JOURNAL OF GEOMETRIC MECHANICS
©American Institute of Mathematical Sciences
Volume 4, Number 4, December 2012

doi:10.3934/jgm.2012.4.421

pp. 421–442

DIRAC STRUCTURES AND HAMILTON–JACOBI THEORY FOR LAGRANGIAN MECHANICS ON LIE ALGEBROIDS

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(Communicated by David Martín de Diego)

Abstract. This paper develops the notion of implicit Lagrangian systems on Lie algebroids and a Hamilton–Jacobi theory for this type of system. The Lie algebroid framework provides a natural generalization of classical tangent bundle geometry. We define the notion of an implicit Lagrangian system on a Lie algebroid E using Dirac structures on the Lie algebroid prolongation $\mathcal{T}^E E^*$. This setting includes degenerate Lagrangian systems with nonholonomic constraints on Lie algebroids.

1. **Introduction.** There is a vast literature on the Lagrangian formalism in mechanics, which is due to the central role played by these systems in the foundations of modern mathematics and physics. In many interesting systems, problems often arise due to their singular nature, which gives rise to constraints that address the fact that the evolution problem is not well-posed (*internal constraints*). Constraints can also manifest a priori restrictions on the states of the system which arise due to physical arguments or from external conditions (*external constraints*). Both cases are of considerable importance.

Systems with *internal constraints* are quite interesting since many dynamical systems are given in terms of presymplectic forms instead of the more habitual symplectic ones. The more frequent case appears in the Lagrangian formalism of singular mechanical systems which are commonplace in many physical theories (as in Yang-Mills theories, gravitation, etc).

2000 *Mathematics Subject Classification.* Primary: 37J60, 53D17, 70H20; Secondary: 17B66, 70F25, 70G45, 70H45.

Key words and phrases. Dirac structures, implicit Lagrangian systems, Lie algebroids, Lagrangian mechanics, nonholonomic systems, Hamilton–Jacobi equation.

The first author is supported by NSF grant DMS-0726263, DMS-1010687, DMS-1065972. The second author is supported by MICINN (Spain) grants MTM2009-13383, MTM2009-08166-E, and Canary government projects SOLSUBC200801000238 and ProID20100210.

Systems subjected to *external constraints* (holonomic and nonholonomic) have a wide range of applications in many different areas: engineering, optimal control theory, mathematical economics (growth economic theory), subriemannian geometry, motion of microorganisms, etc. Interconnected and implicit systems play a key role in, for example, controlled mechanical systems like robots. An important class of implicit mechanical systems is those with nonholonomic constraints, which has a long and rich history (see, for instance, [4] and [35]). The Lagrangian and Hamiltonian approaches for such systems have been extensively developed (see [23, 38, 40, 41]), including symmetry and reduction (see [3, 6, 22, 24, 29]).

Some authors have given descriptions of L-C circuits and nonholonomic systems in the context of Poisson structures (see [33, 38]) and later in the general context of Dirac structures (see [5, 39]) from a Hamiltonian point of view. Inspired by these works, Yoshimura and Marsden in [43, 44] have developed a Lagrangian formalism making use of the framework of Dirac structures.

Recent investigations have lead to a unifying geometric framework covering a plethora of particular situations. It is precisely the underlying structure of a Lie algebroid on the phase space which allows a unified treatment. This idea was first introduced by Weinstein [42] in order to define a Lagrangian formalism which is general enough to account for different types of systems. The geometry and dynamics on Lie algebroids have been extensively studied during the past years. In particular, in [30], E. Martínez developed a geometric formalism of mechanics on Lie algebroids similar to Klein's formalism of ordinary Lagrangian mechanics and, more recently, a description of the Hamiltonian dynamics on a Lie algebroid was given in [27, 31]. The key concept in this theory is the prolongation, $\mathcal{T}^E E$, of the Lie algebroid over the fiber projection τ (for the Lagrangian formalism) and the prolongation, $\mathcal{T}^E E^*$, over the dual fiber projection $\tau^* : E^* \to Q$ (for the Hamiltonian formalism). See [27] for more details. Of course, when the Lie algebroid is $E = TQ$ we obtain that $\mathcal{T}^E E = T(TQ)$ and $\mathcal{T}^E E^* = T(T^*Q)$, recovering the classical case. Another approach to the theory was discussed in [17].

The notion of nonholonomic systems on a Lie algebroid was introduced in [9] when studying mechanical control systems and an approach to mechanical systems on Lie algebroids subject to linear constraints was presented in [34]. A recent comprehensive treatment of nonholonomic systems on a Lie algebroid has been develop in [10], where the authors identify suitable conditions guaranteeing that the system admits a unique solution and show that many of the properties that standard nonholonomic systems enjoy have counterparts in the Lie algebroid setting.

On the other hand, singular or degenerate Lagrangian systems and vakonomic mechanics on Lie algebroids (obtained through the application of a constrained variational principle) also have been studied. In [21], the authors introduce a constraint algorithm for presymplectic Lie algebroids which generalizes the well-known Gotay-Nester-Hinds algorithm (see [15]) and applies it to singular Lagrangian systems on Lie algebroids. Moreover, they develop a geometric description of vakonomic mechanics on Lie algebroids using again the constraint algorithm.

As a consequence of all these investigations, one deduces that there are several reasons for discussing unconstrained (constrained) Mechanics on Lie algebroids:

i) The inclusive nature of the Lie algebroid framework. In fact, under the same umbrella, one can consider standard unconstrained (constrained) mechanical systems, (nonholonomic and vakonomic) Lagrangian systems on Lie algebras, unconstrained (constrained) systems evolving on semidirect products or (nonholonomic and vakonomic) Lagrangian systems with symmetries.

ii) The reduction of a (nonholonomic or vakonomic) mechanical system on a Lie algebroid is a (nonholonomic or vakonomic) mechanical system on a Lie algebroid. However, the reduction of a standard unconstrained (constrained) system on the tangent (cotangent) bundle of the configuration manifold is not, in general, a standard unconstrained (constrained) system.

iii) The theory of Lie algebroids gives a natural interpretation of the use of quasi-coordinates (velocities) in Mechanics (particularly, in nonholonomic and vakonomic mechanics).

On the other hand, Hamilton–Jacobi theory has been studied for different type of systems for many years. For degenerate Lagrangian systems, some work have been done on extending Hamilton–Jacobi theory, using Dirac's theory of constraints (see, e.g., [18]) and from a geometric point of view (see [7]). For nonholonomic systems, in [20], Iglesias-Ponte, de León and Martín de Diego generalized the geometric Hamilton–Jacobi theorem (see Theorem 5.2.4. in [1]) to nonholonomic systems, which has been studied further (see [8, 36, 37]). More recently, in [25], the authors have presented a Hamilton–Jacobi theory which can deal with both degeneracy and nonholonomic constraints. In the context of Lie algebroids, de León, Marrero and Martín de Diego have developed a more general formalism which is also valid for for nonholonomic systems on a Lie algebroid (see [26]), and, in [2], the authors have presented a Hamilton–Jacobi equation for a Hamiltonian system on a skew-symmetric algebroid.

The goal of this paper is to generalize Hamilton–Jacobi theory to implicit Lagrangian systems on a Lie algebroid based on Dirac structures. We introduce the notion of an implicit Lagrangian system on a Lie algebroid E using the induced generalized Dirac structure $\mathcal{D}_{\mathcal{U}}$ on the Lie algebroid prolongation $\mathcal{T}^E E^*$ that is naturally induced by a vector subbundle $\mathcal{U}$ of E and we obtain the Hamilton–Jacobi theorem for this kind of systems. This setting includes degenerate Lagrangian systems with nonholonomic constraints.

The paper is organized as follows. In Section 2, we collect some preliminary notions and geometric objects on Lie algebroids, including differential calculus, morphism and prolongations. We also recall the definition and some properties of (generalized) Dirac structures on vector spaces, vector bundles and manifolds. In Section 3, first we introduce and study the generalized Dirac structure $\mathcal{D}_{\mathcal{U}}$ on $\mathcal{T}^E E^*$ induced by a vector subbundle $\mathcal{U}$ of the Lie algebroid E. The main goal of this section is to define implicit Lagrangian systems in terms of induced Dirac structures. In Section 4, we develop a Hamilton–Jacobi theory for implicit Lagrangian systems on a Lie algebroid. We apply the results obtained to some particular cases, in Section 5, recovering some known results. The paper ends with our conclusions and a description of future research directions.

2. **Preliminaries.**

2.1. **Lie algebroids.**

Let E be a vector bundle of rank n over a manifold Q of dimension m and $\tau : E \to Q$ be the vector bundle projection. Denote by $\Gamma(E)$ the $C^\infty(Q)$-module of sections of $\tau : E \to Q$. A *Lie algebroid structure* $(\llbracket\cdot,\cdot\rrbracket, \rho)$ on E

is a Lie bracket $[\![\cdot,\cdot]\!]$ on the space $\Gamma(E)$ and a bundle map $\rho : E \to TQ$, called *the anchor map*, such that if we also denote by $\rho : \Gamma(E) \to \mathfrak{X}(Q)$ the homomorphism of $C^\infty(Q)$-modules induced by the anchor map, then

$$[\![X, fY]\!] = f[\![X, Y]\!] + \rho(X)(f)Y,$$

for $X, Y \in \Gamma(E)$ and $f \in C^\infty(Q)$. The triple $(E, [\![\cdot,\cdot]\!], \rho)$ is called *a Lie algebroid over Q* (see [28]).

If $(E, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid over Q, then the anchor map $\rho : \Gamma(E) \to \mathfrak{X}(Q)$ is a homomorphism between the Lie algebras $(\Gamma(E), [\![\cdot,\cdot]\!])$ and $(\mathfrak{X}(Q), [\cdot,\cdot])$.

Standard examples of Lie algebroids are real Lie algebras of finite dimension and the tangent bundle TQ of an arbitrary manifold Q. In more detail, let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Then, consider the vector bundle $\tau : \mathfrak{g} \to \{$ one point $\}$. The section of this vector bundle can be identified with the elements of $\mathfrak{g}$ and, therefore, we can consider the Lie bracket given by the Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}}$ on $\mathfrak{g}$ and the anchor map ρ given by the null map. So, $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, 0)$ is a Lie algebroid over a point. On the other hand, let Q a manifold. The sections of the tangent bundle $\tau_E = \tau_Q : E = TQ \to Q$ may be identified with the vector fields on Q, the Lie bracket on $\Gamma(\tau_E) = \mathfrak{X}(Q)$ is the usual vector fields bracket and the anchor map is the identity on TQ. Then, the triple $(TQ, [\cdot,\cdot], Id)$ is a Lie algebroid over Q.

Another example of a Lie algebroid may be constructed as follows. Let $\pi : P \to Q$ be a principal bundle with structure group G. Denote by $\Phi : G \times P \to P$ the free action of G on P and by $T\Phi : G \times TP \to TP$ the tangent lifted action of G on TP. Then, one may consider the quotient vector bundle $\tau_P|G : TP/G \to Q = P/G$ and the sections of this vector bundle may be identified with the vector fields on P which are invariant under the action Φ. Using the fact that every G-invariant vector field on P is π-projectable and the fact that the standard Lie bracket on vector fields is closed with respect to G-invariant vector fields, we can induce a Lie algebroid structure on TP/G. The resultant Lie algebroid is called *the Atiyah (gauge) algebroid associated with the principal bundle $\pi : P \to Q$* (see [27, 28]).

Now, let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid, then one may define *the differential* of E, $d^E : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^{k+1} E^*)$, as follows

$$\begin{aligned} d^E\mu(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i \rho(X_i)(\mu(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \mu([\![X_i, X_j]\!], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k), \end{aligned}$$

for $\mu \in \Gamma(\wedge^k E^*)$ and $X_0, \dots, X_k \in \Gamma(E)$. It follows that $(d^E)^2 = 0$. Moreover, if $X \in \Gamma(E)$, one may introduce, in a natural way, *the Lie derivative with respect to X*, as the operator $\pounds^E_X : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^k E^*)$ given by $\pounds^E_X = i_X \circ d^E + d^E \circ i_X$.

Note that if $E = TQ$ and $X \in \Gamma(E) = \mathfrak{X}(Q)$ then d^{TQ} and $\pounds^{TQ}_X$ are the usual differential and the usual Lie derivative with respect to X, respectively.

If we take local coordinates (x^i) on an open subset U of Q and a local basis $\{e_\alpha\}$ of sections of E defined on U, then we have the corresponding local coordinates (x^i, y^α) on E, where $y^\alpha(e)$ is the α-th coordinate of $e \in E$ in the given basis. Such coordinates determine local functions ρ^i_α, $\mathcal{C}^\gamma_{\alpha\beta}$ on Q which contain local information about the Lie algebroid structure and, accordingly, they are called *the structure* *functions of the Lie algebroid.* They are given by

$$[\![e_\alpha, e_\beta]\!] = \mathcal{C}^\gamma_{\alpha\beta} e_\gamma \quad \text{and} \quad \rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}.$$

These functions should satisfy the relations

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma \mathcal{C}^\gamma_{\alpha\beta},$$

$$\sum_{cyclic(\alpha,\beta,\gamma)} \Big(\rho^i_\alpha \frac{\partial \mathcal{C}^\delta_{\beta\gamma}}{\partial x^i} + \mathcal{C}^\delta_{\alpha\nu} \mathcal{C}^\nu_{\beta\gamma} \Big) = 0,$$

which are usually called *the structure equations.*

If $f \in C^\infty(Q)$, we have that

$$d^E f = \frac{\partial f}{\partial x^i} \rho^i_\alpha e^\alpha, \tag{1}$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$. On the other hand, if $\theta \in \Gamma(E^*)$ and $\theta = \theta_\gamma e^\gamma$, it follows that

$$d^E \theta = \Big(\frac{\partial \theta_\gamma}{\partial x^i} \rho^i_\beta - \frac{1}{2} \theta_\alpha \mathcal{C}^\alpha_{\beta\gamma} \Big) e^\beta \wedge e^\gamma.$$

In particular,

$$d^E x^i = \rho^i_\alpha e^\alpha, \qquad d^E e^\alpha = -\frac{1}{2} \mathcal{C}^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma.$$

2.2. **Morphisms.** Let $(E, [\![\cdot,\cdot]\!], \rho)$ and $(E', [\![\cdot,\cdot]\!]', \rho')$ be Lie algebroids over Q and Q', respectively. A morphism of vector bundles (F, f) from E to E'

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\tau"'] & E' \arrow[d, "\tau'"] \\ Q \arrow[r, "f"] & Q' \end{tikzcd}$$

is a *Lie algebroid morphism* if

$$d^E((F,f)^*\phi') = (F,f)^*(d^{E'}\phi'), \quad \text{for } \phi' \in \Gamma(\wedge^k (E')^*). \tag{2}$$

Note that $(F,f)^*\phi'$ is the section of the vector bundle $\wedge^k E^* \to Q$ defined by

$$((F,f)^*\phi')_x(a_1, \ldots, a_k) = \phi'_{f(x)}(F(a_1), \ldots, F(a_k)),$$

for $x \in Q$ and $a_1, \ldots, a_k \in E_x$, where E_x denotes the fiber of E at the point $x \in Q$. We remark that (2) holds if and only if

$$d^E(g' \circ f) = (F,f)^*(d^{E'} g'), \qquad \text{for } g' \in C^\infty(Q'),$$

$$d^E((F,f)^*\alpha') = (F,f)^*(d^{E'}\alpha'), \qquad \text{for } \alpha' \in \Gamma((E')^*).$$

If (F,f) is a Lie algebroid morphism, f is an injective immersion and $F_{|E_x} : E_x \to E'_{f(x)}$ is injective, for all $x \in Q$, then $(E, [\![\cdot,\cdot]\!], \rho)$ is said to be a *Lie subalgebroid* of $(E', [\![\cdot,\cdot]\!]', \rho')$.

If $Q = Q'$ and $f = id : Q \to Q$ then, it is easy prove that the pair (F, id) is a Lie algebroid morphism if and only if

$$F[\![X,Y]\!] = [\![FX, FY]\!]', \qquad \rho'(FX) = \rho(X),$$

for $X, Y \in \Gamma(E)$.

2.3. **Poisson structure on E^*.** Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over Q and E^* be the dual bundle to E. Then, E^* admits a linear Poisson structure Π_{E^*}, that is, Π_{E^*} is a 2-vector on E^* such that

$$[\Pi_{E^*}, \Pi_{E^*}] = 0,$$

and if f and f' are linear functions on E^*, we have that $\Pi_{E^*}(d^{TE^*}f, d^{TE^*}f')$ is also a linear function on E^*. If (x^i) are local coordinates on Q, $\{e_\alpha\}$ is a local basis of $\Gamma(E)$ and (x^i, p_α) are the corresponding local coordinates on E^*, then the local expression for Π_{E^*} is

$$\Pi_{E^*} = \rho^i_\alpha \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_\alpha} - \frac{1}{2}\mathcal{C}^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\alpha} \wedge \frac{\partial}{\partial p_\beta},$$

where ρ^i_α and $\mathcal{C}^\gamma_{\alpha\beta}$ are the structure functions of E with respect to the coordinates (x^i) and to the basis $\{e_\alpha\}$. The Poisson structure Π_{E^*} induces a linear Poisson bracket of functions on E^* which we will denote by $\{\ ,\ \}_{E^*}$. In fact, if $F, G \in C^\infty(E^*)$ then

$$\{F, G\}_{E^*} = \Pi_{E^*}(d^{TE^*}F, d^{TE^*}G). \tag{3}$$

(For more details, see [27]).

2.4. **The prolongation of a Lie algebroid over a fibration.** Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold Q of dimension m and $\pi : P \to Q$ be a fibration, that is, a surjective submersion.

We consider the subset $\mathcal{T}^E P$ of $E \times TP$ defined by $\mathcal{T}^E P = \bigcup_{p\in P} \mathcal{T}^E_p P$, where

$$\mathcal{T}^E_p P = \{(b, v) \in E_{\pi(p)} \times T_p P \,|\, \rho(b) = (T_p\pi)(v)\},$$

and $T\pi : TP \to TQ$ is the tangent map to π.

Denote by $\tau^\pi : \mathcal{T}^E P \to P$ the map given by

$$\tau^\pi(b, v) = \tau_P(v),$$

for $(b, v) \in \mathcal{T}^E P$, where $\tau_P : TP \to P$ is the canonical projection. Then, if m' is the dimension of P, one may prove that

$$\dim \mathcal{T}^E_p P = n + m' - m.$$

Thus, we conclude that $\mathcal{T}^E P$ is a vector bundle over P of rank $n + m' - m$ with the vector bundle projection $\tau^\pi : \mathcal{T}^E P \to P$.

A section $\tilde{X}$ of $\tau^\pi : \mathcal{T}^E P \to P$ is said to be *projectable* if there exists a section X of $\tau : E \to Q$ and a vector field U on P which is π-projectable to the vector field $\rho(X)$ and such that $\tilde{X}(p) = (X(\pi(p)), U(p))$, for all $p \in P$. For such a projectable section $\tilde{X}$, we will use the following notation $\tilde{X} \equiv (X, U)$. It is easy to prove that one may choose a local basis of projectable sections of the space $\Gamma(\mathcal{T}^E P)$.

The vector bundle $\tau^\pi : \mathcal{T}^E P \to P$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]^\pi, \rho^\pi)$. In fact,

$$[\![(X_1, U_1), (X_2, U_2)]\!]^\pi = ([\![X_1, X_2]\!], [U_1, U_2]), \qquad \rho^\pi(X_1, U_1) = U_1.$$

The Lie algebroid $(\mathcal{T}^E P, [\![\cdot,\cdot]\!]^\pi, \rho^\pi)$ is called *the prolongation of E over π or the E-tangent bundle to P*. Note that if $\mathrm{pr}_1 : \mathcal{T}^E P \to E$ is the canonical projection on the first factor, then the pair (pr_1, π) is a morphism between the Lie algebroids $(\mathcal{T}^E P, [\![\cdot,\cdot]\!]^\pi, \rho^\pi)$ and $(E, [\![\cdot,\cdot]\!], \rho)$ (for more details, see [27]).

Example 2.1. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold Q of dimension m and $\tau : E \to Q$ be the vector bundle projection. Consider the prolongation $\mathcal{T}^E E$ of E over τ,

$$\mathcal{T}^E E = \{(e, v) \in E \times TE \mid \rho(e) = (T\tau)(v)\}.$$

$\mathcal{T}^E E$ is a Lie algebroid over E of rank $2n$ with Lie algebroid structure $([\![\cdot,\cdot]\!]^\tau, \rho^\tau)$.

If (x^i) are local coordinates on an open subset U of Q and $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \to U$, then $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ is a basis of sections of the vector bundle $(\tau^\tau)^{-1}(\tau^{-1}(U)) \to \tau^{-1}(U)$, where $\tau^\tau : \mathcal{T}^E E \to E$ is the vector bundle projection and

$$\mathcal{X}_\alpha(e) = \Big(e_\alpha(\tau(e)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|e}\Big), \qquad \mathcal{V}_\alpha(e) = \Big(0, \frac{\partial}{\partial y^\alpha}_{|e}\Big), \tag{4}$$

for $e \in \tau^{-1}(U)$. Here, ρ^i_α are the components of the anchor map with respect to the basis $\{e_\alpha\}$ and (x^i, y^α) are the local coordinates on E induced by the local coordinates (x^i) and the basis $\{e_\alpha\}$. Using the local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, one may introduce, in a natural way, local coordinates $(x^i, y^\alpha; s^\alpha, w^\alpha)$ on $\mathcal{T}^E E$. If ω is a point of $(\tau^\tau)^{-1}(\tau^{-1}(U))$, then (x^i, y^α) are the coordinates of the point $\tau^\tau(\omega) \in \tau^{-1}(U)$ and

$$\omega = s^\alpha \mathcal{X}_\alpha(\tau^\tau(\omega)) + w^\alpha \mathcal{V}_\alpha(\tau^\tau(\omega)).$$

On the other hand, we have that

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^\tau = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \qquad [\![\mathcal{X}_\alpha, \mathcal{V}_\beta]\!]^\tau = [\![\mathcal{V}_\alpha, \mathcal{V}_\beta]\!]^\tau = 0,$$
$$\rho^\tau(\mathcal{X}_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}, \qquad \rho^\tau(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha},$$

for all α and β, where $\mathcal{C}^\gamma_{\alpha\beta}$ are the structure functions of the Lie bracket $[\![\cdot,\cdot]\!]$ with respect to the basis $\{e_\alpha\}$.

The vector subbundle $(\mathcal{T}^E E)^V$ of $\mathcal{T}^E E$ whose fiber at the point $e \in E$ is

$$(\mathcal{T}^E_e E)^V = \{(0, v) \in E \times T_e E \mid (T_e\tau)(v) = 0\}$$

is called *the vertical subbundle*. Note that $(\mathcal{T}^E E)^V$ is locally generated by the sections $\{\mathcal{V}_\alpha\}$.

Two canonical objects on $\mathcal{T}^E E$ are *the Euler section* Δ and *the vertical endomorphism* S. Δ is the section of $\mathcal{T}^E E \to E$ locally defined by

$$\Delta = y^\alpha \mathcal{V}_\alpha,$$

and S is the section of the vector bundle $(\mathcal{T}^E E) \otimes (\mathcal{T}^E E)^* \to E$ locally characterized by the following conditions

$$S(\mathcal{X}_\alpha) = \mathcal{V}_\alpha, \quad S(\mathcal{V}_\alpha) = 0, \quad \text{for all } \alpha. \tag{5}$$

Finally, a section ξ of $\mathcal{T}^E E \to E$ is said to be a *second-order differential equation* (SODE) on E if $S(\xi) = \Delta$ or, alternatively, $\mathrm{pr}_1(\xi(e)) = e$, for all $e \in E$ (for more details, see [27]).

Example 2.2. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank n over a manifold Q of dimension m and $\tau^* : E^* \to Q$ be the vector bundle projection of the dual bundle E^* to E.

We consider the prolongation $\mathcal{T}^E E^*$ of E over τ^*,

$$\mathcal{T}^E E^* = \{(e', v) \in E \times TE^* \mid \rho(e') = (T\tau^*)(v)\}.$$

$\mathcal{T}^E E^*$ is a Lie algebroid over E^* of rank $2n$ with Lie algebroid structure $([\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$.

If (x^i) are local coordinates on an open subset U of Q, $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau^{-1}(U) \to U$ and $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$, then $\{\mathcal{Y}_\alpha, \mathcal{P}^\alpha\}$ is a basis of sections of the vector bundle $(\tau^{\tau^*})^{-1}((\tau^*)^{-1}(U)) \to (\tau^*)^{-1}(U)$, where $\tau^{\tau^*} : \mathcal{T}^E E^* \to E^*$ is the vector bundle projection and

$$\mathcal{Y}_\alpha(e^*) = \Big(e_\alpha(\tau^*(e^*)), \rho^i_\alpha \frac{\partial}{\partial x^i}_{|e^*}\Big), \qquad \mathcal{P}^\alpha(e^*) = \Big(0, \frac{\partial}{\partial p_\alpha}_{|e^*}\Big), \tag{6}$$

for $e^* \in (\tau^*)^{-1}(U)$. Here, (x^i, p_α) are the local coordinates on E^* induced by the local coordinates (x^i) and the basis $\{e^\alpha\}$ of $\Gamma(E^*)$. Using the local basis $\{\mathcal{Y}_\alpha, \mathcal{P}^\alpha\}$, one may introduce, in a natural way, local coordinates $(x^i, p_\alpha; z^\alpha, u_\alpha)$ on $\mathcal{T}^E E^*$. If ω^* is a point of $(\tau^{\tau^*})^{-1}((\tau^*)^{-1}(U))$, then (x^i, p_α) are the coordinates of the point $\tau^{\tau^*}(\omega^*) \in (\tau^*)^{-1}(U)$ and

$$\omega^* = z^\alpha \mathcal{Y}_\alpha(\tau^{\tau^*}(\omega^*)) + u_\alpha \mathcal{P}^\alpha(\tau^{\tau^*}(\omega^*)).$$

On the other hand, we have that

$$\begin{aligned} [\![\mathcal{Y}_\alpha, \mathcal{Y}_\beta]\!]^{\tau^*} &= \mathcal{C}^\gamma_{\alpha\beta} \mathcal{Y}_\gamma, \qquad & [\![\mathcal{Y}_\alpha, \mathcal{P}^\beta]\!]^{\tau^*} &= [\![\mathcal{P}^\alpha, \mathcal{P}^\beta]\!]^{\tau^*} = 0, \\ \rho^{\tau^*}(\mathcal{Y}_\alpha) &= \rho^i_\alpha \frac{\partial}{\partial x^i}, & \rho^{\tau^*}(\mathcal{P}^\alpha) &= \frac{\partial}{\partial p_\alpha}, \end{aligned} \tag{7}$$

for all α and β. Thus, if $\{\mathcal{Y}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{Y}_\alpha, \mathcal{P}^\alpha\}$, then

$$\begin{aligned} d^{\mathcal{T}^E E^*} f &= \rho^i_\alpha \frac{\partial f}{\partial x^i} \mathcal{Y}^\alpha + \frac{\partial f}{\partial p_\alpha} \mathcal{P}_\alpha, \\ d^{\mathcal{T}^E E^*} \mathcal{Y}^\gamma &= -\frac{1}{2} \mathcal{C}^\gamma_{\alpha\beta} \mathcal{Y}^\alpha \wedge \mathcal{Y}^\beta, \\ d^{\mathcal{T}^E E^*} \mathcal{P}_\gamma &= 0, \end{aligned}$$

for $f \in C^\infty(E^*)$.

We may introduce a canonical section λ_E of the vector bundle $(\mathcal{T}^E E^*)^* \to E^*$ as follows. If $e^* \in E^*$ and $(\tilde{e}, v)$ is a point of the fiber of $\mathcal{T}^E E^*$ over e^*, then

$$\lambda_E(e^*)(\tilde{e}, v) = \langle e^*, \tilde{e} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between E^* and E. λ_E is called *the Liouville section* of $(\mathcal{T}^E E^*)^*$.

Now, *the canonical symplectic section* Ω_E is the nondegenerate closed 2-section defined by

$$\Omega_E = -d^{\mathcal{T}^E E^*} \lambda_E.$$

Then, we have that the map $\Omega_E^\flat : \mathcal{T}^E E^* \to (\mathcal{T}^E E^*)^*$ defined as

$$\Omega_E^\flat(X) = i_X \Omega_E, \tag{8}$$

for all $X \in \mathcal{T}^E E^*$, where i_X denote the contraction by X, is a vector bundles isomorphism.

In local coordinates,

$$\begin{aligned} \lambda_E(x^i, p_\alpha) &= p_\alpha \mathcal{Y}^\alpha, \\ \Omega_E(x^i, p_\alpha) &= \mathcal{Y}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} \mathcal{C}^\gamma_{\alpha\beta} p_\gamma \mathcal{Y}^\alpha \wedge \mathcal{Y}^\beta. \end{aligned} \tag{9}$$

Remark 1. The linear Poisson bracket $\{\cdot, \cdot\}_{E^*}$ on E^* induced by the Lie algebroid structure on E (see (3)) can be also defined in terms of the canonical symplectic 2-section Ω_E. In fact, for $F, G \in C^\infty(E^*)$, we have that

$$\{F, G\}_{E^*} = \Omega_E((\Omega_E^\flat)^{-1}(d^{\mathcal{T}^E E^*} F), (\Omega_E^\flat)^{-1}(d^{\mathcal{T}^E E^*} G)).$$

2.5. **Dirac structures.** In this section we briefly recall the definition and some properties of Dirac structures on vector spaces, vector bundles and manifolds (see [11, 12]). The construction of a Dirac structure will be reviewed, which will be important for defining implicit Lagrangian systems.

Let V be an n-dimensional vector space, V^* be its dual space, and let $\langle\cdot,\cdot\rangle$ be the natural pairing between V^* and V. A *Dirac structure* on V is a subspace $\mathcal{D} \subset V \oplus V^*$ such that $\mathcal{D} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ is the orthogonal complement of $\mathcal{D}$, that is,

$$\mathcal{D}^\perp = \{(u,\beta) \in V \oplus V^* \,|\, \langle\beta, v\rangle + \langle\alpha, u\rangle = 0, \text{ for all } (v,\alpha) \in \mathcal{D}\}.$$

It is easy to prove that a vector subspace $\mathcal{D} \subset V \oplus V^*$ is a Dirac structure on V if and only if $\dim \mathcal{D} = n$ and $\langle\alpha, \bar{v}\rangle + \langle\bar{\alpha}, v\rangle = 0$, for all $(v,\alpha), (\bar{v},\bar{\alpha}) \in \mathcal{D}$. From the definition of a Dirac structure, for each $(v,\alpha) \in \mathcal{D}$, we have that $\langle\alpha, v\rangle = 0$.

If V is a vector bundle over a manifold Q, let $V \oplus_Q V^*$ be the Whitney sum bundle over Q, that is, it is the bundle over the base Q and with fiber over the point $x \in Q$ equal to $V_x \times V_x^*$, where V_x (respectively, V_x^*) is the fiber of V (respectively, V^*) at the point x. A *Dirac structure* on V is a subbundle $\mathcal{D} \subset V \oplus_Q V^*$ that is a Dirac structure in the sense of vector spaces at each point $x \in Q$.

Now, let M be a smooth differentiable manifold and $\tau_M : TM \to M$ its tangent bundle. An *almost* (in the terminology of [43]) *or generalized* (in the terminology of [13]) *Dirac structure* on M is a subbundle $\mathcal{D} \subset TM \oplus_M T^*M$ which is a Dirac structure in the sense of vector bundles.

In geometric mechanics, almost Dirac structures provide a simultaneous generalization of both 2-forms (not necessarily closed and possibly degenerate) as well as almost Poisson structures (that is, bracket that need not satisfy the Jacobi identity). A *Dirac structure* on M is an almost Dirac structure that additionally satisfies the following integrability condition

$$\langle \pounds_{X_1}\alpha_2, X_3\rangle + \langle \pounds_{X_2}\alpha_3, X_1\rangle + \langle \pounds_{X_3}\alpha_1, X_2\rangle = 0,$$

for all $(X_1,\alpha_1), (X_2,\alpha_2), (X_3,\alpha_3) \in \mathcal{D}$, and where $\pounds_X$ denotes the usual Lie derivative with respect to the vector field X. This generalizes closedness for the symplectic form, and the Jacobi identity for Poisson structures. For the remainder of this paper, we will primarily be concerned with almost Dirac structures, since it allows one to incorporate nonholonomic constraints.

Two constructions of almost Dirac structures on a manifold are given as follows. The first construction is induced by a distribution and a 2-form on the manifold. Let M be a manifold, Ω be a 2-form on M and Δ_M be a distribution on M. Denote by $\Omega^\flat$ the associated flat map and by $\Delta_M^\circ \subset T^*M$ the annihilator of Δ_M. Then, from Theorem 2.3 in [43], we have that $\mathcal{D}_M \subset TM \oplus_M T^*M$ defined, for each $x \in M$, by

$$\begin{aligned}\mathcal{D}_M(x) =&\{(v_x,\alpha_x) \in T_xM \times T_x^*M \,|\, v_x \in \Delta_M(x) \text{ and} \\ &\alpha_x - \Omega^\flat(x)(v_x) \in \Delta_M^\circ(x)\}\end{aligned}$$

is an almost Dirac structure on M (see also Theorem 3.2 in [13]).

For the case when $M = T^*Q$ and $\Omega = \Omega_{T^*Q}$ is the canonical symplectic 2-form, this almost Dirac structure was used to introduce the notion of implicit Lagrangian systems in standard mechanics (see [43, 44]).

The second almost Dirac structure is induced by a codistribution and a skew-symmetric 2-tensor on the manifold. Let M a manifold, $\Pi : T^*M \times T^*M \to \mathbb{R}$ be a skew-symmetric 2-tensor and Δ_M^* a codistribution on M. Denote by $\sharp_\Pi : T^*M \to TM$ the associated sharp map and by $\ker \Delta_M^*$ the distribution on M defined as

$$\ker \Delta_M^* = \{X \in TM \mid \alpha(X) = 0, \text{ for all } \alpha \in \Delta_M^*\}.$$

Then, from Theorem 2.4 in [43], we have that $\mathcal{D}_M \subset TM \oplus_M T^*M$ defined, for each $x \in M$, by

$$\begin{aligned} \mathcal{D}_M(x) =& \{(v_x, \alpha_x) \in T_xM \times T_x^*M \mid \alpha_x \in \Delta_M^*(x) \text{ and} \\ & v_x - \sharp_\Pi(x)(\alpha_x) \in \ker \Delta_M^\circ(x)\} \end{aligned}$$

is an almost Dirac structure on M (see also Theorem 3.1 in [13]).

For the case when $M = T^*Q$ and $\Pi = \Pi_{T^*Q}$ is the canonical Poisson structure on T^*Q, this almost Dirac structure coincides with the almost Dirac structure described before which was used to introduce the notion of implicit Lagrangian systems in standard mechanics (see [43, 44]).

3. Implicit Lagrangian systems on a Lie algebroid.

3.1. Induced almost Dirac structure.

First, we introduce the notion of an *induced almost Dirac structure* on the Lie algebroid prolongation $\mathcal{T}^E E^*$ of a Lie algebroid $\tau : E \to Q$. This almost Dirac structure is induced by a vector subbundle $\mathcal{U}$ of E, that is, $\mathcal{U} \subset E$ such that $\tau_{\mathcal{U}} = \tau_{|\mathcal{U}} : \mathcal{U} \to Q$ is a vector bundle.

Consider the dual vector bundle $\tau^* : E^* \to Q$ of $\tau : E \to Q$. We can define its prolongation to the corresponding prolongation Lie algebroids $\mathcal{T}\tau^* : \mathcal{T}^E E^* \to \mathcal{T}^E Q$ as the identity in the first component and the tangent map of τ^* in the second, that is, $\mathcal{T}\tau^* = (id, T\tau^*)$. It is easy to prove that it is a Lie algebroid morphism between $\mathcal{T}^E E^* \to E^*$ and $\mathcal{T}^E Q \to Q$ (see [32] for a general definition of the prolongation of a map). Moreover, we can identify $\mathcal{T}^E Q$ with E and then $\mathcal{T}\tau^* \equiv \mathrm{pr}_1$, $\mathrm{pr}_1 : \mathcal{T}^E E^* \to E$ being the projection on the first factor.

The vector subbundle $\mathcal{U}$ can be lifted to a vector subbundle $\mathcal{U}_{\mathcal{T}^E E^*} \subset \mathcal{T}^E E^*$ as follows

$$\mathcal{U}_{\mathcal{T}^E E^*} = (\mathrm{pr}_1)^{-1}(\mathcal{U}). \tag{10}$$

Denote by $\mathcal{U}^\circ_{\mathcal{T}^E E^*} \subset (\mathcal{T}^E E^*)^*$ its annihilator. Then, we have the following result.

Theorem 3.1. *Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold Q and $\mathcal{U}$ be a vector subbundle of E. For each $e^* \in E^*$, let*

$$\begin{aligned} \mathcal{D}_{\mathcal{U}}(e^*) =& \{(X_{e^*}, \alpha_{e^*}) \in \mathcal{T}^E_{e^*} E^* \times (\mathcal{T}^E_{e^*} E^*)^* \mid X_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*) \text{ and} \\ & \alpha_{e^*} - \Omega_E^\flat(e^*)(X_{e^*}) \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)\}. \end{aligned} \tag{11}$$

Then, $\mathcal{D}_{\mathcal{U}} \subset \mathcal{T}^E E^* \oplus_{E^*} (\mathcal{T}^E E^*)^*$ is an almost Dirac structure on $\mathcal{T}^E E^*$.

Proof. First, it is not difficult to prove that, since $\mathcal{U}_{\mathcal{T}^E E^*}$ is a vector subbundle of $\mathcal{T}^E E^*$, $\mathcal{D}_{\mathcal{U}}$ is a vector subbundle of $\mathcal{T}^E E^* \oplus_{E^*} (\mathcal{T}^E E^*)^*$.

Second, the orthogonal of $\mathcal{D}_{\mathcal{U}} \subset \mathcal{T}^E E^* \oplus_{E^*} (\mathcal{T}^E E^*)^*$ is given at $e^* \in E^*$ by

$$\begin{aligned} \mathcal{D}^\perp_{\mathcal{U}}(e^*) =& \{(Y_{e^*}, \beta_{e^*}) \in \mathcal{T}^E_{e^*} E^* \times (\mathcal{T}^E_{e^*} E^*)^* \mid \alpha_{e^*}(Y_{e^*}) + \beta_{e^*}(X_{e^*}) = 0, \\ & \text{for all } X_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*) \text{ and } \alpha_{e^*} - \Omega_E^\flat(e^*)(X_{e^*}) \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)\}. \end{aligned}$$

To check that $\mathcal{D}_{\mathcal{U}}(e^*) \subset \mathcal{D}^\perp_{\mathcal{U}}(e^*)$, we consider $(X_{e^*}, \alpha_{e^*}) \in \mathcal{D}_{\mathcal{U}}(e^*)$ and then, for any $(X'_{e^*}, \alpha'_{e^*}) \in \mathcal{D}_{\mathcal{U}}(e^*)$, we have that

$$\alpha_{e^*}(X'_{e^*}) + \alpha'_{e^*}(X_{e^*}) = \Omega_E(e^*)(X_{e^*}, X'_{e^*}) + \Omega_E(e^*)(X'_{e^*}, X_{e^*}) = 0,$$

by the skew-symmetry of Ω_E. This implies that $(X_{e^*}, \alpha_{e^*}) \in \mathcal{D}_{\mathcal{U}}^{\perp}(e^*)$. Therefore,

$$\mathcal{D}_{\mathcal{U}}(e^*) \subset \mathcal{D}_{\mathcal{U}}^{\perp}(e^*). \tag{12}$$

Now, to prove that $\mathcal{D}_{\mathcal{U}}^{\perp}(e^*) \subset \mathcal{D}_{\mathcal{U}}(e^*)$, let $(Y_{e^*}, \beta_{e^*}) \in \mathcal{D}_{\mathcal{U}}^{\perp}(e^*)$. Then, we have that

$$\alpha_{e^*}(Y_{e^*}) + \beta_{e^*}(X_{e^*}) = 0, \tag{13}$$

for all $(X_{e^*}, \alpha_{e^*}) \in \mathcal{T}^E_{e^*}E^* \times (\mathcal{T}^E_{e^*}E^*)^*$ such that $X_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*)$ and $\alpha_{e^*} - \Omega_E^\flat(e^*)(X_{e^*}) \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)$. If we choose $X_{e^*} = 0$ and $\alpha_{e^*} \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)$, then $(X_{e^*}, \alpha_{e^*}) \in \mathcal{D}_{\mathcal{U}}(e^*)$. Therefore, using (13), we obtain that $\alpha_{e^*}(Y_{e^*}) = 0$, for all $\alpha_{e^*} \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)$. Then, we conclude $Y_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*)$. On the other hand, let $X_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*)$ be arbitrary and suppose that $\alpha_{e^*}(Z_{e^*}) = \Omega_E(e^*)(X_{e^*}, Z_{e^*})$, for all $Z_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*)$. Since $Y_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*)$, we have $\alpha_{e^*}(Y_{e^*}) = \Omega_E(e^*)(X_{e^*}, Y_{e^*})$ and, from (13), we deduce that

$$\Omega_E(e^*)(X_{e^*}, Y_{e^*}) + \beta_{e^*}(X_{e^*}) = 0,$$

for all $X_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*)$. This implies that $\beta_{e^*} - \Omega_E^\flat(Y_{e^*}) \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)$. Therefore, $(Y_{e^*}, \beta_{e^*}) \in \mathcal{D}_{\mathcal{U}}(e^*)$ and thus

$$\mathcal{D}_{\mathcal{U}}^{\perp}(e^*) \subset \mathcal{D}_{\mathcal{U}}(e^*). \tag{14}$$

Given (12) and (14), we conclude that $\mathcal{D}_{\mathcal{U}}^{\perp}(e^*) = \mathcal{D}_{\mathcal{U}}(e^*)$, and the result follows. $\square$

In what follows, we will obtain a local representation of the almost Dirac structure $\mathcal{D}_{\mathcal{U}}$ induced on $\mathcal{T}^E E^*$ by a vector subbundle $\mathcal{U}$ of E. Consider local coordinates (x^i) on Q, a local basis $\{e_\alpha\}$ of sections of E and the corresponding local coordinates (x^i, y^α) on E. Let $\{\mathcal{Y}_\alpha, \mathcal{P}^\alpha\}$ be the local basis of $\tau^{\tau^*} : \mathcal{T}^E E^* \to E^*$ defined by (6) induced by the local coordinates (x^i) on Q and the local basis $\{e_\alpha\}$ of E and $(x^i, p_\alpha; z^\alpha, u_\alpha)$ be the induced local coordinates on $\mathcal{T}^E E^*$.

Thus, we can locally represent the fiber of $\mathcal{U}_{\mathcal{T}^E E^*}$ at a point $(x^i, p_\alpha) \in E^*$ as

$$\mathcal{U}_{\mathcal{T}^E E^*}(x^i, p_\alpha) = \{(x^i, p_\alpha; z^\alpha, u_\alpha) \,|\, (x^i, z^\alpha) \in \mathcal{U}(x^i)\}.$$

If we denote by $(x^i, p_\alpha; r_\alpha, v^\alpha)$ the corresponding local coordinates induced on $(\mathcal{T}^E E^*)^*$ by the dual basis $\{\mathcal{Y}^\alpha, \mathcal{P}_\alpha\}$ of $\{\mathcal{Y}_\alpha, \mathcal{P}^\alpha\}$, then the annihilator of $\mathcal{U}_{\mathcal{T}^E E^*}$ is locally given by

$$\mathcal{U}^\circ_{\mathcal{T}^E E^*}(x^i, p_\alpha) = \{(x^i, p_\alpha; r_\alpha, v^\alpha) \,|\, v^\alpha = 0 \text{ and } (x^i, r_\alpha) \in \mathcal{U}^\circ(x^i)\}.$$

From (9), we have that

$$\Omega_E^\flat(x^i, p_\alpha)(x^i, p_\alpha; z^\alpha, u_\alpha) = (x^i, p_\alpha; -u_\alpha - \mathcal{C}^\gamma_{\alpha\beta} p_\gamma z^\beta, z^\alpha) \tag{15}$$

and then the condition $\alpha_{e^*} - \Omega_E^\flat(e^*)(X_{e^*}) \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)$ can be written locally as

$$v^\alpha = z^\alpha \quad \text{and} \quad (x^i, r_\alpha + u_\alpha + \mathcal{C}^\gamma_{\alpha\beta} p_\gamma z^\beta) \in \mathcal{U}^\circ(x^i),$$

where $X_{e^*} \equiv (x^i, p_\alpha; z^\alpha, u_\alpha)$ and $\alpha_{e^*} \equiv (x^i, p_\alpha; r_\alpha, v^\alpha)$.

Finally, we obtain that

$$\begin{aligned}\mathcal{D}_{\mathcal{U}}(e^*) =& \{(X_{e^*}, \alpha_{e^*}) \in \mathcal{T}^E_{e^*}E^* \times (\mathcal{T}^E_{e^*}E^*)^* \,|\, X_{e^*} \in \mathcal{U}_{\mathcal{T}^E E^*}(e^*) \text{ and} \\ &\qquad \alpha_{e^*} - \Omega_E^\flat(e^*)(X_{e^*}) \in \mathcal{U}^\circ_{\mathcal{T}^E E^*}(e^*)\} \\ =& \{((x^i, p_\alpha; z^\alpha, u_\alpha), (x^i, p_\alpha; r_\alpha, v^\alpha)) \,|\, (x^i, z^\alpha) \in \mathcal{U}(x^i), \\ &\qquad v^\alpha = z^\alpha \text{ and } (x^i, r_\alpha + u_\alpha + \mathcal{C}^\gamma_{\alpha\beta} p_\gamma z^\beta) \in \mathcal{U}^\circ(x^i)\}.\end{aligned} \tag{16}$$

Remark 2. One of the advantages of working in the Lie algebroids setting is that we can construct a local basis $\{e_\alpha\}$ of sections of E as follows. We take a local basis $\{e_a\}$ of sections of the vector bundle $\tau_{\mathcal{U}} : \mathcal{U} \to Q$ and complete it to a basis $\{e_a, e_A\}$ of local sections of E. In this way, we have coordinates $(x^i, y^\alpha) = (x^i, y^a, y^A)$ on E. In this set of coordinates, the equations which define the subbundle $\mathcal{U}$ are $y^A = 0$. So, we can consider (x^i, y^a) as local coordinates on $\mathcal{U}$. Moreover, if $\{e^a, e^A\}$ is the dual basis of $\{e_a, e_A\}$ of E^*, then $\{e^A\}$ is a local basis of sections of $\mathcal{U}^\circ$. So, from the definition of $\mathcal{U}_{\mathcal{T}^E E^*}$, we deduce that $\{\mathcal{Y}_a, \mathcal{P}^a, \mathcal{P}^A\}$ is a local basis of sections of $\mathcal{U}_{\mathcal{T}^E E^*} \to E^*$ and, if $\{\mathcal{Y}^a, \mathcal{Y}^A, \mathcal{P}_a, \mathcal{P}_A\}$ is the dual basis of $\{\mathcal{Y}_a, \mathcal{Y}_A, \mathcal{P}^a, \mathcal{P}^A\}$, then $\{\mathcal{Y}^A\}$ is a local basis of $\mathcal{U}^\circ_{\mathcal{T}^E E^*}$. Therefore, a local representation for the almost Dirac structure $\mathcal{D}_{\mathcal{U}}$ is

$$\begin{aligned}\mathcal{D}_{\mathcal{U}}(x^i, p_\alpha) =& \{((x^i, p_\alpha; z^\alpha, u_\alpha), (x^i, p_\alpha; r_\alpha, v^\alpha)) \,|\, z^A = v^A = 0,\ v^a = z^a, \\ & \text{and } r_a = -u_a - \mathcal{C}^\gamma_{ab} p_\gamma z^b\}.\end{aligned}$$

We have used the canonical symplectic section Ω_E on $\mathcal{T}^E E^*$ together with a vector subbundle $\mathcal{U} \subset E$ to define the almost Dirac structure $\mathcal{D}_{\mathcal{U}}$. However there is a dual version of the above construction in which the almost Dirac structure is defined by a Poisson structure on $\mathcal{T}^E E^*$ together with a vector subbundle $\mathcal{U} \subset E$.

Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid and $\mathcal{U}$ be a vector subbundle of E. Consider the projection $\pi^2 : (\mathcal{T}^E E^*)^* \to E$ defined as $\pi^2 = \mathrm{pr}_1 \circ (\Omega^\flat_E)^{-1}$, where $\mathrm{pr}_1 : \mathcal{T}^E E^* \to E$ is the projection on the first factor. If we consider local coordinates as before, using (15), we have that

$$\pi^2(x^i, p_\alpha; r_\alpha, v^\alpha) = (x^i, v^\alpha).$$

Now, we define the induced vector subbundle $\mathcal{U}^*_{\mathcal{T}^E E^*}$ of $(\mathcal{T}^E E^*)^*$ by

$$\mathcal{U}^*_{\mathcal{T}^E E^*} = (\pi^2)^{-1}(\mathcal{U}).$$

Note that $\mathcal{U}^*_{\mathcal{T}^E E^*} = \Omega^\flat_E(\mathcal{U}_{\mathcal{T}^E E^*})$, from the definition of $\mathcal{U}_{\mathcal{T}^E E^*}$ (see (10)). Locally, $\mathcal{U}^*_{\mathcal{T}^E E^*}$ is given by

$$\mathcal{U}^*_{\mathcal{T}^E E^*}(x^i, p_\alpha) = \{(x^i, p_\alpha; r_\alpha, v^\alpha) \,|\, (x^i, v^\alpha) \in \mathcal{U}(x^i)\}.$$

The annihilator of $\mathcal{U}^*_{\mathcal{T}^E E^*}$ is given, for each $e^* \in E^*$, by

$$\begin{aligned}(\mathcal{U}^*_{\mathcal{T}^E E^*})^\circ(e^*) =& \{X_{e^*} \in \mathcal{T}^E_{e^*} E^* \,|\, \alpha_{e^*}(X_{e^*}) = 0, \text{ for all } \alpha_{e^*} \in \mathcal{U}^*_{\mathcal{T}^E E^*}(e^*)\} \\ =& \{X_{(x^i, p_\alpha)} = (x^i, p_\alpha; z^\alpha, u_\alpha) \,|\, z^\alpha = 0 \text{ and } (x^i, u_\alpha) \in \mathcal{U}^\circ(x^i)\}.\end{aligned}$$

On the other hand, we introduce the section Π of the vector bundle $\wedge^2(\mathcal{T}^E E^*) \to E^*$ defined by

$$\Pi(\alpha, \beta) = \Omega_E((\Omega^\flat_E)^{-1}(\alpha), (\Omega^\flat_E)^{-1}(\beta)),$$

for $\alpha, \beta \in (\mathcal{T}^E E^*)^*$. Π is the algebraic Poisson structure on the vector bundle $\mathcal{T}^E E^* \to E^*$ associated with the symplectic section Ω_E. Denote by $\sharp_\Pi : (\mathcal{T}^E E^*)^* \to \mathcal{T}^E E^*$ the vector bundles morphism given by

$$\sharp_\Pi(\alpha) = -i_\alpha \Pi, \text{ for } \alpha \in (\mathcal{T}^E E^*)^*.$$

Note that $\sharp_\Pi = (\Omega^\flat_E)^{-1}$.

Then, using the above notation, the induced almost Dirac structure $\mathcal{D}_{\mathcal{U}}$ on $\mathcal{T}^E E^*$ is given, for $e^* \in E^*$, by

$$\begin{aligned}\mathcal{D}_{\mathcal{U}}(e^*) =& \{(X_{e^*}, \alpha_{e^*}) \in \mathcal{T}^E_{e^*} E^* \times (\mathcal{T}^E_{e^*} E^*)^* \,|\, \alpha_{e^*} \in \mathcal{U}^*_{\mathcal{T}^E E^*}(e^*) \text{ and} \\ & X_{e^*} - \sharp_\Pi(e^*)(\alpha_{e^*}) \in (\mathcal{U}^*_{\mathcal{T}^E E^*})^\circ(e^*)\},\end{aligned}$$

whose local representation is

$$\begin{aligned}\mathcal{D}_{\mathcal{U}}(x^i,p_\alpha) =\{ & ((x^i,p_\alpha;z^\alpha,u_\alpha),(x^i,p_\alpha;r_\alpha,v^\alpha)) \mid (x^i,v^\alpha)\in\mathcal{U}(x^i),\\ & v^\alpha = z^\alpha \text{ and } (x^i, r_\alpha + u_\alpha + \mathcal{C}^\gamma_{\alpha\beta}p_\gamma z^\beta)\in\mathcal{U}^\circ(x^i)\},\end{aligned}$$

which coincides with (16).

3.2. **Implicit Lagrangian systems on a Lie algebroid.** In this section, an implicit Lagrangian system on a Lie algebroid E is defined in the context of the induced almost Dirac structure $\mathcal{D}_{\mathcal{U}}$ on $\mathcal{T}^E E^*$. As we shall see, the notion of implicit Lagrangian systems that is developed here can handle systems with degenerate Lagrangians as well as systems with nonholonomic constraints. Another description to address these systems was recently presented by Grabowska and Grabowski in [16], where they use the notion of a Lie algebroid as a double vector bundle morphism.

Let $L : E \to \mathbb{R}$ be a Lagrangian function on the Lie algebroid $(E, [\![\cdot,\cdot]\!], \rho)$.

First of all, we will recall the definition of the Legendre transformation in the context of Lie algebroids. Given a Lagrangian function $L : E \to \mathbb{R}$, one can consider the Poincaré-Cartan 1-section associated with L, $\theta_L \in \Gamma((\mathcal{T}^E E)^*)$, which is given

$$\theta_L(e)(Z_e) = (d^{\mathcal{T}^E E} L(e))(S_e(Z_e)) = \rho^\tau(S_e(Z_e))(L),$$

for $e \in E$ and $Z_e \in \mathcal{T}^E_e E$, $S : \mathcal{T}^E E \to \mathcal{T}^E E$ being the vertical endomorphism defined in (5). So, *the Legendre transformation associated with* L is defined as the smooth map $\mathbb{F}L : E \to E^*$ defined by

$$\mathbb{F}L(e)(e') = \theta_L(e)(Z),$$

for $e, e' \in E$, where $Z \in \mathcal{T}^E_e E$ such that $\mathrm{pr}_1(Z) = e'$, $\mathrm{pr}_1 : \mathcal{T}^E E \to E$ being the canonical projection over the first factor. For more details see [27].

The map $\mathbb{F}L$ is well-defined and its local expression in fiber coordinates on E and E^* is

$$\mathbb{F}L(x^i, y^\alpha) = \Big(x^i, \frac{\partial L}{\partial y^\alpha}\Big).$$

Now, we consider the isomorphism $A_E : \mathcal{T}^E E^* \to (\mathcal{T}^E E)^*$ between the vector bundles $\mathrm{pr}_1 : \mathcal{T}^E E^* \to E$ and $(\tau^\tau)^* : (\mathcal{T}^E E)^* \to E$ introduced in [27] and whose local expression is

$$A_E(x^i, p_\alpha; z^\alpha, u_\alpha) = (x^i, z^\alpha; u_\alpha + \mathcal{C}^\gamma_{\alpha\beta} p_\gamma z^\beta, p_\alpha). \tag{17}$$

Then, we define the map $\gamma_E : (\mathcal{T}^E E)^* \to (\mathcal{T}^E E^*)^*$ as $\gamma_E = \Omega^\flat_E \circ A_E^{-1}$ which is an isomorphism between the vector bundles $(\tau^\tau)^* : (\mathcal{T}^E E)^* \to E$ and $\mathrm{pr}_1^* : (\mathcal{T}^E E^*)^* \to E$. From (15) and (17), we deduce that the local expression of this isomorphism is

$$\gamma_E(x^i, y^\alpha; s_\alpha, w_\alpha) = (x^i, w_\alpha; -s_\alpha, y^\alpha). \tag{18}$$

Now, define a differential operator $\mathbf{D}$ acting on the Lagrangian $L : E \to \mathbb{R}$, which we shall call the *Dirac differential* of L by

$$\mathbf{D}L : E \to (\mathcal{T}^E E^*)^*, \quad \mathbf{D}L = \gamma_E \circ d^{\mathcal{T}^E E} L,$$

where $d^{\mathcal{T}^E E} L$ is the differential of L on the Lie algebroid $\mathcal{T}^E E$ which is a section of $(\tau^\tau)^* : (\mathcal{T}^E E)^* \to E$.

Using (1), (4) and (18), we conclude that $\mathbf{D}L$ is represented in local coordinates by

$$\mathbf{D}L(x^i, y^\alpha) = \Big(x^i, \frac{\partial L}{\partial y^\alpha}; -\rho^i_\alpha \frac{\partial L}{\partial x^i}, y^\alpha\Big). \tag{19}$$

Now, we have all the ingredients to define an implicit Lagrangian system on a Lie algebroid.

Definition 3.2. Let $L : E \to \mathbb{R}$ be a given Lagrangian function (possibly degenerate) on a Lie algebroid $(E, [\![\cdot,\cdot]\!], \rho)$ and $\mathcal{U} \subset E$ be a given vector subbundle of $\tau : E \to Q$. Denote by $\mathcal{D}_{\mathcal{U}}$ the induced almost Dirac structure on the Lie algebroid prolongation $\mathcal{T}^E E^*$ that is given by (11) and $\mathbf{D}L : E \to (\mathcal{T}^E E^*)^*$ the Dirac differential of L. Let $P = \mathbb{F}L(\mathcal{U}) \subset E^*$ be the image of $\mathcal{U}$ under the Legendre transformation.

An **implicit Lagrangian system** is a triple $(L, \mathcal{U}, X)$, where X is a section of the Lie algebroid prolongation $\tau^{\tau^*} : \mathcal{T}^E E^* \to E^*$ defined at the points of P, together with the condition

$$(X, \mathbf{D}L) \in \mathcal{D}_{\mathcal{U}}.$$

In other words, as $P = \mathbb{F}L(\mathcal{U}) \subset E^*$, X can be seen as a section of $\mathcal{T}^E E^* \to E \oplus_Q E^*$ defined at the points of $\mathcal{U} \oplus_Q P$ and thus, we require that for each point $e \in \mathcal{U}$ and with $e^* = \mathbb{F}L(e) \in P$, we have

$$(X(e, e^*), \mathbf{D}L(e)) \in \mathcal{D}_{\mathcal{U}}(e^*).$$

Definition 3.3. A **solution curve** of an implicit Lagrangian system $(L, \mathcal{U}, X)$ is a curve $(x(t), y(t)) \in \mathcal{U}(x(t))$, $t_1 \leq t \leq t_2$, such that $\mathbb{F}L(x(t), y(t))$ is an integral curve of the vector field $\rho^{\tau^*}(X)$ on E^*, ρ^{τ^*} being the anchor map of the Lie algebroid $\tau^{\tau^*} : \mathcal{T}^E E^* \to E^*$.

Remark 3. One can consider the map $i_E : E \to E \oplus_Q E^*$ defined as the direct sum of the identity map on E, $id : E \to E$, and the Legendre transformation $\mathbb{F}L : E \to E^*$. Denote by $\mathcal{K}$ the submanifold of $E \oplus_Q E^*$ defined as the image of $\mathcal{U}$ under i_E. Thus, $\mathcal{K}$ is locally given by

$$\mathcal{K} = \{(x^i, y^\alpha, p_\alpha) \in E_x \times E_x^* \,|\, (x^i, y^\alpha) \in \mathcal{U}(x^i), \quad p_\alpha = \frac{\partial L}{\partial y^\alpha}\}.$$

Another way to define the submanifold $\mathcal{K}$ is the following. Consider the map $\rho_{(\mathcal{T}^E E)^*} : (\mathcal{T}^E E)^* \to E \oplus_Q E^*$ defined as the direct sum of the maps $(\tau^\tau)^* : (\mathcal{T}^E E)^* \to E$ and $\tau^{\tau^*} \circ (A_E)^{-1} : (\mathcal{T}^E E)^* \to E^*$. Recall that $(\tau^\tau)^* : (\mathcal{T}^E E)^* \to E$ is the projection of the dual vector bundle of the Lie algebroid prolongation of E over the fibration τ, $A_E : \mathcal{T}^E E^* \to (\mathcal{T}^E E)^*$ is the vector bundle isomorphism defined in (17) and $\tau^{\tau^*} : \mathcal{T}^E E^* \to E$ is the projection of the Lie algebroid prolongation of E over τ^*. If we consider local coordinates introduced in Section 2.4, the map $\rho_{(\mathcal{T}^E E)^*}$ is given by

$$\rho_{(\mathcal{T}^E E)^*}(x^i, y^\alpha, s_\alpha, w_\alpha) = (x^i, y^\alpha, w_\alpha). \tag{20}$$

Note that when E is the standard Lie algebroid, that is, $E = TQ$, then this map is the map $\rho_{T^*TQ} : T^*TQ \to TQ \oplus_Q T^*Q$ defined in [43] (see Section 4.10 in [43]).

Then, we can construct the map i_E between E and $E \oplus_Q E^*$ by the composition

$$i_E = \rho_{(\mathcal{T}^E E)^*} \circ A_E \circ (\Omega_E^\flat)^{-1} \circ \mathbf{D}L : E \to (\mathcal{T}^E E^*)^* \to \mathcal{T}^E E^* \to (\mathcal{T}^E E)^* \to E \oplus_Q E^*,$$

$\mathbf{D}L$ being the Dirac differential of the Lagrangian function L and $\Omega_E^\flat$ being the flat map defined by the canonical symplectic section Ω_E (see (8)).

From (15), (17), (19) and (20), we have that the local expression of i_E is

$$i_E(x^i, y^\alpha) = \Big(x^i, y^\alpha, \frac{\partial L}{\partial y^\alpha}\Big).$$

Then, the submanifold $\mathcal{K} \subset E \oplus_Q E^*$ can be defined as $\mathcal{K} = i_E(\mathcal{U})$.

Then, a solution of an implicit Lagrangian system $(L, \mathcal{U}, X)$ may be equivalently defined to be a curve $(x(t), y(t), p(t))$, where $t_1 \leq t \leq t_2$, whose image lies in the submanifold $\mathcal{K} \subset E \oplus_Q E^*$ and such that $(x(t), p(t))$ is an integral curve of $\rho^{\tau^*}(X)$ and such that

$$(X(x(t), y(t), p(t)), \mathbf{D}L(x(t), y(t))) \in \mathcal{D}_{\mathcal{U}}(x(t), p(t)).$$

Locally, using the preceding notation, (7), (16) and (19), we deduce that a solution curve $(x^i(t),\ y^\alpha(t), p_\alpha(t))$ for an implicit Lagrangian system $(L, \mathcal{U}, X)$ must satisfy the following equations

$$\begin{cases} (x^i, y^\alpha) \in \mathcal{U}(x^i), \quad \dot{x}^i = \rho^i_\alpha y^\alpha, \quad p_\alpha = \dfrac{\partial L}{\partial y^\alpha}, \\ \left(x^i, \dot{p}_\alpha + \mathcal{C}^\gamma_{\alpha\beta} p_\gamma y^\beta - \rho^i_\alpha \dfrac{\partial L}{\partial x^i}\right) \in \mathcal{U}^\circ(x^i). \end{cases} \tag{21}$$

Remark 4. If we consider the local coordinates on E introduced in Remark 2, the implicit Lagrangian equations reduce to

$$y^A = 0, \quad \dot{x}^i = \rho^i_a y^a, \quad p_\alpha = \frac{\partial L}{\partial y^\alpha} \quad \text{and} \quad \dot{p}_a = -\mathcal{C}^\gamma_{ab} \frac{\partial L}{\partial y^\gamma} y^b + \rho^i_a \frac{\partial L}{\partial x^i}.$$

3.3. **Conservation of energy.** Define the *generalized energy* $E_L : E \oplus_Q E^* \to \mathbb{R}$ by

$$E_L(x, e, e^*) = \langle e^*, e \rangle - L(x, e),$$

where $(x, e) \in \mathcal{U}$ and $(x, e^*) \in P$.

Proposition 1. *Let $(x(t), y(t))$, $t_1 \leq t \leq t_2$, be an integral curve of a given implicit Lagrangian system $(L, \mathcal{U}, X)$ on a Lie algebroid E. Then, the function $E_L(x(t), y(t), p(t))$ is constant in time, where $p(t) = \frac{\partial L}{\partial y}(x(t), y(t))$.*

Proof. We give the proof using local coordinates. Then, from the definition of the generalized energy E_L, we have that

$$\frac{dE_L}{dt} = \dot{y}^\alpha p_\alpha + y^\alpha \dot{p}_\alpha - \frac{\partial L}{\partial x^i} \dot{x}^i - \frac{\partial L}{\partial y^\alpha} \dot{y}^\alpha.$$

As $p_\alpha(t) = \dfrac{\partial L}{\partial y^\alpha}(x^i(t), y^\beta(t))$, we deduce that

$$\begin{aligned} \frac{dE_L}{dt} &= y^\alpha \dot{p}_\alpha - \frac{\partial L}{\partial x^i} \dot{x}^i \\ &= y^\alpha \left(\dot{p}_\alpha + \mathcal{C}^\gamma_{\alpha\beta} \frac{\partial L}{\partial y^\gamma} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} \right) - y^\alpha \mathcal{C}^\gamma_{\alpha\beta} \frac{\partial L}{\partial y^\gamma} y^\beta + y^\alpha \rho^i_\alpha \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial x^i} \dot{x}^i. \end{aligned}$$

Now, as $(x^i(t), y^\alpha(t))$ satisfies the implicit Lagrangian equations (21), we know that

$$(x^i, y^\alpha) \in \mathcal{U}(x^i), \quad \dot{x}^i = \rho^i_\alpha y^\alpha \quad \text{and} \quad \left(x^i, \dot{p}_\alpha + \mathcal{C}^\gamma_{\alpha\beta} \frac{\partial L}{\partial y^\gamma} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i}\right) \in \mathcal{U}^\circ(x^i).$$

Moreover, as $\mathcal{C}^\gamma_{\alpha\beta} = -\mathcal{C}^\gamma_{\beta\alpha}$, the term $y^\alpha \mathcal{C}^\gamma_{\alpha\beta} \dfrac{\partial L}{\partial y^\gamma} y^\beta = 0$. So, we conclude that

$$\frac{dE_L}{dt} = 0.$$

$\square$

4. **Hamilton–Jacobi theory for implicit Lagrangian systems.** Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold Q with projection $\tau : E \to Q$ and $(L, \mathcal{U}, X)$ be an implicit Lagrangian system on E.

Theorem 4.1. *Let $\tilde{\gamma} : Q \to E \oplus_Q E^*$ be a section of the canonical projection $\nu : E \oplus_Q E^* \to Q$ such that*

$$\tilde{\gamma}(Q) \subset \mathcal{K}, \tag{22}$$

and

$$d^E(\mathrm{pr}_{E^*} \circ \tilde{\gamma})_{|\mathcal{U}\times\mathcal{U}} = 0. \tag{23}$$

Denote by $\sigma \in \Gamma(E)$ the section $\sigma = \mathrm{pr}_1 \circ X \circ \mathrm{pr}_{E^*} \circ \tilde{\gamma}$, where $\mathrm{pr}_1 : \mathcal{T}^E E^* \to E$ is the projection on the first factor and $\mathrm{pr}_{E^*} : E \oplus_Q E^* \to E^*$ the projection over the second component. Then, the following conditions are equivalent:

1. *For every curve $c : I \to Q$ in Q such that*

$$\dot{c}(t) = \rho(\sigma)(c(t)), \ \ \textit{for all } t, \tag{24}$$

the curve $\tilde{\gamma} \circ c$ is a solution of the implicit Lagrangian system $(L, \mathcal{U}, X)$.

2. *$\tilde{\gamma}$ satisfies the **Hamilton–Jacobi equation for implicit Lagrangian systems**:*

$$d^E(E_L \circ \tilde{\gamma}) \in \mathcal{U}^\circ. \tag{25}$$

Proof. We give the proof using local coordinates. We consider local coordinates (x^i) on an open subset V of Q and a local basis $\{e_\alpha\}$ of sections of E defined on V, then we have the corresponding local coordinates (x^i, y^α) on E. Denote by ρ^i_α and $\mathcal{C}^\gamma_{\alpha\beta}$ the structure functions of the Lie algebroid E with respect to (x^i) and $\{e_\alpha\}$.

Suppose that $\tilde{\gamma}(x^i) = (x^i, \gamma^\alpha(x^j), \bar{\gamma}_\alpha(x^j))$. Then, the condition (22) means that

$$(x^i, \gamma^\alpha(x^j)) \in \mathcal{U}(x^i) \quad \text{and} \quad \bar{\gamma}_\alpha(x^i) = \frac{\partial L}{\partial y^\alpha}(x^i, \gamma^\alpha(x^j)), \tag{26}$$

and the condition (23) can be written locally as

$$\frac{\partial \bar{\gamma}_\delta}{\partial x^i}\rho^i_\beta v^\beta w^\delta = \Big(\frac{\partial \bar{\gamma}_\beta}{\partial x^i}\rho^i_\delta + \bar{\gamma}_\alpha \mathcal{C}^\alpha_{\beta\delta}\Big) v^\beta w^\delta, \tag{27}$$

for all $v, w \in \mathcal{U}$ given locally by $v = v^\beta e_\beta$ and $w = w^\delta e_\delta$.

If $c(t) = (c^i(t))$, it is easy to prove that equation (24) can be rewritten in local coordinates as

$$\dot{c}^i(t) = \gamma^\alpha(c(t))\rho^i_\alpha(c(t)). \tag{28}$$

Using the hypothesis (22) (see its local expression (26)), we also have that equation (25) is locally written as

$$\Big(\gamma^\beta \frac{\partial \bar{\gamma}_\beta}{\partial x^i} - \frac{\partial L}{\partial x^i}\Big)\rho^i_\alpha v^\alpha = 0, \tag{29}$$

for all $v = v^\alpha e_\alpha \in \mathcal{U}$.

(i) $\Rightarrow$ (ii) Assume that (i) holds. Therefore

$$\begin{cases} (c^i(t), \gamma^\alpha(c(t))) \in \mathcal{U}(c(t)), \\ \dot{c}^i(t) = \gamma^\alpha(c(t))\rho^i_\alpha(c(t)), \\ \bar{\gamma}_\alpha(c(t)) = \dfrac{\partial L}{\partial y^\alpha}(c(t), \gamma^\beta(c(t))), \\ \Big(\dfrac{\partial \bar{\gamma}_\alpha}{\partial x^j}(c(t))\dot{c}^j(t) + \mathcal{C}^\delta_{\alpha\beta}(c(t))\bar{\gamma}_\delta(c(t))\gamma^\beta(c(t)) \\ \qquad\qquad -\rho^j_\alpha(c(t))\dfrac{\partial L}{\partial x^j}(c(t), \gamma^\beta(c(t)))\Big)e^\alpha(c(t)) \in \mathcal{U}^\circ(c(t)). \end{cases} \tag{30}$$

Then, using (27) and (30), we have that at $c(t)$, for $w = w^\alpha e_\alpha(c(t)) \in \mathcal{U}(c(t))$,

$$\begin{aligned}
0 &= \Big(\frac{\partial \bar\gamma_\alpha}{\partial x^j}\gamma^\beta \rho^j_\beta + \mathcal{C}^\delta_{\alpha\beta}\bar\gamma_\delta\gamma^\beta - \rho^j_\alpha \frac{\partial L}{\partial x^j}\Big) w^\alpha \\
&= \Big(\frac{\partial \bar\gamma_\beta}{\partial x^j}\gamma^\beta \rho^j_\alpha + \bar\gamma_\delta \mathcal{C}^\delta_{\beta\alpha}\gamma^\beta + \mathcal{C}^\delta_{\alpha\beta}\bar\gamma_\delta\gamma^\beta - \rho^j_\alpha \frac{\partial L}{\partial x^j}\Big) w^\alpha \\
&= \Big(\frac{\partial \bar\gamma_\beta}{\partial x^j}\gamma^\beta - \frac{\partial L}{\partial x^j}\Big)\rho^j_\alpha w^\alpha.
\end{aligned}$$

Then, we conclude that (ii) holds (see (29)).

(ii) $\Rightarrow$ (i) Suppose that (ii) holds, that is, condition (29) is satisfied. Let $c : I \to Q$ a curve such that $\dot c(t) = \rho(\sigma)(c(t))$. Then, we have that

$$\dot c^i(t) = \gamma^\alpha(c(t))\rho^i_\alpha(c(t)).$$

Moreover, from (26), we also know that

$$(c^i(t), \gamma^\alpha(c(t))) \in \mathcal{U}(c(t)) \quad \text{and} \quad \bar\gamma_\alpha(c(t)) = \frac{\partial L}{\partial y^\alpha}(c(t), \gamma^\alpha(c(t))), \tag{31}$$

Moreover, using (26) and (27), we deduce that at $c(t)$, for all $w = w^\alpha e_\alpha(c(t)) \in \mathcal{U}(c(t))$,

$$\begin{aligned}
&\Big(\frac{\partial \bar\gamma_\alpha}{\partial x^j}\dot c^j + \mathcal{C}^\delta_{\alpha\beta}\bar\gamma_\delta\gamma^\beta - \rho^j_\alpha \frac{\partial L}{\partial x^j}\Big) w^\alpha \\
&\qquad = \Big(\frac{\partial \bar\gamma_\beta}{\partial x^j}\gamma^\beta \rho^j_\alpha + \bar\gamma_\delta \mathcal{C}^\delta_{\beta\alpha}\gamma^\beta + \mathcal{C}^\delta_{\alpha\beta}\bar\gamma_\delta\gamma^\beta - \rho^j_\alpha \frac{\partial L}{\partial x^j}\Big) w^\alpha \\
&\qquad = \Big(\frac{\partial \bar\gamma_\beta}{\partial x^j}\gamma^\beta - \frac{\partial L}{\partial x^j}\Big)\rho^j_\alpha w^\alpha = 0.
\end{aligned} \tag{32}$$

Finally, from (31) and (32), we conclude that $\tilde\gamma \circ c$ is an integral curve of X.

□

5. **Examples.**

5.1. **The case $\mathcal{U} = E$.**

This is perhaps the simplest case in which one has no constraints but the Lagrangian may be degenerate. In this case, the induced almost Dirac structure $\mathcal{D}_\mathcal{U} = \mathcal{D}_E$ is given by

$$\mathcal{D}_E(e^*) = \{(X_{e^*}, \alpha_{e^*}) \in \mathcal{T}^E_{e^*}E^* \times (\mathcal{T}^E_{e^*}E^*)^* \,|\, \alpha_{e^*} = \Omega^\flat_E(e^*)(X_{e^*})\}.$$

So, locally the equations defining the almost Dirac structure in this case are

$$v^\alpha = z^\alpha \quad \text{and} \quad r_\alpha = -u_\alpha - \mathcal{C}^\gamma_{\alpha\beta}p_\gamma z^\beta,$$

where $X_{e^*} \equiv (x^i, p_\alpha; z^\alpha, u_\alpha)$ and $\alpha_{e^*} \equiv (x^i, p_\alpha; r_\alpha, v^\alpha)$. Then, a curve $(x^i(t), y^\alpha(t))$ in E is a solution of the implicit Lagrangian system if and only if

$$p_\alpha = \frac{\partial L}{\partial y^\alpha},\ \dot x^i = \rho^i_\alpha y^\alpha \quad \text{and} \quad \dot p_\alpha = \rho^i_\alpha \frac{\partial L}{\partial x^i} - \mathcal{C}^\gamma_{\alpha\beta}p_\gamma y^\beta.$$

This means that in this case, the condition of an implicit Lagrangian system is equivalent to the Euler–Lagrange equations for L (see Equations (2.40) in [27]).

In the usual formulation of Lagrangian systems on a Lie algebroid, one must to restrict to admissible curves on the Lie algebroid E, that is, curves $c(t)$ in E such that $(c(t), \dot c(t)) \in \mathcal{T}^E_{c(t)}E$ or, locally, if $c(t) = (x^i(t), y^\alpha(t))$ then $\dot x^i = \rho^i_\alpha y^\alpha$. Notice that integral curves of an implicit Lagrangian system automatically satisfy this condition.

In this case we can reformulate the Theorem 4.1 as follows.

Corollary 1. *Let $\gamma : Q \to E$ be a section of the Lie algebroid $\tau : E \to Q$ such that*

$$d^E(\mathbb{F}L \circ \gamma) = 0.$$

Denote by $\sigma \in \Gamma(E)$ the section $\sigma = \mathrm{pr}_1 \circ X \circ \mathbb{F}L \circ \gamma$, where $\mathrm{pr}_1 : \mathcal{T}^E E^* \to E$ is the projection on the first factor. Then, the following conditions are equivalent:

1. *For every curve $c : I \to Q$ in Q such that*
$$\dot{c}(t) = \rho(\sigma)(c(t)), \ \text{for all } t,$$
the curve $\gamma \circ c$ is a solution of the implicit Lagrangian system (L, E, X), or equivalently, $\gamma \circ c$ is a solution of the Euler–Lagrange equations for the Lagrangian L.
2. *γ satisfies*
$$d^E(\varepsilon_L \circ \gamma) = 0,$$
where $\varepsilon_L : E \to \mathbb{R}$ is the energy function associated with L (see (2.39) in [27]) which is defined as $\varepsilon_L = \rho^\tau(\Delta)(L) - L$, $\Delta \in \Gamma(\mathcal{T}^E E)$ is the Euler section and ρ^τ is the anchor map of the Lie algebroid $\tau^\tau : \mathcal{T}^E E \to E$ (see Example 2.1).

This result can be viewed as the Lagrangian version of the Theorem 3.16 in [27].

5.2. **The case $E = TQ$.** Let E be the standard Lie algebroid $\tau_{TQ} : TQ \to Q$. In this case, the sections of this vector bundle can be identified with vector fields on Q, the Lie bracket of sections is just the usual Lie bracket of vector fields and the anchor map is the identity map $id : TQ \to TQ$. A vector subbundle $\mathcal{U}$ of TQ is just a distribution Δ_Q on Q.

Moreover, in this case, the Lie algebroid $(\mathcal{T}^E E^*, [\![\cdot,\cdot]\!]^{\tau^*}, \rho^{\tau^*})$ is the standard Lie algebroid $(TT^*Q, [\cdot,\cdot], id)$. So, the lift of the vector subbundle $\mathcal{U} = \Delta_Q$ to $\mathcal{T}^E E^* = TT^*Q$ is just the distribution Δ_{T^*Q} on T^*Q defined as

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q),$$

$\pi_Q : T^*Q \to Q$ being the canonical projection. Moreover, $\Omega_E = \Omega_{TQ}$ is the canonical symplectic 2-form on T^*Q. Then, the induced almost Dirac structure $\mathcal{D}_\mathcal{U}$ defined in Theorem 3.1 is given, for each point $z \in T^*Q$, by

$$\begin{aligned}\mathcal{D}_{\Delta_Q} =& \{(v_z, \alpha_z) \in T_zT^*Q \times T^*_zT^*Q | v_z \in \Delta_{T^*Q}(z) \text{ and} \\ & \alpha_z - \Omega^\flat_{TQ}(z)(v_z) \in \Delta^\circ_{T^*Q}(z)\}.\end{aligned}$$

This almost Dirac structure coincides with the induced almost Dirac structure introduced by Yoshimura and Marsden in [43]. Thus, if we apply the results of Section 3.2 to this particular case we recover the formulation of implicit Lagrangian systems develop in [43].

Moreover, applying the Theorem 4.1 to this particular case one recover the result develop in [25] for standard implicit Lagrangian systems.

Example 5.1. We are going to consider a simple example: the case of Euler–Poincaré reduction. In this case, we consider the particular case when the manifold Q is a Lie group G and the distribution Δ_Q is just $TQ = TG$ (that is, the cases when $\mathcal{U} = E$ and $E = TG$). Let $L : TG \to \mathbb{R}$ be a left-invariant Lagrangian. Then, we have that $(g(t), v(t)) \in T_{g(t)}G$, $t_1 \leq t \leq t_2$, is a solution curve of the implicit Lagrangian system (L, TG, X) if and only if $g(t)$ is a solution of the Euler–Lagrange equations for L on G and $\dot{g}(t) = v(t)$, for $t_1 \leq t \leq t_2$.

On the other hand, let $\mathfrak{g}$ be the Lie algebra associated with G which is a Lie algebroid over a point. As, L is a left-invariant function, we can consider the reduced Lagrangian $l : \mathfrak{g} \to \mathbb{R}$, $l = L_{|\mathfrak{g}}$. Taking $\mathcal{U} = \mathfrak{g}$, we have that a curve $\xi(t) \in \mathfrak{g}$ is a solution of the implicit Lagrangian system $(l, \mathfrak{g}, Y)$ if and only if it is a solution of the Euler–Poincaré equations on $\mathfrak{g}$. Moreover, as well known, $g(t)$ satisfies the Euler–Lagrange equations for L on G if and only if $\xi(t) = g(t)^{-1}\dot{g}(t)$ satisfies the Euler–Poincaré equations on $\mathfrak{g}$. Then, we conclude that $(g(t), v(t))$ is a solution curve of the implicit Lagrangian system (L, TG, X) if and only if $\xi(t) = g(t)^{-1}\dot{g}(t)$ is a solution curve of the implicit Lagrangian system $(l, \mathfrak{g}, Y)$ and $\dot{g}(t) = v(t)$.

5.3. **Nonholonomic mechanics on Lie algebroids.** Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid. Nonholonomic constraints on the Lie algebroids setting are given by a vector subbundle $\mathcal{U}$ of E. In [10], the authors introduced the notion of a nonholonomically constrained Lagrangian system on a Lie algebroid E as a pair $(L, \mathcal{U})$, where $L : E \to \mathbb{R}$ is a Lagrangian function on E and $\mathcal{U}$ is the constraint subbundle, that is, it is a vector subbundle of E.

If we consider local coordinates as in Remark 2, then a solution curve $(x^i(t), y^\alpha(t))$, $t_1 \le t \le t_2$, on E for a nonholonomic system must satisfy the differential equations (see Equations (3.7) in [10])

$$\begin{cases} \dot{x}^i = \rho^i_a y^a, \\ \dfrac{d}{dt}\Big(\dfrac{\partial L}{\partial y^a}\Big) + \dfrac{\partial L}{\partial y^\gamma}\mathcal{C}^\gamma_{ab} y^b - \rho^i_a \dfrac{\partial L}{\partial x^i} = 0, \\ y^A = 0. \end{cases}$$

So, a nonholonomic system is locally represented by an implicit Lagrangian system $(L, \mathcal{U}, X)$ together with the condition

$$p_\alpha(t) = \frac{\partial L}{\partial y^\alpha}(x^i(t), y^\alpha(t)),$$

since $(x^i(t), p_\alpha(t)) = \mathbb{F}L(x^i(t), y^\alpha(t))$, where $\mathbb{F}L$ is the Legendre transformation.

Example 5.2. Consider the situation of Example 5.1 but with a non-trivial left-invariant distribution on G. Then, we have $Q = G$ a Lie group, $L : TG \to \mathbb{R}$ a left-invariant Lagrangian and $\mathcal{U} = \Delta_G$, $\Delta_G \ne TG$ and $\Delta_G \ne \{0\}$, a left-invariant distribution on G, that is, a standard nonholonomic LL system on G. As we have proved in general, $(g(t), v(t), p(t))$, $t_1 \le t \le t_2$, is a solution curve of the implicit Lagrangian system (L, TG, X) if and only if $(g(t), v(t))$ is a solution of the Lagrange-d'Alembert equations for L on G and $p(t) = \partial L/\partial v(g(t), v(t))$, for $t_1 \le t \le t_2$.

On the other hand, this type of nonholonomic system on G may be reduced to a nonholonomic system on the Lie algebra $\mathfrak{g}$ associated with G. The reduced Lagrangian $l : \mathfrak{g} \to \mathbb{R}$ is $l = L_{|\mathfrak{g}}$ and the vector subspace $\mathfrak{d}$ of $\mathfrak{g}$ is given by $\mathfrak{d} = \Delta_G(e)$. Then, one has a constrained system $(l, \mathfrak{d})$ on $\mathfrak{g}$. So, a curve $\xi(t) \in \mathfrak{g}$ is a solution of the implicit Lagrangian system $(l, \mathfrak{g}, Y)$ if and only if it is a solution of the constrained Euler–Poincaré equations (or the so-called Euler–Poincaré–Suslov equations, see [14]) on $\mathfrak{g}$.

6. **Conclusion and future work.** In this paper, we introduced the notion of an induced almost Dirac structure, and show how it leads to implicit Lagrangian systems on Lie algebroids. This provides a generalization of Lagrangian mechanics on Lie algebroids that can address degenerate Lagrangians as well as holonomic and nonholonomic constraints. Furthermore, we have obtained a Hamilton–Jacobi theory for such systems.

In future research, we aim to study the possibility of obtaining a Hamilton–Jacobi equation, as in the general case described in [26], using the notion of Dirac algebroids given in [16]. In this case, the theory will include all important cases of Lagrangian and Hamiltonian systems, including systems with and without constraints, and autonomous and non-autonomous systems.

Another interesting direction would be to generalize to Dirac algebroids the work in [25] that relates the Hamilton–Jacobi theory of a Dirac mechanical system with symmetry and the Hamilton–Jacobi theory of the associated reduced Dirac system. Furthermore, the relationship between the various Hamilton–Jacobi theories for reduction of Dirac mechanical systems formulated on Dirac, Courant, and Lie algebroids, and the formulations based on Lagrange–Poincaré bundles [45] also remains to be studied.

Acknowledgments. This material is based upon work supported by the National Science Foundation under the applied mathematics grant DMS-0726263, the Faculty Early Career Development (CAREER) award DMS-1010687, the FRG award DMS-1065972, MICINN (Spain) grants MTM2009-13383 and MTM2009-08166-E, and the projects of the Canary government SOLSUBC200801000238 and ProID20100210. We also want to thank the two anonymous referees for their helpful comments and suggestions.

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Received June 2012; revised November 2012.

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