Homework #9

1. (a) Solve the following linear system using Gaussian elimination with partial pivoting and back substitution under 3-digit rounding

$$\begin{cases} 0.002x - 4y &= -2 \\ x + 4y &= 3. \end{cases}$$

   (b) Solve the following linear system using Gaussian elimination with partial pivoting and back substitution under 3-digit rounding

$$\begin{cases} 2x - 4000y &= -2000 \\ x + 4y &= 3. \end{cases}$$

   (c) Which of the two answers is closer to the exact solution $x = 500/501$, $y = 1003/2004$?

2. (a) Write out each step of Gaussian elimination with partial pivoting on the augmented matrix for the linear system

$$\begin{cases} 2x_1 + x_2 + x_3 - x_4 &= 4 \\ -4x_1 - 2x_2 + x_3 + 2x_4 &= -2 \\ 2x_1 + 2x_2 - x_3 - 2x_4 &= -1 \\ -x_1 + 4x_2 - 2x_3 + x_4 &= 0. \end{cases}$$

   (b) Now use back substitution to solve for x_1, x_2, x_3, x_4.

3. (a) Find the PLU factorization ($PA = LU$) of the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 & -1 \\ -1 & 2 & 0 & 1 \\ 4 & -1 & -2 & -4 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

   using Gaussian elimination with partial pivoting.

   (b) Use the PLU factorization and forward and back substitution to solve the linear system $A\vec{x} = \vec{b}$, with $\vec{b} = [1, 0, 1, 0]^t$.

4. Perform three iterations of Jacobi method with starting guess $\vec{x}^{(0)} = [0, 0, 0]^t$ for the linear system

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & -2 & 1 \\ 2 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix}.$$

5. Perform two iterations of the Gauss-Seidel method with starting guess $\vec{x}^{(0)} = [0, 0, 0]^t$ for the linear system

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & -2 & 1 \\ 2 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix}.$$

6. Consider a linear system $Ax = b$ where A is an $n \times n$ matrix with nonzero diagonal elements. Suppose we know A has only m nonzero elements and we know their locations and we can avoid operations at other locations. Count the number of flops needed for one iteration of Jacobi method (your result should be roughly $2m - n$). How does this result compare to Gaussian elimination with partial pivoting on a regular $n \times n$ matrix (flop count $2n^3/3 + \mathcal{O}(n^2)$) when $n = 10^6$ and $m = 7 \cdot 10^6$?

7. Consider linear systems with the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   (a) Find the spectral radius of the iteration matrix for Jacobi's method and show Jacobi's method converges if and only if $\left|\frac{bc}{ad}\right| < 1$.

   (b) Find the spectral radius of the iteration matrix for Gauss-Seidel and show Gauss-Seidel converges if and only if $\left|\frac{bc}{ad}\right| < 1$.

   (c) When $\left|\frac{bc}{ad}\right| < 1$, which has faster convergence?

8. (Matlab)

   (a) Write a Matlab function that inputs the matrix A, right hand side vector $\vec{b}$, size n, and number of iterations N, and outputs Jacobi method's approximation $x^{(N)}$ using the zero vector as initial guess. Use the formula

$$x_i^{(k+1)} = \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}\right) / a_{ii}$$

   for Jacobi method. Print out or write out your function.

   (b) Apply your method to problem #4, running 20 iterations and print out or write out your answer.

9. (Math 274)

   (a) Prove Jacobi method converges for any initial guess when A is strictly diagonally dominant.

   (b) Prove Gauss-Seidel converges for any initial guess when A is strictly diagonally dominant.