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ASYMPTOTIC EXPANSION OF A SERIES OF RAMANUJAN

by BRUCE C. BERNDT* and RONALD J. EVANS

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An asymptotic expansion is given for the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!} \frac{(a+n)^{n-r}}{(x+a+n)^{n+s}}$$

as $x \to \infty$ in the sector $|\operatorname{Arg} x| \leqq \pi/2-\delta$. Here δ, $\operatorname{Re}(a)$, and $\operatorname{Re}(s)$ are positive and r is a positive integer. In the case $a=r=s=1$, this yields the nontrivial result

$$e^x \sum_{k=1}^{\infty} \frac{1}{k^2(1+x/k)^k} - \frac{e^x}{x} = -\frac{2}{x^2} + \frac{16}{3x^3} - \frac{56}{3x^4} + \frac{3712}{45x^5} + 0\left(\frac{1}{x^6}\right)$$

stated by Ramanujan in his notebooks [6].

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1. Introduction

The primary object of this paper is to prove Theorem 1 below, which gives an asymptotic expansion as $x \to \infty$ in the sector $|\operatorname{Arg} x| \leqq \pi/2-\delta$ for the series

$$T(x) := \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!} \frac{(a+n)^{n-r}}{(x+a+n)^{n+s}}, \tag{1.1}$$

where here and in the sequel $\delta > 0$ is fixed and arbitrarily small, r is a fixed positive integer, and a and s are fixed complex numbers with positive real parts.

Theorem 1 *Let N be an integer with $N \geqq 1$. Then as $x \to \infty$ in the sector $|\operatorname{Arg} x| \leqq \pi/2-\delta$,*

$$T(x) = \sum_{k=0}^{r-1} A_k x^{-k-s} - e^{-x}\left(\sum_{m=0}^{N-1} \frac{C_m(x)}{(a+x/2)^{m+1}} + 0(x^{-1-r-N/2})\right), \tag{1.2}$$

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where

$$A_k = \sum_{j=0}^{k} (-1)^{k-j}\,\Gamma(s+k)\binom{k}{j}(a+j)^{k-r}, \tag{1.3}$$

and where the functions $C_m(x)$ (*defined in* (*2.9*)) *have the estimate*

$$C_m(x) = 0(x^{[m/2]-r}). \tag{1.4}$$

Observe that (1.2) is a genuine asymptotic expansion, in view of (1.4). Note also that x can be replaced by $x+b$ in (1.2), for any constant b. Thus, e.g., if the sign of a is reversed in the denominator of (1.1), then $C_m(x)/(a+x/2)^{m+1}$ is replaced by $2^{m+1}C_m(x-2a)/x^{m+1}$.

Theorem 1 was inspired by Ramanujan, who stated the case $r=s=1$ in the unorganized pages of his second notebook [**6**, p. 272, eq. (5)]. Ramanujan found that for $r=s=1$, $C_0(x)=x^{-1}$ and each $C_m(x)$ with $m \geqq 1$ is a polynomial in x such that, for $k \geqq 1$,

$$C_{2k-1}(x) = 0 \tag{1.5}$$

and

$$C_{2k}(x) = \left(-\frac{1}{12}\right)^k \frac{(2k)!}{k!}x^{k-1} - \cdots - \frac{(2k+1)}{2}B_{2k}x - B_{2k}, \tag{1.6}$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers defined by the generating function [**1**, p. 804]

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} \frac{B_m}{m!}t^m, \quad |t| < 2\pi. \tag{1.7}$$

In fact, he computed the first five leading coefficients and the last six trailing coefficients of the polynomials $C_m(x)$ $(m \geqq 1,\ r=s=1)$; see [**6**, pp. 272–273], [**2**]. Ramanujan [**6**, p. 271, eq. (2)] also stated (in a different form) the following:

Corollary 2. *As* $x \to \infty$ *with* $|\operatorname{Arg} x| \leqq \pi/2 - \delta$,

$$\sum_{k=1}^{\infty} \frac{k^{k-2}}{(k+x)^k} = \frac{1}{x} - e^{-x}\left(\frac{2}{x^2} - \frac{16}{3x^3} + \frac{56}{3x^4} - \frac{3712}{45x^5} + 0(x^{-6})\right). \tag{1.8}$$

Proof. This follows from Theorem 1 with $a=r=s=1$ and $N=8$, upon substitution of the values

$$C_2=-\frac{1}{6},\quad C_4=\frac{x}{12}+\frac{1}{30},\quad C_6=-\frac{5x^2}{72}-\frac{x}{12}-\frac{1}{42} \tag{1.9}$$

given, e.g., by (1.6).

Writing the first few terms of the expansion in Theorem 1 in more explicit form, we obtain the following generalization of Corollary 2:

Corollary 3. *As $x\to\infty$ with $|\operatorname{Arg} x|\leqq\pi/2-\delta$,*

$$\sum_{n=0}^{\infty}\frac{(a+n)^{n-1}}{(x+a+n)^{n+1}}=\frac{1}{ax}-e^{-x}$$

$$\times\left(\frac{2}{x^2}-\frac{(12a+4)}{3x^3}+\frac{(24a^2+24a+8)}{3x^4}-\frac{(720a^3+1440a^2+1200a+352)}{45x^5}+0(x^{-6})\right) \tag{1.10}$$

In Corollaries 2 and 3, the asymptotic series are expressed explicitly in descending powers of x. The general asymptotic series in Theorem 1 could also be expressed in this way if an asymptotic expansion could be given for each $C_m(x)$ in descending powers of x. This is indeed possible and we show how this can be accomplished in Section 6. If s is a positive integer, we prove the stronger result that $C_m(x)$ is a Laurent polynomial in x.

Ramanujan [**6**, p. 270, eq. (1)] also found the following interesting exact formula for the series in Corollary 2:

$$\sum_{k=1}^{\infty}\frac{k^{k-2}}{(x+k)^k}=\frac{1}{x}+e^{-x}\left(-\frac{1}{x}+\sum_{k=1}^{\infty}k^{k-2}e^{-k}\sum_{j=1}^{k}\frac{(x+k)^{-j}}{(k-j)!}\right), \tag{1.11}$$

where $\operatorname{Re}(x)>0$. For a proof, see [**2**].

In Section 2, we discuss confluent hypergeometric functions and introduce further notation. The goal of Section 3 is to prove the integral representation (3.8) for $T(x)$. In Section 4, we prove Lemma 4, which provides bounds for the derivatives with respect to t of the function $f(t,x)$ defined in (2.8). The proof of Theorem 1 is given in Section 5 and is based on the results of the previous sections. Finally, in Section 6, we show that $C_m(x)$ possesses an asymptotic expansion in descending powers of x, and that moreover $C_m(x)$ is a Laurent polynomial in x when s is an integer.

2. Confluent hypergeometric functions

Consider the confluent hypergeometric function

$${}_1F_1(s,s+r;\,z)=\sum_{m=0}^{\infty}\frac{(s)_m z^m}{(s+r)_m m!},\quad |z|<\infty, \tag{2.1}$$

with the usual notation

$$(s)_m = \Gamma(s+m)/\Gamma(s), \quad m \geqq 0. \tag{2.2}$$

This function is related to $U(s, s+r; z)$, the confluent hypergeometric function of the second kind, by

$$\begin{aligned} {}_1F_1(s, s+r; z) &= \frac{\Gamma(s+r)}{\Gamma(r)} e^{i\pi s} U(s, s+r; z) \\ &+ \frac{\Gamma(s+r)}{\Gamma(s)} (-1)^r e^z U(r, s+r; -z), \quad \frac{\pi}{2} < \arg z < \frac{3\pi}{2}; \end{aligned} \tag{2.3}$$

see [5, p. 257, eq. (10.09)], [4, p. 270, eq. (9.12.4)]. In many books (e.g., [4, p. 263]), U is designated by Ψ. As $z \to \infty$ with $|\arg z| \leqq 3\pi/2 - \delta$, we have the asymptotic expansion [5, p. 256]

$$U(r, s+r; z) \sim \sum_{m=0}^{\infty} \frac{(-1)^m (r)_m (1-s)_m}{m!\, z^{m+r}}. \tag{2.4}$$

Since r is a positive integer, $U(s, s+r; z)$ can be expressed as a Laguerre polynomial; see [3, p. 189, eq. (14)], [3, p. 188, eq. (7)]. Thus

$$U(s, s+r; z) = \sum_{k=0}^{r-1} \frac{(-1)^k (s)_k (1-r)_k}{z^{k+s}}. \tag{2.5}$$

For brevity, write, for $t \geqq 0$,

$$w = w(t) = t/(1-e^{-t}), \tag{2.6}$$

so that by (1.7),

$$w = \sum_{m=0}^{\infty} \frac{B_m}{m!} (-t)^m, \quad |t| < 2\pi. \tag{2.7}$$

For $t \geqq 0$, $\operatorname{Re}(x) > 0$, define

$$f(t, x) = e^{x(1-w+t/2)} (-t)^{r-1} w^s U(r, s+r; wx). \tag{2.8}$$

Finally, the functions $C_m(x)$ in Theorem 1 are defined by

$$C_m(x) = f^{(m)}(0, x), \quad \operatorname{Re}(x) > 0, \tag{2.9}$$

where the superscript m denotes the mth derivative with respect to t.

We remark that in the case $r=1$,

$$f(t,x)=x^{-s}e^{x+xt/2}\Gamma(s,wx) \tag{2.10}$$

for the incomplete gamma function

$$\Gamma(s,z)=\int_z^\infty e^{-t}t^{s-1}\,dt, \quad \operatorname{Re} s>0. \tag{2.11}$$

This follows from (2.8) and the formula [3, p. 136, eq. (15)]

$$\Gamma(s,z)=e^{-z}z^{s}U(1,s+1;\,z). \tag{2.12}$$

3. Integral representation of the series $T(x)$

Define, for each integer $m \geqq 0$,

$$\mu_m=\sum_{n=0}^{\infty}\frac{\Gamma(m+s+n)}{n!(a+n)^{m+s+r}}. \tag{3.1}$$

From Euler's integral representation of the gamma function,

$$\frac{1}{(a+n)^{m+s+r}}=\frac{1}{\Gamma(m+s+r)}\int_0^\infty e^{-nt-at}\,t^{m+s+r-1}\,dt. \tag{3.2}$$

Thus

$$\mu_m=\frac{\Gamma(m+s)}{\Gamma(m+s+r)}\int_0^\infty e^{-at}t^{m+s+r-1}\sum_{n=0}^{\infty}\frac{\Gamma(m+s+n)}{\Gamma(m+s)n!}e^{-tn}\,dt, \tag{3.3}$$

where absolute convergence justifies the interchange of integration and summation. The sum on n in (3.3) equals $(1-e^{-t})^{-m-s}$, and so

$$\mu_m=\frac{\Gamma(m+s)}{\Gamma(m+s+r)}\int_0^\infty e^{-at}t^{r-1}w^{m+s}\,dt, \tag{3.4}$$

where w is defined in (2.6).

Recall that $T(x)$ is defined in (1.1) for $\operatorname{Re} x>0$. Assuming for the moment that $|x|<|a|$, we find that

$$T(x)=\sum_{n=0}^{\infty}\frac{\Gamma(n+s)}{n!(a+n)^{s+r}}\left(1+\frac{x}{a+n}\right)^{-n-s}$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(n+s)}{n!(a+n)^{s+r}}\sum_{m=0}^{\infty}\frac{\Gamma(m+s+n)}{m!\Gamma(n+s)}\left(\frac{-x}{a+n}\right)^{m}. \tag{3.5}$$

By (3.1) and (3.5),

$$T(x)=\sum_{m=0}^{\infty}\frac{(-x)^m}{m!}\mu_m,\quad |x|<|a|, \tag{3.6}$$

where absolute convergence justifies the interchange of summations. Ramanujan [**6**, p. 271, eq. (3)] gave the case $r=s=1$ of (3.6).

Put (3.4) in (3.6) to obtain for $|x|<|a|$,

$$T(x)=\sum_{m=0}^{\infty}\frac{(-x)^m}{m!}\frac{\Gamma(m+s)}{\Gamma(m+s+r)}\int_0^{\infty}e^{-at}t^{r-1}w^{m+s}\,dt$$

$$=\int_0^{\infty}e^{-at}t^{r-1}w^{s}\sum_{m=0}^{\infty}\frac{(-xw)^m}{m!}\frac{\Gamma(m+s)}{\Gamma(m+s+r)}\,dt, \tag{3.7}$$

where the interchange of integration and summation can be justified by absolute convergence. By (2.1), (2.2), and (3.7),

$$T(x)=\frac{\Gamma(s)}{\Gamma(s+r)}\int_0^{\infty}e^{-at}t^{r-1}w^{s}\,{}_1F_1(s,s+r;\,-wx)\,dt, \tag{3.8}$$

for $|x|<|a|$.

As $x\to\infty$ with $|\operatorname{Arg}x|\leqq\pi/2-\delta$, $-wx\to\infty$ with $\pi/2+\delta\leqq\arg(-wx)\leqq3\pi/2-\delta$. Thus by (2.3)–(2.5), the integral in (3.8) is convergent and analytic in each variable a,x in the right half plane. From (1.1), $T(x)$ is also seen to be analytic in each of a,x in the right half plane. Thus (3.8) holds for all x with $\operatorname{Re}x>0$.

4. Bounds for derivatives of $f(t,x)$

The proof of Lemma 4 below makes heavy use of Faa di Bruno's formula [**7**, p. 36]; [**8**],

$$\frac{d^n}{dt^n}h(g(t))=\sum\frac{n!h_k(g(t))}{k_1!\dots k_n!}\left(\frac{g_1}{1!}\right)^{k_1}\cdots\left(\frac{g_n}{n!}\right)^{k_n}, \tag{4.1}$$

where the sum is over all integers $k_1,k_2,\dots,k_n$ for which

$$n=k_1+2k_2+\cdots+nk_n,\quad k_i\geqq0, \tag{4.2}$$

and where $k=k_1+\cdots+k_n$,

$$h_k(z)=\frac{d^k}{dz^k}\,h(z), \quad \text{and} \quad g_i=g_i(t)=\frac{d^i}{dt^i}\,g(t). \tag{4.3}$$

Lemma 4. *Fix $N\geqq 1$. As $x\to\infty$ with $|\operatorname{Arg} x|\leqq \pi/2-\delta$,*

$$f^{(N)}(t,x)=0\left(x^{-r+[N/2]}\sum_{j=0}^{N}|xt|^j\right), \tag{4.4}$$

uniformly for $t\in[0,1]$.

Proof. Let $0\leqq t\leqq 1$ and $n\geqq 0$. We will obtain uniform estimates for nth derivatives of each factor $(-t)^{r-1}$, w^s, $e^{x(1-w+t/2)}$, and $U(r,s+r;\,wx)$ of $f(t,x)$ in (2.8), and then combine them to deduce (4.4) from Leibniz's rule.

First, for each $n\geqq 0$,

$$\frac{d^n}{dt^n}\,(-t)^{r-1}=0(1), \tag{4.5}$$

since r is a positive integer. Next, by (2.7), we have, for each $k\geqq 0$,

$$\frac{d^k}{dt^k}\,w=0(1). \tag{4.6}$$

Consequently, by (4.1) with $h(z)=z^s$ and $g(t)=w$,

$$\frac{d^n}{dt^n}\,w^s=0(1). \tag{4.7}$$

For $|\operatorname{Arg} z|\leqq \pi/2-\delta$, $U(r,s+r;\,z)$ is analytic (see [**5**, p. 257, eq. (10.04)]) and so by [**5**, pp. 9, 10, Theorem 4.2] we can differentiate in (2.4) to obtain, for $k\geqq 0$ and large $|z|$,

$$\frac{d^k}{dz^k}\,U(r,s+r;\,z)\sim\sum_{m=0}^{\infty}\frac{(r)_{m+k}(-1)^{m+k}(1-s)_m}{m!\,z^{m+r+k}}=0(z^{-k-r}). \tag{4.8}$$

Now apply (4.1) with $h(z)=U(r,s+r;z)$ and $g(t)=wx$ to deduce from (4.6) and (4.8) that as $x\to\infty$ with $|\operatorname{Arg} x|\leqq \pi/2-\delta$,

$$\frac{d^n}{dt^n}\,U(r,r+s;\,wx)=0(x^{-r}), \tag{4.9}$$

uniformly for $0\leqq t\leqq 1$.

A final application of (4.1) with $h(z)=e^{zx}$ and $g(t)=(1+t/2-w)$ yields

$$\frac{d^n}{dt^n}\, e^{x(1+t/2-w)} = e^{xg(t)} \sum B(k_1,\dots,k_n) g_1^{k_1}\dots g_n^{k_n} x^{k_1+\cdots+k_n}, \tag{4.10}$$

where the sum is over integers k_i satisfying (4.2), where the coefficients $B(k_1,\dots,k_n)$ are independent of x, t, and where g_i is defined by (4.3). By (2.7),

$$g(t) = -\sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!}\, t^{2m}, \tag{4.11}$$

and so

$$g_i = O(t) \quad \text{for all odd} \quad i \geqq 1 \tag{4.12}$$

and

$$g_j = O(1) \quad \text{for all} \quad j \geqq 1. \tag{4.13}$$

Since $g(t) \leqq 0$ for $0 \leqq t \leqq 1$,

$$e^{xg(t)} = O(1). \tag{4.14}$$

By (4.2),

$$k_2 + k_4 + k_6 + \cdots \leqq \tfrac{1}{2}(k_1 + 2k_2 + \cdots + nk_n) = n/2. \tag{4.15}$$

Combining (4.10) and (4.12)–(4.15), we see that

$$\frac{d^n}{dt^n}\, e^{x(1+t/2-w)} \ll \sum |x|^{k_1+k_2+\cdots+k_n} t^{k_1+k_3+k_5+\cdots}$$

$$\ll \sum |xt|^{k_1+k_3+k_5+\cdots} |x|^{k_2+k_4+k_6+\cdots} \ll x^{[n/2]} \sum_{i=0}^{n} |xt|^i. \tag{4.16}$$

The result now follows from (4.5), (4.7), (4.9), (4.16) and Leibniz's rule.

5. Proof of Theorem 1

By (2.3) and (3.8),

$$T(x) = A(x) - B(x), \tag{5.1}$$

where

$$A(x)=\frac{\Gamma(s)}{\Gamma(r)}\int_0^\infty e^{-at}t^{r-1}(-w)^s U(s,s+r;\ -wx)\,dt \tag{5.2}$$

and

$$B(x)=\int_0^\infty e^{-at}(-t)^{r-1}\,w^s e^{-wx}\,U(r,s+r;\ wx)\,dt, \tag{5.3}$$

with $\pi/2<\arg(-wx)<3\pi/2$. We first examine $A(x)$, which yields the dominant part of the asymptotic expansion of $T(x)$. Using (2.5) in (5.2), we find that

$$\begin{aligned}A(x)&=\frac{\Gamma(s)}{\Gamma(r)}\sum_{k=0}^{r-1}(-1)^k(s)_k(1-r)_k\int_0^\infty e^{-at}t^{r-1}(-w)^s(-wx)^{-s-k}\,dt\\&=\frac{\Gamma(s)}{\Gamma(r)}\sum_{k=0}^{r-1}\frac{(-1)^k(s)_k(1-r)_k}{x^{s+k}}\int_0^\infty e^{-at}t^{r-k-1}(e^{-t}-1)^k\,dt\\&=\frac{\Gamma(s)}{\Gamma(r)}\sum_{k=0}^{r-1}\sum_{j=0}^{k}\frac{(-1)^j(s)_k(1-r)_k}{x^{s+k}}\binom{k}{j}\int_0^\infty e^{-t(a+j)}t^{r-k-1}\,dt,\end{aligned} \tag{5.4}$$

where we have expanded $(e^{-t}-1)^k$ by the binomial theorem. It follows easily from (5.4) that

$$A(x)=\sum_{k=0}^{r-1}A_k x^{-k-s}, \tag{5.5}$$

in agreement with (1.2) and (1.3).

Now, (1.4) follows by putting $t=0$ in (4.4). Thus, by (1.2), (2.8), (5.1) and (5.3), it remains to show that

$$\int_0^\infty e^{-t(a+x/2)}f(t,x)\,dt=\sum_{m=0}^{N-1}\frac{C_m(x)}{(a+x/2)^{m+1}}+0(x^{-1-r-N/2}). \tag{5.6}$$

By (2.4) and (2.8),

$$f(t,x)\ll e^{x(1-w+t/2)}t^{r-1}w^s(wx)^{-r}, \tag{5.7}$$

and so

$$e^{-t(a+x/2)}f(t,x)\ll e^{-at}e^{x(1-w)}x^{-r}t^{s-1} \tag{5.8}$$

uniformly for $t\geqq 1$. Since

$$1-w<-1/2 \quad \text{for} \quad t\geqq 1, \tag{5.9}$$

it follows from (5.8) that

$$\int_1^\infty e^{-t(a+x/2)} f(t,x)\,dt \ll e^{-x/2}x^{-r}\int_1^\infty e^{-t\operatorname{Re}(a)}t^{\operatorname{Re}(s)-1}\,dt \ll e^{-x/2}. \tag{5.10}$$

In view of (5.6) and (5.10), it remains to show that

$$\int_0^1 e^{-t(a+x/2)} f(t,x)\,dt = \sum_{m=0}^{N-1}\frac{C_m(x)}{(a+x/2)^{m+1}} + 0(x^{-1-r-N/2}). \tag{5.11}$$

Integrating by parts N times, we obtain

$$\int_0^1 e^{-t(a+x/2)} f(t,x)\,dt = \sum_{m=0}^{N-1}\frac{f^{(m)}(0,x)-f^{(m)}(1,x)e^{-(a+x/2)}}{(a+x/2)^{m+1}}$$

$$+(a+x/2)^{-N}\int_0^1 e^{-t(a+x/2)} f^{(N)}(t,x)\,dt. \tag{5.12}$$

By Lemma 4,

$$e^{-(a+x/2)}f^{(m)}(1,x) \ll e^{-(a+x/2)}x^{3m/2} \ll e^{-x/3}. \tag{5.13}$$

Thus, to prove (5.11), it remains to prove that

$$\int_0^1 e^{-t(a+x/2)} f^{(N)}(t,x)\,dt = 0(x^{N/2-r-1}). \tag{5.14}$$

Again by Lemma 4,

$$\int_0^1 e^{-t(a+x/2)} f^{(N)}(t,x) \ll x^{N/2-r}\int_0^1 e^{-t\operatorname{Re}(a+x/2)}\sum_{j=0}^{N}|xt|^j\,dt$$

$$\ll x^{N/2-r}\sum_{j=0}^{N}|x|^j\int_0^\infty e^{-t\operatorname{Re}(a+x/2)}t^j\,dt$$

$$= x^{N/2-r}\sum_{j=0}^{N}\frac{|x|^j j!}{(\operatorname{Re}(a+x/2))^{j+1}}$$

$$\ll x^{N/2-r}\sum_{j=0}^{N}\frac{|x|^j j!}{|x|^{j+1}} \ll x^{N/2-r-1}. \tag{5.15}$$

6. Asymptotic expansion of $C_m(x)$

As promised following Corollary 3, we show here that $C_m(x)$ possesses an asymptotic expansion in descending powers of x.

As in Section 4, we will estimate $C_m(x)=f^{(m)}(0,x)$ by combining Leibniz's rule with formulas for the nth derivatives of $(-t)^{r-1}$, w^s, $e^{x(1-w+t/2)}$, and $U(r,s+r;\,wx)$. The nth derivatives of $(-t)^{r-1}$ and w^s at $t=0$ are constants. Since the function $g(t)$ in (4.10) satisfies $g(0)=0$, the nth derivative of $e^{x(1+t/2-w)}$ at $t=0$ is, by (4.10), a polynomial in x. It remains to show that the nth derivative of $U(r,s+r;\,wx)$ at $t=0$ has an asymptotic expansion in descending powers of x. By (4.1) with $h(z)=U(r,s+r;\,z)$ and $g(t)=wx$, we have

$$\frac{d^n}{dt^n}\,U(r,s+r;\,wx)\bigg|_{t=0}=\sum_{k=0}^{n}E_k x^k\frac{d^k}{dz^k}\,U(r,s+r;\,z)\bigg|_{z=x} \tag{6.1}$$

for some constants E_k. Using the asymptotic formula (4.8) in (6.1), we obtain the desired result.

If s is a positive integer, the stronger result holds that $C_m(x)$ is a Laurent polynomial. To see this, note that when s is an integer,

$$U(r,s+r;\,z)=\sum_{k=0}^{s-1}\frac{(-1)^k(r)_k(1-s)_k}{z^{k+r}} \tag{6.2}$$

by (2.5) with r and s interchanged. Thus, $U(r,s+r;\,z)$ and its derivatives with respect to z are Laurent polynomials in z, and the result follows from (6.1) as before.

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Department of Mathematics
1409 West Green Street
University of Illinois
Urbana, IL 61801

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112