is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|,$$

for all $i = 1, \ldots, n$. Prove Gaussian elimination (without pivoting) on a strictly diagonally dominant matrix produces a strictly diagonally dominant upper triangular matrix.