Some results on polarity graphs

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$\mathrm{ex}(n, C_4)$

In 1938, Erdős asked how many edges an n-vertex graph with no C_4 may have. This quantity is denoted by

$$\mathrm{ex}(n, C_4)$$

and is called the Turán number for C_4.

Many problems in extremal combinatorics can be phrased as (hypergraph) Turán problems for the appropriate family of excluded subgraphs.

Beginnings

Bounds for $\mathrm{ex}(n, C_4)$

A double counting and convexity argument of Kővári, Sós, and Turán gives

$$\mathrm{ex}(n, C_4) \leq \frac{1}{2}n^{3/2} + \frac{1}{2}n.$$

Theorem (Brown, Erdős-Rényi-Sós, 1966)

For any prime power q

$$\mathrm{ex}(q^2 + q + 1, C_4) \geq \frac{1}{2}q(q+1)^2.$$

Bounds for ex(n, C_4)

The graphs constructed giving the lower bound are called Erdős-Rényi polarity graphs and are denoted by ER_q.

By results on the distribution of primes, gives the asymptotic formula

$$\mathrm{ex}(n, C_4) \sim \frac{1}{2} n^{3/2}.$$

Exact results?

Exact results

Theorem (Füredi 1983, 1996)

For q a prime power

$$\mathrm{ex}(q^2+q+1, C_4) = \frac{1}{2}q(q+1)^2.$$

Theorem (Firke-Kosek-Nash-Williford, 2013)

For q a power of 2,

$$\mathrm{ex}(q^2+q, C_4) = \frac{1}{2}q(q+1)^2 - q.$$

Exact results for $n \leq 31$ by computer search.
No other exact results known. Later we will discuss bounds.

Overview

1. ER_q and other polarity graphs
2. History
3. $\mathrm{ex}(n, C_4)$
4. The chromatic number of ER_q

ER_q

The graph constructed by Brown and Erdős, Rényi, Sós is called the Erdős-Rényi Polarity graph and denoted by ER_q.

$$V(ER_q) = \text{one dimensional subspaces of } \mathbb{F}_q^3.$$

$$(x_0, x_1, x_2) \sim (y_0, y_1, y_2) \;\Leftrightarrow\; x_0y_0 + x_1y_1 + x_2y_2 = 0.$$

ER_q is an example of a polarity graph.

ER_q

Let $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a finite geometry.

Definition

A polarity of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a bijection $\pi : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ such that

- Points are sent to lines and lines are sent to points.
- π^2 is the identity function.
- π preserves incidence.

An example

Let $\mathcal{P}$ be the one-dimensional subspaces of $\mathbb{F}_q^3$ and $\mathcal{L}$ be the two-dimensional subspaces.

Define $\mathcal{I}$ by containment. i.e. $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a finite projective plane of order q.

Define a map π that sends points and lines to their orthogonal complements.

π is a polarity.

Polarity graphs

Given a geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ and a polarity π, one can construct a polarity graph.

$V(G_\pi) = \mathcal{P}$

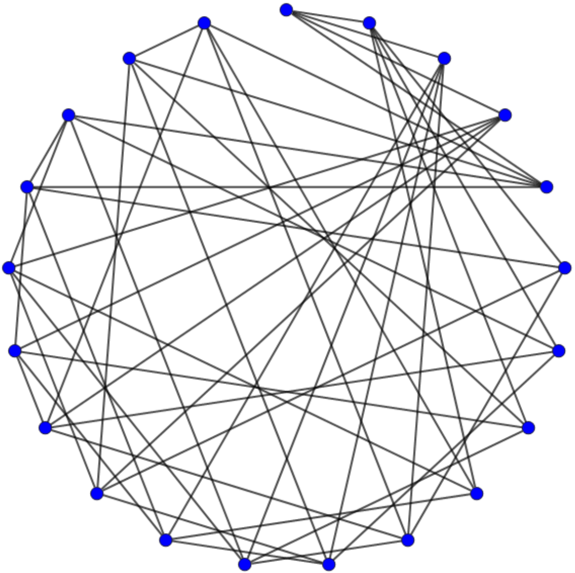
$E(G_\pi) = \{\{p, q\} : p \neq q \in \mathcal{P}, (p, \pi(q)) \in \mathcal{I}\}$.

If $(p, \pi(p)) \in \mathcal{I}$ then p is called an *absolute point*.

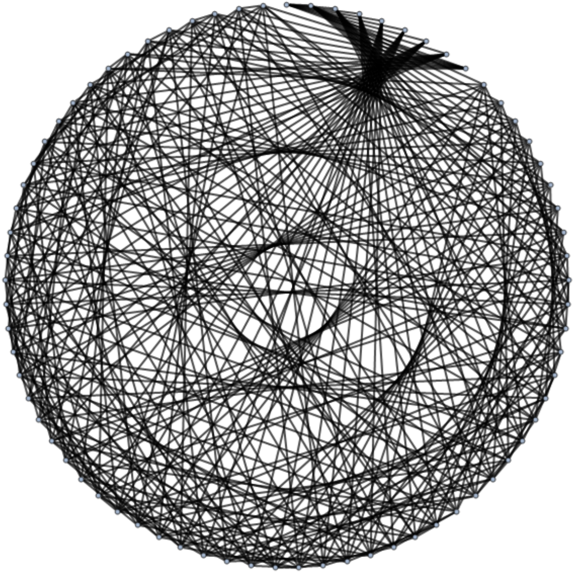
ER_q is the polarity graph obtained by the previous example.

$$(x_0, x_1, x_2) \sim (y_0, y_1, y_2) \Leftrightarrow x_0y_0 + x_1y_1 + x_2y_2 = 0.$$

ER_4



ER_8



Precedent

Sudakov asked about the independence number of ER_q.

- Mubayi and Williford (2007) determined $\alpha(ER_q) = \Theta\left(q^{3/2}\right)$.
- Godsil and Newman (2008) improved the upper bound given by Hoffman's bound. This was refined (2009) using the Lovász theta function.
- Hobart and Williford (2013) gave upper bounds for the independence number of general polarity graphs for q even using coherent configurations.

Abreu, Balbuena, and Labbate (2010) gave an explicit way to construct the adjacency matrix of ER_q using latin squares.

Applications

Polarity graphs have been applied to other areas in mathematics.

- $\mathrm{ex}(n, C_4)$.
- $\mathrm{ex}(n, C_{2k})$ (Lazebnik-Ustimenko-Woldar).
- Constructive lower bounds for Ramsey numbers (Kostochka-Pudlák-Rödl 2010).
- Multicolor Ramsey numbers (Lazebnik-Woldar 2000 and Lazebnik-Mubayi 2003).
- Dense 3-uniform hypergraphs of girth 5 (Lazebnik-Verstraëte 2003).
- Cops and robbers game?

C_4-free graphs

Theorem (Füredi)

For q a prime power

$$\mathrm{ex}(q^2+q+1, C_4) = \frac{1}{2}q(q+1)^2.$$

Moreover, for $q > 13$, any extremal graph is an orthogonal polarity graph of a projective plane.

Theorem (Firke, Kosek, Nash, Williford)

For q a power of 2

$$\mathrm{ex}(q^2+q, C_4) = \frac{1}{2}q(q+1)^2 - q$$

Moreover, for q large enough, the unique extremal graphs are orthogonal polarity graphs with a degree of vertex q deleted.

Lower bounds

McCuaig's Conjecture

Any extremal graph that is C_4 free with $\text{ex}(n, C_4)$ edges is an induced subgraph of some polarity graph.

This suggests that a good way to get lower bounds for $\text{ex}(n, C_4)$ is to take a polarity graph and remove an appropriate set of vertices.

Abreu, Balbuena, and Labbate used this technique successfully to give the best-known lower bounds for $\text{ex}(n, C_4)$ for certain values n.

Lower bounds

Conjecture (Abreu-Balbuena-Labbate 2010)

Let q be a prime power.

$$\mathrm{ex}(q^2 - q - 2, C_4) = \begin{cases} (\frac{1}{2}q - 1)(q^2 - 1) & \text{if } q \text{ is odd;} \\ \frac{1}{2}q^3 - q^2 & \text{if } q \text{ is even.} \end{cases}$$

Theorem (MT and Timmons)

Abreu-Balbuena-Labbate conjecture false for odd q large enough.

Proof: Take a Cayley sum graph of a particular Sidon set. Remove a dense subgraph.

Dense Subgraphs

The denser the subgraph you remove is, the more edges are left over.

This converts the problem of lower bounding $\text{ex}(n, C_4)$ to the problem of finding a dense subgraph in a polarity graph.

Dense Subgraphs

Theorem (MT and Timmons)

Let Π be a projective plane of order q that contains an oval and has a polarity π. If $m \in \{1, 2, ..., q+1\}$, then the polarity graph G_π contains a subgraph on $m + \binom{m}{2}$ vertices that induces at least

$$2\binom{m}{2} + \frac{m^4}{8q} - O\left(\frac{m^4}{q^{3/2}} + m\right)$$

edges.

Corollary

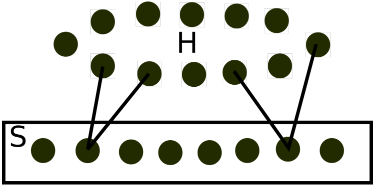
Let q be a prime power. Then

$$\mathrm{ex}(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2 + \frac{3}{2}q - O(q^{1/2}).$$

Proof sketch: preliminaries

Let H be an oval in the projective plane (a set of points of size $q+1$ with no 3 on a line).

- Any two points have a unique common neighbor.
- No vertex has more than 2 neighbors in H.



Let S be the set of vertices secant to H (with exactly 2 neighbors in H).

Proof sketch: preliminaries

Polarity graphs of projective planes are pseudorandom. Given a subset of vertices of size $\delta|V(G)|$, it induces

$$(\delta^2 + o(1))|E(G)|$$

edges.

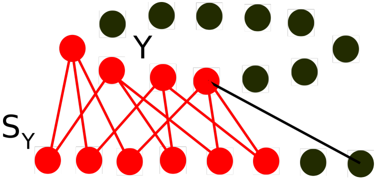
Proof: $A^2 = J + qI$. The eigenvalues of A are $q+1$ and $\pm\sqrt{q}$.

Given a set S, the Expander-Mixing Lemma gives that

$$\left| e(S) - \frac{(q+1)|S|^2}{q^2+q+1} \right| \leq \sqrt{q}|S|.$$

Proof sketch: Building a bipartite subgraph

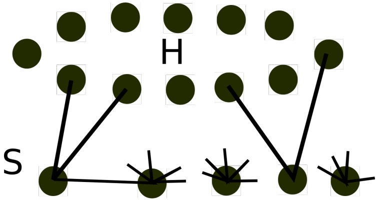
Let $Y \subset H$ and let S_Y be the secants to Y.



Biregular graph with degrees 2 and $|Y| - 1$.
This is not quite enough to disprove the Abreu-Balbuena-Labbate conjecture. To complete the proof, we show that we can choose Y so that S_Y induces enough edges.

Proof sketch: edges in S_Y

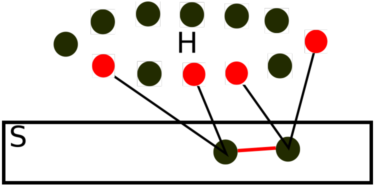
Recall that pseudo-randomness gives that the secants to H induce $(\frac{1}{8} + o(1))q^3$ edges.



Choose Y by randomly selecting m vertices from H.

Proof sketch: edges in S_Y

Given an edge in S, there are at most 4 vertices in H that must be chosen so that the edge is induced by S_Y.



$$\mathbb{P}(e \in E\left(G\left[S_Y\right]\right) \geq \frac{\binom{q-3}{m-4}}{\binom{q+1}{m}}.$$

S_Y induces at least $\frac{m^4}{8q} - O\left(\frac{m^4}{q^{3/2}}\right)$ edges in expectation. □

Difficulties

Can we improve this construction? Can we find a polarity graph with a subgraph that is denser?

There are two situations where we can substantially improve our construction, but they indicate the general problem is difficult.

Difficulties

Let q be a square. The classical projective plane of order q has a subplane of order $\sqrt{q}$, which implies $ER_{\sqrt{q}}$ is a subgraph of ER_q.

$ER_{\sqrt{q}}$ is a graph on $q + \sqrt{q} + 1$ vertices with $\frac{1}{2}\sqrt{q}(\sqrt{q} + 1)^2$ edges. This is much better than our result, which gives a subgraph with the number of edges linear in q.

Difficulties

Our theorem gives

$$\mathrm{ex}(q^2 - q + 1, C_4) \geq \frac{1}{2}q^3 - q^2 + \frac{9}{2}q - O\left(q^{1/2}\right).$$

Let q be a power of 2 and $q-1$ prime (a Mersenne prime). Then ER_{q-1} is a graph on

$$(q-1)^2 + (q-1) + 1 = q^2 - q + 1$$

vertices. That contains

$$\frac{1}{2}(q-1)q^2$$

edges. This improves our bound by a factor of $(\frac{1}{2} + o(1))q^2$.

Future

Problem

Can one find denser subgraphs by studying specific non-desarguesian planes?

$\chi(ER_q)$

Recall that Mubayi and Williford showed that $\alpha\left(ER_q\right) = \Theta\left(q^{3/2}\right)$. This implies that $\chi\left(ER_q\right) = \Omega\left(q^{1/2}\right)$. Is this the correct order of magnitude?

Theorem (Peng, MT, Timmons)

Let $q = p^{2r}$ where p is an odd prime. Then

$$\chi\left(ER_q\right) \leq 2\sqrt{q} + O\left(\sqrt{q}/\log q\right).$$

$\chi(ER_q)$

We had difficulty when q is an odd power of a prime:

Theorem (Peng, MT, Timmons)

Let q be an odd power of an odd prime. If there is a $\mu \in \mathbb{F}_q$ such that $x^{2r+1} - \mu$ is irreducible in $\mathbb{F}_q[x]$, then

$$\chi\left(ER_{q^{2r+1}}\right) \leq \frac{2r+5}{3} q^{\frac{4r}{3}+1} + (2r+1)q^{r+1} + 1.$$

There is also a technical hurdle when q is even.

Conjecture

Let p be a prime. Then

$$\chi\left(ER_{p^{2r+1}}\right) = O\left(p^{r+1}\right).$$

Proof sketch: A large independent set

Most of the graph is isomorphic to

$$(x_1, x_2) \sim (y_1, y_2) \Leftrightarrow (x_1 + y_1)^2 = x_2 + y_2$$

We give an explicit isomorphism that uses 2^{-1}.

Partition $\mathbb{F}^*_{\sqrt{q}}$ into sets $\mathbb{F}^+_{\sqrt{q}}$ and $\mathbb{F}^-_{\sqrt{q}}$ such that

$$a \in \mathbb{F}^+_{\sqrt{q}} \text{ if and only if } -a \in \mathbb{F}^-_{\sqrt{q}}.$$

Let $\theta \in \mathbb{F}_q \setminus \mathbb{F}_{\sqrt{q}}$ and $\mathbb{F}_q = \{a\theta + b : a, b \in \mathbb{F}_{\sqrt{q}}\}$. Then

$$\{(x, y\theta + z) : x, z \in \mathbb{F}_{\sqrt{q}}, y \in \mathbb{F}^+_{\sqrt{q}}\}$$

is an independent set.

$$(x_1 + x_2)^2 \neq (y_1\theta + z_1) + (y_2\theta + z_2).$$

When the exponent is odd, a similar construction is not quite as good.

Many large independent sets

Lemma

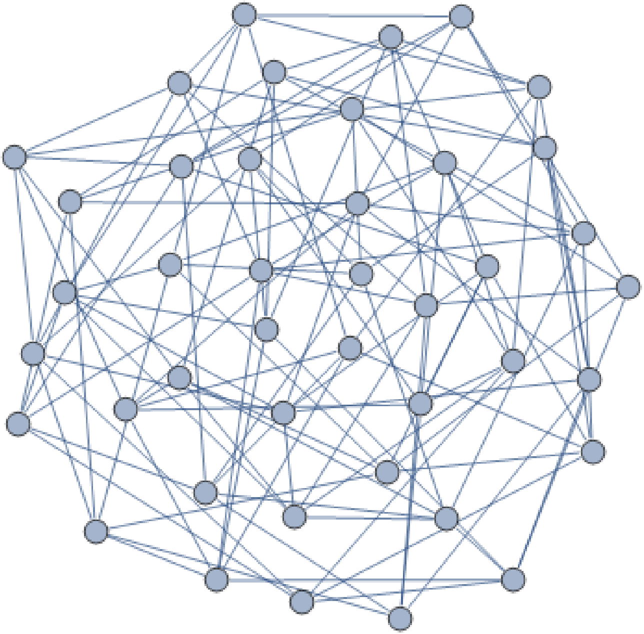
Let $k \in \mathbb{F}^*_{\sqrt{q}}$. Then the map

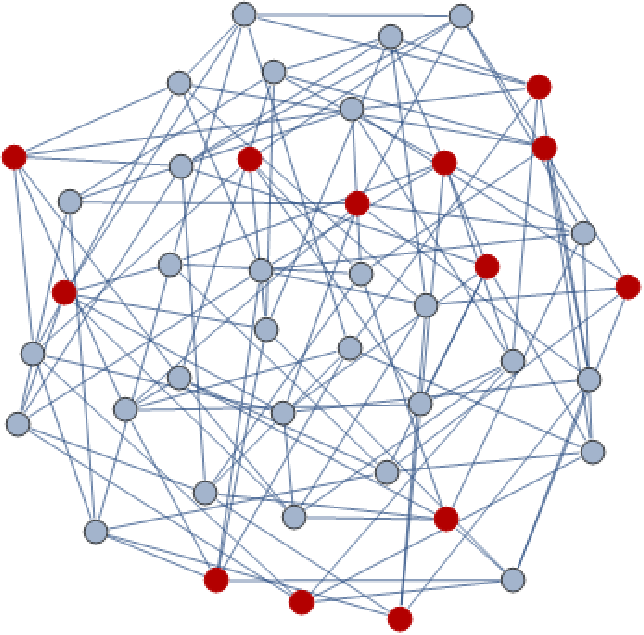
$$\psi_k((x,y)) = (x+k, y+4kx+2k^2)$$

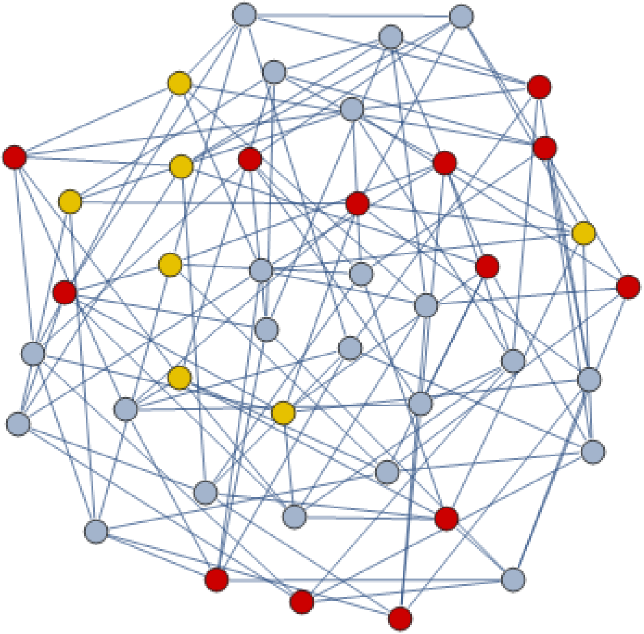
is an automorphism.

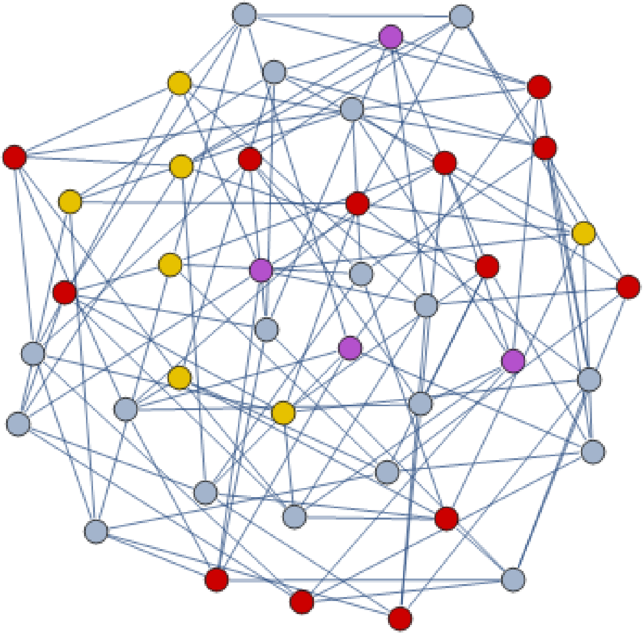
When q is even, the maps ψ_k don't "move" the vertices enough.

Color most of the vertices with these large independent sets.









Completing the coloring

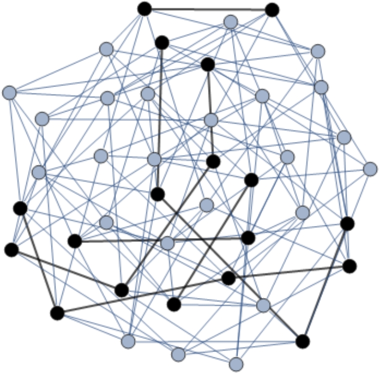
Let H be the graph induced by the uncolored vertices.

Lemma

$$\Delta(H) \leq 2\sqrt{q} - 1.$$

When the exponent is odd we had a very hard time bounding the maximum degree of H.

Completing the coloring



$$\chi(H) = O\left(\frac{\sqrt{q}}{\log q}\right)$$

Open Problems

Problem 1

Give an effective upper bound on $\Delta(H)$ when the exponent is odd.

Problem 2

Find a proof that works when q is even.

Open Problems

Problem 3 (easier)

If G is any polarity graph of a projective plane of order q, determine if

$$\alpha(G) = \Omega\left(q^{3/2}\right)$$

Problem 4 (harder)

If G is any polarity graph of a projective plane of order q, determine if

$$\chi(G) = O\left(q^{1/2}\right).$$

Open Problems: The unitary polarity graph

Let q be a square. The unitary polarity graph U_q has

$$(x_0, x_1, x_2) \sim (y_0, y_1, y_2) \Leftrightarrow x_0 y_0^{\sqrt{q}} + x_1 y_1^{\sqrt{q}} + x_2 y_2^{\sqrt{q}} = 0.$$

The absolute points form an independent set of size $q^{3/2} + 1$. Mubayi and Williford asked what is the order of magnitude of the largest independent set not containing absolute points.

Problem 5 (easier)

Is there an independent set of non-absolute points in U_q of size $\varepsilon q^{3/2}$?

Problem 6 (harder)

Determine if

$$\chi(U_q) \leq C q^{1/2}.$$