Homework #9

3. (a) Putting the L, U into one matrix, and keeping track of the permuted rows:

$$\begin{bmatrix} -2 & 0 & 1 & -1 \\ -1 & 2 & 0 & 1 \\ 4 & -1 & -2 & -4 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \to \begin{bmatrix} 4 & -1 & -2 & -4 \\ -1 & 2 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \to$$
$$\begin{bmatrix} 4 & -1 & -2 & -4 \\ -1/4 & 7/4 & -1/2 & 0 \\ -1/2 & -1/2 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \to \begin{bmatrix} 4 & -1 & -2 & -4 \\ -1/4 & 7/4 & -1/2 & 0 \\ -1/2 & -2/7 & -1/7 & -3 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} \to$$
$$\begin{bmatrix} 4 & -1 & -2 & -4 \\ -1/4 & 7/4 & -1/2 & 0 \\ -1/2 & -2/7 & -1/7 & -3 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix} \to \begin{bmatrix} 4 & -1 & -2 & -4 \\ -1/4 & 7/4 & -1/2 & 0 \\ 0 & 0 & 2 & 0 \\ -1/2 & -2/7 & -1/7 & -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix} \to$$
$$\begin{bmatrix} 4 & -1 & -2 & -4 \\ -1/4 & 7/4 & -1/2 & 0 \\ 0 & 0 & 2 & 0 \\ -1/2 & -2/7 & -1/14 & -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix}.$$

So

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/2 & -2/7 & -1/14 & 1 \end{bmatrix}, U = \begin{bmatrix} 4 & -1 & -2 & -4 \\ 0 & 7/4 & -1/2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

6. Let m_i denote the number of nonzero elements in row i of A. Now, for Jacobi method to work, every diagonal element of A has to be nonzero, so for row i, we count 1 nonzero on the diagonal and $m_i - 1$ nonzeros off the diagonal. Then, in the formula

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1, j \neq i}^{n} a_{ij} x_j^{(k)} \right) / a_{ii},$$

the summation sign involves elements of A that are off the diagonal, of which there are $m_i - 1$ nonzeros. So, ignoring zeros, we count $m_i - 1$ elements being summed under the summation sign, each of which has one multiplication. This adds up to $m_i - 1$ multiplications under the summation sign. In total, for each i, there are m_i multiplications/divisions. Furthermore, the summations sign adds or subtracts $m_i - 1$ elements, and then subtracts the result from b_i. Thus the numerator has m_i elements being added or subtracted, which requires $m_i - 1$ additions/subtractions. So, for each i, there are $m_i + m_i - 1 = 2m_i - 1$ flops. Thus, there is a total of $\sum_{i=1}^{n}(2m_i - 1) = 2\sum_{i=1}^{n} m_i - n = 2m - n$ flops.

7. (a) Jacobi's method's iteration matrix is

$$T = D^{-1}(L+U) = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ -c/d & 0 \end{bmatrix}.$$

For eigenvalues, the characteristic equation is

$$0 = \det(T - \lambda I) = \det \begin{bmatrix} -\lambda & -b/a \\ -c/d & -\lambda \end{bmatrix} = \lambda^2 - bc/(ad),$$

so the eigenvalues are $\lambda = \pm\sqrt{bc/(ad)}$, which may or may not have imaginary parts. So $|\lambda| = \sqrt{|bc/(ad)|}$, and $\rho(T) = \sqrt{|bc/(ad)|}$. Now

$$\rho(T) < 1 \Leftrightarrow \sqrt{|bc/(ad)|} < 1 \Leftrightarrow |bc/(ad)| < 1,$$

so Jacobi method converges if and only if $|bc/(ad)| < 1$.

(a) Gauss-Seidel's iteration matrix is

$$T = (D-L)^{-1}U = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \frac{1}{ad} \begin{bmatrix} d & 0 \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ 0 & bc/(ad) \end{bmatrix}.$$

For eigenvalues, the characteristic equation is

$$0 = \det(T - \lambda I) = \det \begin{bmatrix} -\lambda & -b/a \\ 0 & bc/(ad) - \lambda \end{bmatrix} = \lambda(\lambda - bc/(ad)),$$

so the eigenvalues are $\lambda = 0, bc/(ad)$. So $|\lambda| = 0, |bc/(ad)|$, and $\rho(T) = |bc/(ad)|$. So

$$\rho(T) < 1 \Leftrightarrow |bc/(ad)| < 1,$$

so Gauss-Seidel converges if and only if $|bc/(ad)| < 1$.

8. (Matlab)

   (a) See "hw9afn.m".

   (b) We get the approximation $[1.000000003884503, 0.999999806026317, 1.999999821080499]^T$.