Homework #8

1. Euler's method gives

$$w_1^* = w_0 + h \sin w_0 = 1 + 0.5 \sin 1 = 1.42073549240395.$$

Then Trapezoid method gives

$$w_1 = w_0 + \frac{h}{2}[\sin w_0 + \sin w_1^*] = 1 + 0.25[\sin 1 + \sin 1.42073549240395] = 1.45755824248413.$$

Continuing, we get

$$\begin{aligned} w_2^* &= w_1 + h \sin w_1 = 1.45755824248413 + 0.5 \sin 1.45755824248413 = 1.95435595062392 \\ w_2 &= w_1 + \frac{h}{2}[\sin w_1 + \sin w_2^*] \\ &= 1.45755824248413 + 0.25 * [\sin 1.45755824248413 + \sin 1.95435595062392] \\ &= 1.93779170093176. \end{aligned}$$

This w_2 is the approximation for $y(1)$.

4. (a) Note

$$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) / u_{ii}.$$

In the numerator, $1 + n - i$ real numbers are being added together at one time, which requires $n - i$ additions/subtractions. Thus, adding together all these additions/subtractions for $i = 1, \ldots, n$, we get

$$\sum_{i=1}^{n} n - i = \sum_{k=0}^{n-1} k,$$

where $k = n - i$. This simplifies to $n(n-1)/2$ additions/subtractions.

(b) Note

$$x_i = \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) / u_{ii}.$$

In the numerator, there are $n - i$ multiplications. Then there is 1 division, for a total of $n - i + 1$ multiplications/divisions. Thus, adding together all these multiplications/divisions for $i = 1, \ldots, n$, we get

$$\sum_{i=1}^{n} n - i + 1 = \sum_{m=1}^{n} m,$$

where $m = n - i + 1$. This simplifies to $n(n+1)/2$ multiplications/divisions.

5. A symmetric means $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$. We will prove $b_{ij} = b_{ji}$ for all $i, j = 2, \ldots n$. We get

$$\begin{aligned} b_{ij} &= a_{ij} - \frac{a_{1j}}{a_{11}} a_{i1} \\ &= a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i} \\ &= a_{ji} - \frac{a_{1i}}{a_{11}} a_{j1} \\ &= b_{ji}. \end{aligned}$$

8. (Matlab)
   (a) See "hw8afn.m".
   (b) For $n = 10$, we get 615 flops; for $n = 20$, we get 5130 flops; for $n = 100$, we get 661650 flops; and for $n = 200$, we get 5313300 flops.

9. (Math 274) We first prove the result for the first step of Gaussian elimination. Let B be the matrix after the first step of Gaussian elimination. Note $b_{1j} = a_{1j}$, for all j, so

$$\sum_{j=2}^{n} |b_{1j}| = \sum_{j=2}^{n} |a_{1j}| < |a_{11}| = |b_{11}|.$$

For $i = 2, \ldots, n$, note $b_{i1} = 0$. So

$$\begin{aligned} \sum_{j=1, j \neq i}^{n} |b_{ij}| &= \sum_{j=2, j \neq i}^{n} |b_{ij}| \\ &= \sum_{j=2, j \neq i}^{n} \left| a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right| \\ &\leq \sum_{j=2, j \neq i}^{n} \left[|a_{ij}| + \frac{|a_{i1}|}{|a_{11}|} |a_{1j}| \right] \\ &= \sum_{j=2, j \neq i}^{n} |a_{ij}| + \frac{|a_{i1}|}{|a_{11}|} \sum_{j=2, j \neq i}^{n} |a_{1j}| \\ &= \sum_{j=1, j \neq i}^{n} |a_{ij}| - |a_{i1}| + \frac{|a_{i1}|}{|a_{11}|} \left(\sum_{j=2}^{n} |a_{1j}| - |a_{1i}| \right) \\ &< |a_{ii}| - |a_{i1}| + \frac{|a_{i1}|}{|a_{11}|} (|a_{11}| - |a_{1i}|) \\ &= |a_{ii}| - \frac{|a_{i1}|}{|a_{11}|} |a_{1i}| \\ &\leq \left| a_{ii} - \frac{a_{i1}}{a_{11}} a_{1i} \right| \\ &= |b_{ii}|. \end{aligned}$$

This proves given a matrix that is strictly diagonally dominant, after one step of Gaussian elimination, the resulting matrix is still strictly diagonally dominant. So, by induction, note, for the base case, A is strictly diagonally dominant. Now suppose $A^{(k)}$, the result of k steps of Gaussian elimination on A, is strictly diagonally dominant. Then $A^{(k+1)}$ is also strictly diagonally dominant since it is one step of Gaussian elimination on $A^{(k)}$. Therefore, by induction, $A^{(k)}$ is strictly diagonally dominant for all $k = 0, \ldots, n-1$. Thus, the upper triangular matrix we are interested in, $A^{(n-1)}$, is strictly diagonally dominant.