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Chapter 3 of Ramanujan's Second Notebook

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In 1929, G. N. Watson and B. M. Wilson agreed to begin the enormous task of editing Ramanujan's notebooks. Apparently, Wilson was assigned the task of editing Chapters 2–12, while Watson was charged with Chapters 13–21 of the second notebook, which is a revised, enlarged edition of the first notebook. For six years, until his death in 1935, Wilson devoted all of his efforts to this undertaking. Watson worked tirelessly on primarily Chapters 16–21 until the late 1930's when his interest in the project evidently waned.

The main purpose of this paper is to prove and expound upon all of the results set forth by Ramanujan in Chapter 3. The paper draws partly upon the notes which Wilson left. These notes were transferred to Watson upon the death of Wilson. After Watson passed away in 1965, the manuscripts of both Watson and Wilson were donated to Trinity College Library, Cambridge. Chapter 3 is considerably more analytical in character than is Chapter 2 which is more elementary and which has also been edited [9]. Although no combinatorial problems are mentioned in Chapter 3, the contents of this chapter belong under the umbrella of combinatorial analysis, as will be made plain in the sequel. Chapter 3 contains the statements of 86 theorems and formulas. As with Chapter 2, Ramanujan very briefly sketches the proofs of some of his findings in Chapter 3.

It has frequently been declared that much of Ramanujan's early work before he departed for England is the rediscovery of the work of others. Hardy [40, p. 10] estimated that two-thirds of Ramanujan's best work is rediscovery. Indeed, some of the results in Chapter 3 can be traced back to Lambert, Lagrange, Euler, Rothe, Abel, and others. On the other hand, much of Ramanujan's work in Chapter 3 has been rediscovered by others unaware of his work. For example, the single variable Bell polynomials were first thoroughly examined in print by Touchard [61] in 1933 and by Bell [7] in

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1934, but Ramanujan had already discovered many properties of these polynomials in Chapter 3. Also, several other results were rediscovered and considerably generalized by Gould [27–32] in the late 1950s and early 1960s.

The first nine sections of Chapter 3 comprise a total of 45 formulas. The majority of these results involve properties of the Bell numbers and single variable Bell polynomials and are not very difficult to establish.

Entry 10 is enormously interesting and is certainly the most difficult result to prove in this chapter. Ramanujan proposes an asymptotic expansion for a wide class of power series and provides a sketch of his proof. His argument, however, is formal and not mathematically rigorous. He then gives three very intriguing applications of this theorem. Unfortunately, for none of these applications are the hypotheses, implied in his formal argument, satisfied. We shall establish Ramanujan's asymptotic formula under much weaker assumptions than those implied by his argument. Ramanujan's three examples are then seen to be special cases of our theorem. As is to be expected, our method of attack is much different from that of Ramanujan, but since his argument is interesting, we shall provide a sketch of it.

The content of Sections 11–17 is not unrelated to that of Sections 1–9. However, the proofs are somewhat more formidable. The key problem is to express certain series as powers of x, where x is a root of a particular equation. This theme appears to have commenced in the work of Lambert [45] and Euler [24] and has had a fairly long history. Entry 13 is central in Ramanujan's theory and is the ground for several variations in the sequel. Example 1 of Entry 15 is an extremely interesting result. Entries 16 and 17 do not seem to have been expanded upon in the literature and would appear to be a basis for further fruitful research.

ENTRY 1. *Let $f(z)$ be analytic on $|z| < R_1$, where $R_1 > 1$, and let $g(z) = \sum_{k=0}^{\infty} Q_k z^k$ be analytic on $|z| < R_2$, where $R_2 > 0$. Define P_k, $0 \leqslant k < \infty$, by $\sum_{k=0}^{\infty} P_k z^k = e^z g(z)$, where $|z| < R_2$. Suppose that $\sum_{j=0}^{\infty} Q_j \sum_{k=0}^{\infty} (f^{(j+k)}(0)/k!)$ converges and that this repeated summation may be replaced by a summation along diagonals, i.e., $j+k=n$, $0 \leqslant n < \infty$. Then*

$$\sum_{n=0}^{\infty} P_n f^{(n)}(0) = \sum_{n=0}^{\infty} Q_n f^{(n)}(1).$$

Proof. Since $R_1 > 1$, we find from Taylor's theorem that

$$f^{(j)}(1) = \sum_{k=0}^{\infty} \frac{f^{(j+k)}(0)}{k!}, \qquad 0 \leqslant j < \infty.$$

Hence,

$$\sum_{j=0}^{\infty} Q_j f^{(j)}(1) = \sum_{j=0}^{\infty} Q_j \sum_{k=0}^{\infty} \frac{f^{(j+k)}(0)}{k!} = \sum_{n=0}^{\infty} P_n f^{(n)}(0),$$

which can readily be seen from the definition of P_n.

COROLLARY 1. *Suppose that the hypotheses of Entry 1 are satisfied for* $f(z) = (1 + xz)^n$, *where* $|x| < 1$ *and* n *is arbitrary. Then*

$$\sum_{k=0}^{\infty} P_k \frac{x^k}{\Gamma(n-k+1)} = \sum_{k=0}^{\infty} Q_k \frac{x^k(1+x)^{n-k}}{\Gamma(n-k+1)}.$$

Proof. Elementary calculations yield $f^{(k)}(0) = \Gamma(n+1)x^k/\Gamma(n-k+1)$ and $f^{(k)}(1) = \Gamma(n+1)\,x^k(1+x)^{n-k}/\Gamma(n-k+1)$, $0 \leqslant k < \infty$. The desired equality now follows.

Corollary 2 is simply an alternative formulation of Entry 1, and so we shall not bother to state this corollary.

Note that the next entry gives concrete examples for P_k and Q_k in Entry 1. Ramanujan indicates two proofs of Entry 2. The first is purely formal, while the second is more easily made rigorous.

ENTRY 2. *For all complex* x *and* z, *define*

$$\varphi(z) = \sum_{k=1}^{\infty} \frac{x^k}{(z+k-1)(k-1)!}.$$

Then

$$\varphi(z) = e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{z(z+1)(z+2)\cdots(z+k-1)}. \tag{2.1}$$

First Proof. By employing the Maclaurin series for e^x and integrating termwise, Ramanujan gets

$$\varphi(z) = \frac{1}{x^{z-1}} \sum_{k=0}^{\infty} \frac{x^{z+k}}{(z+k)k!} = \frac{1}{x^{z-1}} \int x^{z-1} e^x \, dx$$
$$= e^x \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{z(z+1)(z+2)\cdots(z+k-1)},$$

upon infinitely many integrations by parts.

Second Proof. An easy calculation gives $z\varphi(z)+x\varphi(z+1)=xe^x$. By employing this recursion formula n times, we obtain

$$\varphi(z)-e^x\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{z(z+1)(z+2)\cdots(z+k-1)}$$
$$=\frac{(-1)^n x^n\varphi(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}. \tag{2.2}$$

From the definition of $\varphi(z)$ and Stirling's formula, it is not hard to see that the right side of (2.2) tends to 0 as n tends to ∞.

COROLLARY 1. *Let f satisfy the hypotheses of Entry 1. Then for all z,*

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{(z+k)k!}=\sum_{k=0}^{\infty}\frac{(-1)^k f^{(k)}(1)}{z(z+1)\cdots(z+k)}.$$

Proof. Use the functions of Entry 2 in Entry 1, and the desired result immediately follows.

COROLLARY 2. *For each complex number x, we have*

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)\frac{x^k}{k!}=e^x\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k!\,k}.$$

Proof. In (2.1) replace x by $-x$ and z by $z+1$ to get

$$\sum_{k=1}^{\infty}\frac{x^k}{(z+1)(z+2)\cdots(z+k)}=e^x\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{(z+k)(k-1)!}. \tag{2.3}$$

Now differentiate both sides of (2.3) with respect to z and then set $z=0$ to achieve the desired formula.

The function

$$\psi(z)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{(z+1)(z+2)\cdots(z+k)},\qquad x\neq 0, \tag{3.1}$$

is a meromorphic function of z with simple poles at $z=-k$, $1\leqslant k<\infty$, and thus has an essential singularity at ∞. For each fixed x, $\psi(z)$ is an inverse factorial series and has its abscissa of convergence equal to $-\infty$. Thus, the series also represents the function asymptotically as z tends to ∞ in the region

$$R_\varepsilon=\{z\colon -\pi+\varepsilon\leqslant\arg z\leqslant\pi-\varepsilon\},\qquad \varepsilon>0. \tag{3.2}$$

In Entry 3, Ramanujan obtains a second asymptotic expansion for $\psi(z)$ valid in R_ε. To describe this expansion, first define, after Ramanujan,

$$f_{-1}(x) \equiv 1, \qquad e^x f_n(x) = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}, \tag{3.3}$$

where x is any complex number and n is a nonnegative integer. In Entry 3, Ramanujan shows that as z tends to ∞ in R_ε,

$$\psi(z) \sim \sum_{k=0}^{\infty} \frac{(-1)^{k-1} f_{k-1}(x)}{z^k}. \tag{3.4}$$

The series in (3.4) is divergent for all values of $x \neq 0$ and z, as can be seen directly from (3.3). It is curious that Ramanujan makes no distinction between this expansion for $\psi(z)$ and the convergent expansion for $\psi(z)$ given in (3.1). Entry 3 is readily seen to be a special case of Example 1 in Section 8, and so we shall defer the proof of Entry 3 until then.

ENTRY 3. *Let ψ and f_n be defined by* (3.1) *and* (3.3), *respectively. Then as z tends to ∞ in R_ε,* (3.4) *holds.*

ENTRY 4. *Let a and x be arbitrary complex numbers. Then*

$$e^{x(e^a-1)} = \sum_{n=0}^{\infty} \frac{a^n}{n!} f_{n-1}(x), \tag{4.1}$$

where $f_n(x)$ is defined by (3.3).

Proof. We have

$$\begin{aligned} e^{xe^a} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x^m (ma)^n}{m!\, n!} \\ &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{x^m m^{n-1}}{(m-1)!} = \sum_{n=0}^{\infty} \frac{a^n}{n!} e^x f_{n-1}(x), \end{aligned}$$

from which the desired result follows.

ENTRY 5. *For each nonnegative integer n,*

$$f_n(x) = x \sum_{k=0}^{n} \binom{n}{k} f_{k-1}(x). \tag{5.1}$$

Proof. Differentiate both sides of (4.1) with respect to a to obtain

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} f_n(x) = xe^a e^{x(e^a-1)} = xe^a \sum_{k=0}^{\infty} \frac{a^k}{k!} f_{k-1}(x). \tag{5.2}$$

If we now equate coefficients of a^n on the extremal sides of (5.2), we readily deduce (5.1).

It is clear from the recursion formula (5.1) that $f_n(x)$ is a polynomial of degree $n+1$ with integral coefficients. Furthermore, for $n \geqslant 0$, $f_n(0)=0$. Thus, following Ramanujan, we define integers $\varphi_1(n),..., \varphi_{n+1}(n)$, $0 \leqslant n < \infty$, by

$$f_n(x) = \sum_{k=1}^{n+1} \varphi_k(n)\, x^k. \tag{6.1}$$

The polynomials $f_n(x)$ appear to have been first systematically studied in the literature by Touchard [61] in 1933 and Bell [7] in 1934, although there is an early reference to these polynomials in Bromwich's book [11, p. 195]. They are now called single variable Bell polynomials and are most often designated by $\varphi_n(x) = f_{n-1}(x)$, $n \geqslant 0$. Touchard [61, 62] and Carlitz [15] have studied these polynomials in detail and have established many arithmetic properties for them. Actually, Bell [7] introduced a much more general class of polynomials, now called Bell polynomials. In addition to Bell's papers [7, 8], extensive discussions of Bell polynomials may be found in the books of Riordan [54] and Andrews [4] which also describe combinatorial applications of Bell polynomials. The coefficients $\varphi_k(n)$ are Stirling numbers of the second kind. In the most frequent contemporary notation, $\varphi_k(n) = S(n+1, k)$. The recursion formula (5.1) is now well known as are the properties of the Stirling numbers of the second kind found in the next three entries.

ENTRY 6. *Let n be a nonnegative integer and let r be a positive integer with $r \leqslant n+1$. Then*

$$\sum_{k=0}^{r-1} \frac{\varphi_{r-k}(n)}{k!} = \frac{r^n}{(r-1)!}.$$

Proof. From (6.1) and (3.3),

$$e^x \sum_{k=1}^{n+1} \varphi_k(n)\, x^k = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}. \tag{6.2}$$

Equating coefficients of x^r on both sides above, we achieve the sought equality.

ENTRY 7. *Let r and n be nonnegative integers with $r \leqslant n$. Then*

$$r!\,\varphi_{r+1}(n) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} (r+1-k)^n.$$

Proof. Multiply both sides of (6.2) by e^{-x} and equate coefficients of x^{r+1} on both sides to reach the desired conclusion.

ENTRY 8. *Let n and r be integers such that $1 \leqslant r \leqslant n+1$. Then*

$$\varphi_r(n+1) = r\varphi_r(n) + \varphi_{r-1}(n),$$

where $\varphi_0(n) = 0$.

Proof. By Entry 7,

$$\begin{aligned}
&\varphi_r(n+1) - \varphi_{r-1}(n)\\
&\quad = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} (r-k)^{n+1}\\
&\qquad - \frac{1}{(r-2)!} \sum_{k=1}^{r-1} (-1)^{k-1} \binom{r-2}{k-1} (r-k)^n\\
&\quad = \frac{1}{(r-1)!} \left\{ r^{n+1} + \sum_{k=1}^{r-1} (-1)^k \right.\\
&\qquad \left. \times \left\{ (r-k)\binom{r-1}{k} + (r-1)\binom{r-2}{k-1} \right\} (r-k)^n \right\}\\
&\quad = \frac{1}{(r-1)!} \left\{ r^{n+1} + r \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} (r-k)^n \right\} = r\varphi_r(n).
\end{aligned}$$

Ramanujan next indicates that the recursion formula in Entry 8 can be employed to calculate $f_n(x)$. In the following corollary, Ramanujan gives $f_n(x)$, $0 \leqslant n \leqslant 6$. In a corollary after Entry 5, Ramanujan inexplicitly indicates that the calculus of finite differences in conjunction with Entry 5 can also be used to calculate $f_n(x)$. Since this is now very well known, we shall forego any further calculations and be content with merely exhibiting the first seven polynomials.

COROLLARY.

$$\begin{aligned}
f_0(x) &= x,\\
f_1(x) &= x + x^2,\\
f_2(x) &= x + 3x^2 + x^3,
\end{aligned}$$

$$f_3(x) = x + 7x^2 + 6x^3 + x^4,$$
$$f_4(x) = x + 15x^2 + 25x^3 + 10x^4 + x^5,$$
$$f_5(x) = x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6,$$
$$f_6(x) = x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7.$$

EXAMPLE 1. Let $\varphi_1(n),..., \varphi_{n+1}(n)$, $0 \leqslant n < \infty$, be defined by (6.1). Let $\{a_k\}$, $1 \leqslant k < \infty$, be any sequence of complex numbers such that

$$\psi(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{(z+1)(z+2) \cdots (z+k)}$$

has abscissa of convergence $\lambda < \infty$. Define, for each nonnegative integer j, $F(j) = \sum_{k=1}^{j+1} a_k \varphi_k(j)$. Let R_ε be defined by (3.2) if $\lambda = -\infty$, but if λ is finite, let

$$R_\varepsilon = \{z: -\tfrac{1}{2}\pi + \varepsilon \leqslant \arg(z - \lambda) \leqslant \tfrac{1}{2}\pi - \varepsilon\}, \qquad \varepsilon > 0.$$

Then as z tends to ∞ in R_ε,

$$\psi(z) \sim \sum_{k=0}^{\infty} \frac{(-1)^k F(k)}{z^{k+1}}.$$

Proof. Using a well-known generating function for Stirling numbers of the second kind [3, Formula 24.1.4B, p. 824], we have

$$\frac{1}{(z+1)(z+2) \cdots (z+k)} = \sum_{j=k}^{\infty} \frac{(-1)^{j+k} \varphi_k(j-1)}{z^j}$$
$$= \sum_{j=k}^{n} \frac{(-1)^{j+k} \varphi_k(j-1)}{z^j} + O(z^{-n-1})$$

as z tends to ∞ in R_ε. Hence,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1} a_k}{(z+1)(z+2) \cdots (z+k)}$$
$$= \sum_{k=1}^{n} (-1)^{k-1} a_k \sum_{j=k}^{n} \frac{(-1)^{j+k} \varphi_k(j-1)}{z^j} + O(z^{-n-1})$$
$$= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{z^j} \sum_{k=1}^{j} a_k \varphi_k(j-1) + O(z^{-n-1})$$
$$= \sum_{j=1}^{n} \frac{(-1)^{j-1} F(j-1)}{z^j} + O(z^{-n-1}). \qquad (8.1)$$

This completes the proof.

EXAMPLE 2. Let r and n be integers with $0 \leqslant r \leqslant n$. Then $r!\varphi_{r+1}(n)$ is the coefficient of $x^n/n!$ in the Maclaurin series of $e^x(e^x-1)^r$.

Proof. By Entry 7, $r!\,\varphi_{r+1}(n)$ is the coefficient of $x^n/n!$ in the expansion of

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} e^{x(r+1-k)} = e^x(e^x-1)^r,$$

and the proof is complete.

EXAMPLE 3. Let n be a positive integer. Then

$$f'_{n-1}(x) = \sum_{k=0}^{n-1} \binom{n}{k} f_{k-1}(x).$$

Proof. From Entry 4,

$$\sum_{n=0}^{\infty} \frac{a^{n+1}}{(n+1)!} f'_n(x) = (e^a-1)\,e^{x(e^a-1)} = (e^a-1) \sum_{k=0}^{\infty} \frac{a^k}{k!} f_{k-1}(x).$$

Equating the coefficients of a^n on both sides, we complete the proof.

Both Examples 2 and 3 are well known.

The next example is the first of many entries in the second notebook that involves the Bernoulli numbers B_n, $0 \leqslant n < \infty$. Ramanujan defines the Bernoulli numbers by

$$\frac{x}{e^x-1} = 1 - \frac{1}{2}x + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} x^{2n}, \qquad |x| < 2\pi.$$

However, today the Bernoulli numbers are more commonly defined by [3, p. 804]

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \qquad |x| < 2\pi, \tag{8.2}$$

and so the latter convention shall be employed here. Moreover, generally, Ramanujan's formulas are more easily stated in the notation (8.2).

EXAMPLE 4. Let n denote an integer greater than or equal to -1. Then

$$\int_0^x f_n(t)\,dt = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k} f_k(x)}{k+1}.$$

Proof. Replace x by t in (4.1) and integrate both sides over $0 \leqslant t \leqslant x$ to obtain, for $|a| < 2\pi$,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a^n}{n!} \int_0^x f_{n-1}(t)\, dt &= \frac{1}{e^a - 1} \{e^{x(e^a - 1)} - 1\} \\ &= \frac{1}{a} \sum_{j=0}^{\infty} \frac{B_j a^j}{j!} \sum_{k=1}^{\infty} \frac{a^k}{k!} f_{k-1}(x) \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{B_{n+1-k} f_{k-1}(x)}{k} \right\} \frac{a^n}{n!},\end{aligned} \tag{8.3}$$

where we have utilized (8.2). Now equate the coefficients of a^{n+1} on the extremal sides of (8.3) to obtain the desired result.

Example 4 was incorrectly stated by Ramanujan. On the right side of his equality, replace n by $n + 1$ everywhere except in the suffixes.

EXAMPLE 5i. For each nonnegative integer n, define A_n by

$$eA_n = ef_n(1) = \sum_{k=1}^{\infty} \frac{k^n}{(k-1)!}. \tag{8.4}$$

Then $A_0 = 1$, $A_1 = 2$, $A_2 = 5$, $A_3 = 15$, $A_4 = 52$, $A_5 = 203$, $A_6 = 877$, $A_7 = 4140$, and $A_8 = 21147$.

Proof. It is not difficult to show that, with $x = 1$, (5.1) can be equivalently expressed by means of difference notation in the form

$$\Delta^n A_n = A_{n-1}, \qquad n \geqslant 1. \tag{8.5}$$

Since $A_0 = 1$, (8.5) can be used to construct a difference table in order to calculate A_n.

The numbers A_n are now called Bell numbers with the nth Bell number $B(n)$ being defined by $B(n) = A_{n-1}$, $n \geqslant 1$. Combinatorially, $B(n)$ is the number of ways of partitioning a set of n elements. According to Gould [34], the earliest known application of these numbers is in an edition of the Japanese Tale of Genji published in the 17th century. The numbers $B(n)$ arose as the number of ways of arranging n incense sticks. As another application, $B(n)$ is the number of ways of rhyming a stanza of n lines [12]. The first explicit appearance of these numbers apparently is in a paper of Kramp [43] in 1796. They are also found in a treatise of Tate [60, p. 45] published in 1845. The formula (8.4) appears as a problem in the *Matematicheskii Sbornik* [51] in 1868. In 1877, Dobinski [19] used (8.4) to calculate $B(1), \dots, B(8)$. In 1885, Cesàro [17] found the numbers to be solutions of the difference equation (8.5). Again, in connection with (8.4), the numbers appear in problems in the texts of Hardy [37, p. 424] and Bromwich [11, p. 197]. Touchard [61, 62], Bell [7], Browne [12], Williams [63], Ginsburg [26], and Balasubrahmanian [6] have established several elementary properties and give lists of varying lengths of the Bell numbers. Carlitz [16] has written a nice paper on Bell numbers, Stirling numbers of the second kind, and some generalizations. For references to other papers of Carlitz on this subject see [16]. Levine and Dalton [46] have calculated the first 74 Bell numbers. Obviously, $B(n)$ grows very rapidly, and Epstein [23] has found an asymptotic formula for $B(n)$. He has also discovered other analytic properties of the Bell numbers, for example, integral representations. For the numbers listed in Example 5i, A_n is even if $n \equiv 1 \pmod 3$, and A_n is odd otherwise. This property persists, and a simple proof of it can be found in the paper of Balasubrahmanian [6]. Actually, more general congruences are known; see [62, 63], for example. The Bell numbers have been rediscovered by many authors and we have listed but a small portion of those papers in which properties of the Bell numbers are proved and combinatorial applications are given. For further references, readers should consult Gould's extremely comprehensive bibliography [34].

EXAMPLE 5ii. For each nonnegative integer n, define C_n by

$$\frac{C_n}{e} = \sum_{k=1}^{\infty} \frac{(-1)^k k^n}{(k-1)!}.$$

Then $C_0 = -1$, $C_1 = 0$, $C_2 = C_3 = 1$, $C_4 = -2$, $C_5 = C_6 = -9$, $C_7 = 50$, and $C_8 = 267$.

Proof. Observe that $C_n = f_n(-1)$, $n \geqslant 0$. Thus, from (5.1) it is readily shown that $\Delta^n C_n = -C_{n-1}$, $n \geqslant 1$. Using this difference equation and the initial value $C_0 = -1$, we may compose a difference table to calculate C_n.

The equalities in the next example are easily verified from Examples 5i and 5ii.

EXAMPLE 6.

(i) $f_3(1) = 3f_2(1) = 15$,

(ii) $f_5(1) + f_2(1) = 4f_4(1) = 208$,

(iii) $f_3(-1) = f_2(-1) = 1$,

(iv) $f_6(-1) = f_5(-1) = -9$,

(v) $f_8(-1) + f_6(-1) + f_5(-1) + f_3(-1) = 5f_7(-1) = 250$.

Let x, a, and b be complex numbers, and let n be a nonnegative integer. Generalizing $f_n(x)$, we define

$$e^x F_n(a, b; x) = e^x F_n(x) = \sum_{k=1}^{\infty} \frac{(a+bk)^n x^k}{(k-1)!}. \tag{9.1}$$

Thus, $f_n(x)$, defined by (3.3), is the particular case of $F_n(x)$ which is obtained by putting $a=0$ and $b=1$. Moreover, $F_n(x)$ can readily be expressed in terms of $f_0(x),..., f_n(x)$, since

$$\begin{aligned} e^x F_n(x) &= \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j k^j \\ &= e^x \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j f_j(x). \end{aligned} \tag{9.2}$$

Expressed in a slightly different way, Entry 9i is a generalization of Entry 3.

ENTRY 9i. *As z tends to* ∞ *in* R_ε, *where* R_ε *is defined by* (3.2),

$$\sum_{k=0}^{n} \frac{(-1)^k F_k(x)}{z^{k+1}} = \sum_{k=1}^{n+1} \frac{(-b)^{k-1} x^k}{(z+a+b)(z+a+2b)\cdots(z+a+kb)} + O(z^{-n-2}).$$

Proof. By Taylor's theorem,

$$\sum_{k=j}^{n} \binom{k}{j} \frac{(-1)^{k+j} a^k}{z^{k+1}} = \frac{a^j}{z^{j+1}} \left(1 + \frac{a}{z}\right)^{-j-1} + O(z^{-n-2}), \tag{9.3}$$

as z tends to ∞. Thus, by (9.2), (9.3), and (8.1).

$$\begin{aligned} \sum_{k=0}^{n} \frac{(-1)^k F_k(x)}{z^{k+1}} &= \sum_{k=0}^{n} \frac{(-1)^k}{z^{k+1}} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} b^j f_j(x) \\ &= \sum_{j=0}^{n} (-b/a)^j f_j(x) \sum_{k=j}^{n} \binom{k}{j} \frac{(-1)^{j+k} a^k}{z^{k+1}} \\ &= \sum_{j=0}^{n} \frac{(-b)^j}{(z+a)^{j+1}} f_j(x) + + O(z^{-n-2}) \\ &= \frac{1}{b} \sum_{k=1}^{n+1} \frac{(-1)^{k-1} x^k}{\left(\frac{z+a}{b}+1\right)\left(\frac{z+a}{b}+2\right)\cdots\left(\frac{z+a}{b}+k\right)} + O(z^{-n-2}), \end{aligned}$$

from which the desired asymptotic formula follows.

The next entry generalizes Entry 4.

ENTRY 9ii. *If a, b, x, and y are complex numbers, we have*

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} F_n(x) = xe^{(a+b)y} e^{x(e^{by}-1)}. \tag{9.4}$$

Proof. By (9.2),

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{y^n}{n!} F_n(x) &= \sum_{n=0}^{\infty} \frac{y^n}{n!} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k f_k(x) \\
&= \sum_{k=0}^{\infty} (b/a)^k f_k(x) \sum_{n=k}^{\infty} \binom{n}{k} \frac{(ay)^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(by)^k e^{ay}}{k!} f_k(x).
\end{aligned}$$

Applying (5.2), with by in place of a, we complete the proof.

ENTRY 9iii. *For each nonnegative integer n, we have*

$$F_{n+1}(x) - (a+b) F_n(x) = bx \sum_{k=0}^{n} \binom{n}{k} b^k F_{n-k}(x).$$

Proof. Differentiating both sides of (9.4) with respect to y, we find that

$$\sum_{n=1}^{\infty} \frac{y^{n-1}}{(n-1)!} F_n(x) = (a + b + bxe^{by}) \sum_{n=0}^{\infty} \frac{y^n}{n!} F_n(x).$$

On equating coefficients of y^n on both sides, we finish the proof.

Entry 9iii is obviously an analog of Entry 5. After Entry 9iv, Ramanujan indicates very briefly how to express Entry 9iii in terms of differences. For each nonnegative integer n, define

$$\psi_n(x) = F_{n+1}(x) - (a+b) F_n(x) = bx(b+F)^n,$$

where in the expansion of $(b+F)^n$, F^k is to be interpreted as meaning $F_k(x)$. Next, define an operator δ by $\delta g(n) = g(n) - bg(n-1)$. So,

$$\delta\psi_n = \psi_n - b\psi_{n-1} = bx(b+F)^{n-1} F.$$

By inducting on k, it can easily be shown that

$$\delta^k \psi_n = bx(b+F)^{n-k} F^k, \qquad 0 \leqslant k \leqslant n.$$

In particular,

$$\delta^n \psi_n = bxF^n = bxF_n(x).$$

Since $F_0(x) = x$, it follows from Entry 9iii, or from (9.2), or from the preceding paragraph, that $F_n(x)$ is a polynomial in x of degree $n+1$. Moreover, $F_n(0) = 0$. Hence, we define $\varphi_1(n),..., \varphi_{n+1}(n)$, $n \geqslant 0$, by

$$F_n(x) = \sum_{k=1}^{n+1} \varphi_k(n)\, x^k. \tag{9.5}$$

The next four results generalize Entries 6–8 and Example 2 of Section 8, respectively. The proofs are completely analogous, and so we omit them.

ENTRY 9iv. *Suppose that r and n are integers such that $0 < r \leqslant n+1$. Then*

$$\sum_{k=0}^{r-1} \frac{\varphi_{r-k}(n)}{k!} = \frac{(a+br)^n}{(r-1)!}.$$

ENTRY 9v. *Let r and n be as in Entry 9iv. Then*

$$(r-1)!\, \varphi_r(n) = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \{a + (r-k)b\}^n.$$

ENTRY 9vi. *Let n and r be integers such that $1 \leqslant r \leqslant n+1$. Then*

$$\varphi_r(n+1) = (a+br)\, \varphi_r(n) + b\varphi_{r-1}(n),$$

where $\varphi_0(n) = 0$.

Using Entry 9vi, Ramanujan next calculates $F_n(x)$, $1 \leqslant n \leqslant 4$. Thus,

$$\begin{aligned}
F_0(x) &= x,\\
F_1(x) &= (a+b)x + bx^2,\\
F_2(x) &= (a+b)^2\, x + b(2a+3b)\, x^2 + b^2x^3,\\
F_3(x) &= (a+b)^3\, x + b\{3(a+b)(a+2b) + b^2\}\, x^2\\
&\quad + 3b^2(a+2b)\, x^3 + b^3x^4,\\
F_4(x) &= (a+b)^4\, x + b(2a+3b)\{2(a+b)(a+2b) + b^2\}\, x^2\\
&\quad + b^2\{6(a+2b)^2 + b^2\}\, x^3 + 2b^3(2a+5b)\, x^4 + b^4x^5.
\end{aligned} \tag{9.6}$$

ENTRY 9vii. *Let r and n be integers with $0 \leqslant r \leqslant n$. Then $r!\, \varphi_{r+1}(n)$ is the coefficient of $x^n/n!$ in the Maclaurin series of $e^{(a+b)x}(e^{bx} - 1)^r$.*

EXAMPLE i.

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^3+(2k+1)^2}{k!} = 0.$$

Proof. From (9.1), we have

$$\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^3+(2k+1)^2}{k!} = -\frac{1}{e}\{F_3(-1,2;-1)+F_2(-1,2;-1)\}. \tag{9.7}$$

But from (9.6), $F_2(-1,2;x) = x+8x^2+4x^3$ and $F_3(-1,2;x) = x+26x^2+36x^3+8x^4$. Hence, $F_2(-1,2;-1) = 3 = -F_3(-1,2;-1)$. Using these values in (9.7), we complete the proof.

EXAMPLE ii.

$$\sum_{k=1}^{\infty} \frac{k^4}{(k-1)!} = 4\sum_{k=0}^{\infty} \frac{(2k+1)^2}{k!}.$$

Proof. The left side above is $52e$ by Example 5i of Entry 8. The right side above is $4eF_2(-1,2;1)$ by (9.1). But from the previous proof, $F_2(-1,2;1) = 13$, and so the proof is complete.

EXAMPLE iii.

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^7+k^6}{(k-1)!} = \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)^4}{(k-1)!}.$$

Proof. By Example 5ii of Entry 8, the left side above is $-41/e$. Now, by (9.6), $F_4(-1,2;x) = x+80x^2+232x^3+128x^4+16x^5$. Thus, the right side above is $F_4(-1,2;-1)/e = -41/e$.

Example iii must be corrected in the second notebook by multiplying either side of the equality by -1. Example iv must be corrected in the second notebook by replacing -4 on the right side of the equality by -8.

EXAMPLE iv.

$$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^4}{k!\,k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k} - 8.$$

Proof. We have

$$\sum_{k=1}^{\infty} (-1)^k \frac{(2k+1)^4}{k!\,k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k} + \sum_{k=1}^{\infty} (-1)^k \frac{8+24k+32k^2+16k^3}{k!}$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k} - \frac{1}{e}\{8F_0(-1,1;-1) + 24F_1(-1,1;-1)$$
$$+ 32F_2(-1,1;-1) + 16F_3(-1,1;-1)\} - 8. \tag{9.8}$$

Now, from (9.6), $F_0(x)=x$, $F_1(x)=x^2$, $F_2(x)=x^2+x^3$, and $F_3(x)=x^2+3x^3+x^4$ when $-a=1=b$. Thus, the expression in curly brackets on the right side of (9.8) is equal to 0. This complete the proof.

Some properties of $F_n(a-1, 1; x)$ have been derived by Manikarnikamma [47].

In preparation for Entry 10, we first define a sequence of nonnegative integers b_{kn}, $k \geqslant 2$, by the equalities

$$b_{kk} = 1,$$
$$b_{kn} = 0, \qquad \text{for} \quad n < k \text{ or } n > 2k-2,$$

and

$$b_{k+1,n+1} = nb_{k,n-1} + (n-k+1)\,b_{kn}, \qquad \text{for} \quad k \leqslant n \leqslant 2k-1. \tag{10.1}$$

A short list of values for b_{kn} is provided in Table I. In fact, $b_{kn} = S_2(n, n+1-k)$, where $S_2(n,k)$ is the 2-associated Stirling number of the second kind [18, pp. 221–222; 54, pp. 74–78].

TABLE I

$k \backslash n$	2	3	4	5	6	7	8	9	10	11	12
2	1										
3		1	3								
4			1	10	15						
5				1	25	105	105				
6					1	56	490	1260	945		
7						1	119	1918	9450	17325	10395

ENTRY 10. *Let $\varphi(x)$ denote a function of at most polynomial growth as x (real) tends to ∞. Suppose that there exists a constant $A \geqslant 1$ and a function $G(x)$ of at most polynomial growth as x tends to ∞ such that for each nonnegative integer m and all sufficiently large x, the derivatives $\varphi^{(m)}(x)$ exist and satisfy*

$$|\varphi^{(m)}(x)/m!| \leqslant G(x)(A/x)^m. \tag{10.2}$$

Put

$$\varphi_\infty(x) = e^{-x} \sum_{k=0}^{\infty}{}' \frac{x^k \varphi(k)}{k!},$$

where the prime on the summation sign indicates that the (finitely many) terms for which $\varphi(k)$ may be undefined are not included in the sum. Then for any fixed positive integer M,

$$\varphi_\infty(x) = \varphi(x) + \sum_{k=2}^{M} \sum_{n=k}^{2k-2} b_{kn} x^{n-k+1} \frac{\varphi^{(n)}(x)}{n!} + O(G(x)\, x^{-M}), \tag{10.3}$$

as x tends to ∞, where the numbers b_{kn} are defined by (10.1).

Before embarking upon a proof of Entry 10, we offer several comments. Examples of functions φ satisfying the conditions of Entry 10 are functions of polynomial growth that are analytic in some right half plane. This follows from Cauchy's integral formula for derivatives. Specific examples will be given upon the conclusion of the proof of Entry 10.

The following result is related to Entry 10. If φ is bounded and continuous on $[0, \infty)$, then [25, pp. 219, 227]

$$e^{-x} \sum_{k=0}^{\infty} \frac{x^k \varphi(k)}{k!} \sim \varphi(x)$$

as x tends to ∞. Observe that the left side above is the expected value $E(\varphi(U))$, where U is a random variable with Poisson distribution of mean x.

The asymptotic formula above has a superficial resemblance to Borel summability [38, pp. 79–80]. However, it is doubtful that Ramanujan was influenced by this. In particular, nothing in the foregoing material of the second notebook pertains to Borel summability.

Formula (10.3) is a more precise version of the formula that Ramanujan gives in his Entry 10. He provides a very brief sketch of his formal "proof" of Entry 10, and because it is instructive, we shall give it below.

Ramanujan tacitly assumes that φ is an entire function. Hence,

$$\begin{aligned} e^x \varphi_\infty(x) &= \sum_{k=0}^{\infty} \frac{x^k \varphi(k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)\, k^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{k=0}^{\infty} \frac{k^{n-1} x^k}{(k-1)!} \\ &= e^x \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)\, f_{n-1}(x)}{n!}, \end{aligned}$$

by (3.3), where it is assumed that the inversion in order of summation above is justified. (There is a misprint in the notebooks in that $f_{n-1}(x)$ is replaced by $f_n(x)$.) Using (6.1) above, we find that

$$\begin{aligned} \varphi_\infty(x) &= \varphi(0) + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \sum_{j=1}^{n} \varphi_j(n-1)\, x^j \\ &= \varphi(0) + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-k+1}(n-1)\, x^{n-k+1}. \end{aligned} \tag{10.4}$$

We now separate $\varphi(0)$ together with the series for $k=1$ in (10.4). These terms are

$$\varphi(0) + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_n(n-1)\, x^n = \varphi(0) + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(0)}{n!} x^n = \varphi(x). \tag{10.5}$$

Here we have used the fact that

$$\varphi_n(n-1) = 1, \qquad n \geqslant 1, \tag{10.6}$$

which is easily proved by induction with the aid of Entry 8.

Next, we examine the series for $k=2$ in (10.4). This series is

$$\sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-1}(n-1)\, x^{n-1} = \frac{x}{2} \sum_{n=2}^{\infty} \frac{\varphi^{(n)}(0)}{(n-2)!} x^{n-2} = \frac{x}{2} \varphi''(x). \tag{10.7}$$

In this calculation we have used the evaluation $\varphi_n(n) = n(n+1)/2$ for $n \geqslant 1$, which again is readily established by induction with the help of Entry 8.

Ramanujan continues to calculate in the fashion indicated above. In fact, using special cases of Lemma 3 below, he finds that

$$\sum_{n=3}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-2}(n-1)\, x^{n-2} = \frac{x}{6} \varphi'''(x) + \frac{x^2}{8} \varphi^{(4)}(x), \tag{10.8}$$

$$\sum_{n=4}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-3}(n-1)\, x^{n-3} = \frac{x}{24}\varphi^{(4)}(x) + \frac{x^2}{12}\varphi^{(5)}(x) + \frac{x^3}{48}\varphi^{(6)}(x), \tag{10.9}$$

$$\sum_{n=5}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-4}(n-1)\, x^{n-4} \tag{10.10}$$

$$= \frac{x}{120}\varphi^{(5)}(x) + \frac{5x^2}{144}\varphi^{(6)}(x) + \frac{x^3}{48}\varphi^{(7)}(x) + \frac{x^4}{384}\varphi^{(8)}(x),$$

and

$$\sum_{n=6}^{\infty} \frac{\varphi^{(n)}(0)}{n!} \varphi_{n-5}(n-1)\, x^{n-5}$$

$$= \frac{x}{720}\varphi^{(6)}(x) + \frac{x^2}{90}\varphi^{(7)}(x) + \frac{7x^3}{576}\varphi^{(8)}(x)$$

$$+ \frac{x^4}{288}\varphi^{(9)}(x) + \frac{x^5}{3840}\varphi^{(10)}(x). \tag{10.11}$$

At this point Ramanujan ceases his calculations and substitutes (10.5) and (10.7)–(10.11) into (10.4). With the help of Table I for b_{kn}, we readily verify that Ramanujan's result agrees with (10.3).

In a corollary, Ramanujan claims that $\varphi_{\infty}(x) = \varphi(x) + (x/2)\,\varphi''(x)$ "very nearly." However, no discussion of the error term is given.

Before commencing the proof of Entry 10, we provide four lemmas.

LEMMA 1. *Let t be fixed, where $0 < t < 1$. Then $e^{-x}\sum_{0 \leqslant k < tx} x^k/k!$ and $e^{-x}\sum_{k > x/t} x^k/k!$ each tend to 0 exponentially as x tends to ∞.*

An easy proof of Lemma 1 has been given by Breusch [48].

LEMMA 2. *Let $2 \leqslant k \leqslant n$. Then $b_{kn} \leqslant (n-1)!$.*

Proof. With the use of (10.1), induct on n, and the result follows easily.

LEMMA 3. *Let $2 \leqslant k \leqslant n$. Then*

$$\varphi_{n+1-k}(n-1) = \sum_{j=k}^{2k-2} b_{kj} \binom{n}{j},$$

where $\varphi_k(n)$ is defined by (6.1).

Proof. The result follows from Entry 8 by induction on k. See also [18, p. 226].

LEMMA 4. *Let $p(x)$ be a polynomial of degree n. Then*

$$p_\infty(x) \equiv e^{-x} \sum_{k=0}^{\infty} \frac{x^k p(k)}{k!} = p(x) + \sum_{j=2}^{n} \sum_{k=2}^{j} b_{kj} x^{j-k+1} \frac{p^{(j)}(x)}{j!}.$$

Proof. By linearity, it suffices to prove this lemma in the case that $p(x) = x^n$, where n is a nonnegative integer. The result is easily proved for $n = 0, 1$, and so we suppose that $n \geqslant 2$. By (6.1),

$$\begin{aligned} p_\infty(x) &= e^{-x} \sum_{k=0}^{\infty} \frac{x^k k^n}{k!} = f_{n-1}(x) = \sum_{k=1}^{n} \varphi_k(n-1)\, x^k \\ &= \varphi_n(n-1)\, x^n + \sum_{k=2}^{n} \varphi_{n-k+1}(n-1)\, x^{n-k+1}. \end{aligned}$$

Using (10.6) and Lemma 3, we deduce that

$$\begin{aligned} p_\infty(x) &= x^n + \sum_{k=2}^{n} \sum_{j=k}^{2k-2} b_{kj} x^{j-k+1} \binom{n}{j} x^{n-j} \\ &= p(x) + \sum_{k=2}^{n} \sum_{j=k}^{2k-2} b_{kj} x^{j-k+1} \frac{p^{(j)}(x)}{j!}. \end{aligned}$$

Since $p^{(j)}(x) = 0$ for $j > n$ and since $b_{kj} = 0$ for $j > 2k-2$, the upper index $2k-2$ on the inner sum may be replaced by n. The result now follows upon inverting the order of summation.

Proof of Entry 10. Throughout the proof we always assume that x is sufficiently large. Fix $t \in (0, 1)$, but we require that t is close enough to 1 so that $3(1-t)A/t < 1$. Define the intervals I_1, I_2, and I_3 by $I_1 = [0, tx)$, $I_2 = [tx, (2-t)x)$, and $I_3 = [(2-t)x, \infty)$.

Consider the Taylor polynomial

$$p(y) = \sum_{r=0}^{N-1} \frac{\varphi^{(r)}(x)}{r!} (y-x)^r, \tag{10.12}$$

where $N = [\sqrt{x}/6A]$. By Taylor's theorem, for each $y \in I_2$,

$$\varphi(y) = p(y) + \frac{\varphi^{(N)}(\xi)}{N!} (y-x)^N,$$

where ξ is some point between x and y. Thus, by (10.2),

$$\begin{aligned} |\varphi(y) - p(y)| &\leqslant |\varphi^{(N)}(\xi)/N!|\, \{x(1-t)\}^N \\ &\leqslant G(\xi)(A/tx)^N \{x(1-t)\}^N \\ &< G(\xi)\, 3^{-N} < 2^{-N}, \end{aligned}$$

for every $y \in I_2$. Therefore, as x tends to ∞,

$$e^{-x} \sum_{k \in I_2} \frac{x^k \varphi(k)}{k!} = e^{-x} \sum_{k \in I_2} \frac{x^k p(k)}{k!} + O(2^{-N}). \tag{10.13}$$

Since $\varphi(x)$ has at most polynomial growth as x tends to ∞, it follows from Lemma 1 that

$$e^{-x} \sum_{k \in I_1}{}' \frac{x^k \varphi(k)}{k!} \to 0 \qquad \text{exponentially as } x \to \infty. \tag{10.14}$$

Also, for some fixed natural number B,

$$e^{-x} \sum_{k \in I_3} \frac{x^k \varphi(k)}{k!} \ll e^{-x} \sum_{k \in I_3} \frac{x^k}{(k-B)!} = x^B e^{-x} \sum_{k \geqslant (2-t)x-B} \frac{x^k}{k!}.$$

So again by Lemma 1,

$$e^{-x} \sum_{k \in I_3} \frac{x^k \varphi(k)}{k!} \to 0 \qquad \text{exponentially as } x \to \infty. \tag{10.15}$$

By (10.2) and (10.12) for $0 \leqslant y \leqslant 2x$,

$$|p(y)| \leqslant \sum_{r=0}^{N-1} G(x)(A/x)^r x^r \ll (A+1)^N, \tag{10.16}$$

and so by Lemma 1,

$$e^{-x} \sum_{k \in I_1} \frac{x^k p(k)}{k!} \to 0 \qquad \text{exponentially as } x \to \infty. \tag{10.17}$$

Write

$$e^{-x} \sum_{k \in I_3} \frac{x^k p(k)}{k!} = S_1 + S_2, \tag{10.18}$$

where

$$S_1 = e^{-x} \sum_{(2-t)x \leqslant k < 2x} \frac{x^k p(k)}{k!} \qquad \text{and} \qquad S_2 = \sum_{k \geqslant 2x} \frac{x^k p(k)}{k!}.$$

By (10.16) and Lemma 1,

$$S_1 \to 0 \qquad \text{exponentially as } x \to \infty. \tag{10.19}$$

For $k \geqslant 2x$, it follows from (10.2) that

$$|p(k)| \leqslant \sum_{r=0}^{N-1} G(x)(A/x)^r(k-x)^r \leqslant G(x)\, NA^N \left(\frac{k-x}{x}\right)^N \ll (k/x)^{x/7}.$$

Also, for $k \geqslant 2x$, $x^k/x! < (xe/k)^k$. Hence,

$$S_2 \ll e^{-x} \sum_{k \geqslant 2x} (xe/k)^k (k/x)^{x/7}. \tag{10.20}$$

The summands in (10.20) are strictly decreasing in k. Thus,

$$\begin{aligned} S_2 &\ll e^{-x} \sum_{j=2}^{\infty} \sum_{jx \leqslant k < (j+1)x} (xe/k)^k (k/x)^{x/7} \\ &\ll xe^{-x} \sum_{j=2}^{\infty} (xe/jx)^{jx} (jx/x)^{x/7} \\ &= x \sum_{j=2}^{\infty} \{e^{j-1} j^{-j+1/7}\}^x \\ &< x \left(\sum_{j=2}^{\infty} \{e^{j-1} j^{-j+1/7}\}^3 \right)^{x/3}. \end{aligned}$$

Since the series in parentheses above converges to a number less than 1,

$$S_2 \to 0 \qquad \text{exponentially as } x \to \infty. \tag{10.21}$$

By (10.18), (10.19), and (10.21),

$$e^{-x} \sum_{k \in I_3} \frac{x^k p(k)}{k!} \to 0 \qquad \text{exponentially as } x \to \infty. \tag{10.22}$$

By (10.13)–(10.15), (10.17), and (10.22),

$$\varphi_\infty(x) - p_\infty(x) = O(2^{-N}) \tag{10.23}$$

as x tends to ∞, where

$$p_\infty(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k p(k)}{k!}.$$

By Lemma 4, (10.12), and (10.1),

$$p_\infty(x) = p(x) + \sum_{j=2}^{N-1} \sum_{k=2}^{j} b_{kj} x^{j-k+1} \frac{p^{(j)}(x)}{j!}$$

$$= \varphi(x) + \sum_{j=2}^{N-1} \sum_{k=2}^{j} b_{kj} x^{j-k+1} \frac{\varphi^{(j)}(x)}{j!}$$

$$= \varphi(x) + \sum_{k=2}^{N-1} \sum_{j=k}^{N-1} b_{kj} x^{j-k+1} \frac{\varphi^{(j)}(x)}{j!}$$

$$= \varphi(x) + \sum_{k=2}^{M} \sum_{j=k}^{2k-2} b_{kj} x^{j-k+1} \frac{\varphi^{(j)}(x)}{j!} + S_3, \qquad (10.24)$$

where

$$S_3 = \sum_{k=M+1}^{N-1} \sum_{j=k}^{N-1} b_{kj} x^{j-k+1} \frac{\varphi^{(j)}(x)}{j!}.$$

In view of (10.23) and (10.24), in order to prove (10.3) it suffices to show that $S_3 = O(G(x)\,x^{-M})$ as x tends to ∞.

By (10.1), (10.2), and Lemma 2,

$$|S_3| \leqslant \sum_{k=M+1}^{N} \sum_{j=k}^{2k-2} b_{kj} x^{j-k+1}\, |\varphi^{(j)}(x)|/j!$$

$$\leqslant \sum_{k=M+1}^{N} \sum_{j=k}^{2k-2} (j-1)!\, G(x)\, A^j x^{1-k}$$

$$\leqslant G(x) \sum_{k=M+1}^{N} (2k)!\, A^{2k} x^{1-k}$$

$$\leqslant G(x)\, x^{-M} \sum_{k=0}^{N} (2k+2M+2)!\, (A/\sqrt{x})^{2k}$$

$$\leqslant G(x)\, x^{-M} \sum_{k=0}^{2N} (k+2M+2)!\, (A/\sqrt{x})^{2k}.$$

Since $N \leqslant \sqrt{x}/6A$, the $(k+1)$th term in the last sum above is less than half of the kth term, for each $k < 2N$. Thus,

$$|S_3| \leqslant G(x)\, x^{-M} (2M+2)! \sum_{k=0}^{2N} 2^{-k} = O(G(x)\, x^{-M}).$$

This completes the proof.

The following four examples give applications of (10.3):

EXAMPLE 1. As x tends to ∞,

$$\log\left(\sum_{k=1}^{\infty} \frac{x^k \sqrt{k}}{k!}\right) = x + \frac{1}{2}\log x - \frac{1}{8x} - \frac{1}{16x^2} + O\left(\frac{1}{x^3}\right).$$

Proof. Letting $\varphi(x)=G(x)=\sqrt{x}$ in (10.3) with $M=3$, we find that

$$e^{-x}\sum_{k=1}^{\infty}\frac{x^k\sqrt{k}}{k!}=\sqrt{x}\left(1-\frac{1}{8x}-\frac{7}{128x^2}+O\left(\frac{1}{x^3}\right)\right),$$

as x tends to ∞. Hence,

$$\log\left(\sum_{k=1}^{\infty}\frac{x^k\sqrt{k}}{k!}\right)=\log\left\{e^x\sqrt{x}\left(1-\frac{1}{8x}-\frac{7}{128x^2}+O\left(\frac{1}{x^3}\right)\right)\right\},$$

and the result easily follows.

Example 1 is actually a special case of a result of Hardy [36, pp. 410–411; 39, pp. 71–72], who derived in the case $\varphi(x)=(x+a)^{-s}$ an asymptotic series which is in a more complicated form than that given by (10.3). In particular, we find that

$$\begin{aligned}e^{-x}\sum_{k=0}^{\infty}{}'\frac{x^k(k+a)^{-s}}{k!}&=(x+a)^{-s}+\frac{s(s+1)}{2}x(x+a)^{-s-2}\\&\quad-\frac{s(s+1)(s+2)}{6}x(x+a)^{-s-3}\\&\quad+\frac{s(s+1)(s+2)(s+3)}{8}x^2(x+a)^{-s-4}\\&\quad+O(x^{-s-3}).\end{aligned}$$

EXAMPLE 2. As x tends to ∞,

$$e^{-x}\sum_{k=1}^{\infty}\frac{x^k\log(k+1)}{k!}=\log x+\frac{1}{2x}+\frac{1}{12x^2}+O\left(\frac{1}{x^3}\right).$$

Proof. Letting $\varphi(x)=\log(x+1)$ in (10.3) with $M=4$, we deduce that

$$\begin{aligned}&e^{-x}\sum_{k=1}^{\infty}\frac{x^k\log(k+1)}{k!}\\&=\log(x+1)-\frac{x}{2(x+1)^2}+\frac{x}{3(x+1)^3}-\frac{3x^2}{4(x+1)^4}+O\left(\frac{1}{x^3}\right)\\&=\log x+\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{2x}\left(1-\frac{2}{x}\right)+\frac{1}{3x^2}-\frac{3}{4x^2}+O\left(\frac{1}{x^3}\right),\end{aligned}$$

as x tends to ∞. The desired asymptotic expansion now readily follows.

EXAMPLE 3. $\log(\sum_{k=0}^{\infty}(100^k\varphi(k)/k!)) = 100 + \log((\varphi(110)+\varphi(90))/2)$ "nearly."

Proof. We have quoted Ramanujan above, who evidently uses the approximation $\varphi_\infty(x) \sim \varphi(x)$, sets $x = 100$, and then replaces $\varphi(100)$ by $\{\varphi(110) + \varphi(90)\}/2$.

EXAMPLE 4. Let $\psi(x) = \sum_{k \leqslant x} 1/k$. Then as x tends to ∞,

$$\sum_{k=1}^{\infty} \frac{x^k \psi(k)}{k!} = e^x(\log x + \gamma + O(1/x)), \tag{10.25}$$

where γ denotes Euler's constant.

Proof. As x tends to ∞ [5, p. 43],

$$\psi(x) = \log x + \gamma + O(1/x). \tag{10.26}$$

Now substitute (10.26) into the left side of (10.25). Then apply (10.3) to $\varphi(x) = \log x$ with $M = 1$. The result now easily follows.

An independent proof of Example 4 can be gotten by employing Corollary 2 of Entry 2. We omit the details. Anticipating his work on divergent series in Chapter 6, Ramanujan calls γ (c in his notation) the "constant" of the series $\sum_{k=1}^{\infty} 1/k$.

ENTRY 11. *Suppose that* $f(x) = \sum_{n=1}^{\infty} A_n x^n/n$ *is analytic for* $|x| < R$. *Define* p_n, $0 \leqslant n < \infty$, *by*

$$\sum_{n=0}^{\infty} p_n x^n = \exp\{f(x)\}, \qquad |x| < R. \tag{11.1}$$

Then

$$np_n = \sum_{k=1}^{n} A_k p_{n-k}, \qquad n \geqslant 1. \tag{11.2}$$

Proof. Taking the derivative of both sides of (11.1), we find that

$$\sum_{n=1}^{\infty} np_n x^{n-1} = \sum_{j=0}^{\infty} p_j x^j \sum_{k=1}^{\infty} A_k x^{k-1}, \qquad |x| < R.$$

Equating the coefficients of x^{n-1} on both sides above, we obtain the desired recursion formula.

COROLLARY. *Let* $\{a_k\}$, $1 \leqslant k < \infty$, *be a sequence of complex numbers such that* $\sum_{k=1}^{\infty} |a_k| < \infty$. *Let* $S_n = \sum_{k=1}^{\infty} a_k^n$, *where* n *is a positive integer.*

For $n \geq 1$, define p_n to be the sum of all products of n distinct terms taken from $\{a_k\}$. Let $p_0 = 1$. Then

$$np_n = \sum_{k=1}^{n} (-1)^{k-1} S_k p_{n-k}, \qquad n \geq 1.$$

Proof. For $|x| < \rho \equiv \inf_n 1/|a_n|$, $a_n \neq 0$,

$$\sum_{n=0}^{\infty} p_n x^n = \prod_{n=1}^{\infty} (1 + a_n x)$$

is analytic and nonzero. Thus, in the notation of Entry 11, for $|x| < \rho$,

$$\sum_{n=1}^{\infty} \frac{A_n}{n} x^n = \sum_{k=1}^{\infty} \log(1 + a_k x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} S_n}{n} x^n.$$

Hence, $A_n = (-1)^{n-1} S_n$, $n \geq 1$. Substituting this in (11.2), we complete the proof.

In the corollary above, Ramanujan assumes that the sequence $\{a_k\}$ is finite, but this is unnecessary.

For integral r and complex n, define

$$F_r(n) = \sum_{k=0}^{\infty} \frac{(n+k)^{r+k}}{a^k k!}. \tag{12.1}$$

By Stirling's formula, the $(k+1)$th term of $F_r(n)$ is asymptotic to $(1/\sqrt{2\pi})\, k^{r-1/2}(e/a)^k$ as k tends to ∞. Hence, $F_r(n)$ converges for $|a| > e$ and also for $|a| = e$ if $r < -\frac{1}{2}$.

ENTRY 12. *For r, n, and a as specified above, we have*

$$F_{r+1}(n) = nF_r(n) + \frac{1}{a} F_{r+1}(n+1).$$

Proof. We have

$$\begin{aligned} F_{r+1}(n) &= nF_r(n) + \sum_{k=0}^{\infty} \frac{k(n+k)^{r+k}}{a^k k!} \\ &= nF_r(n) + \frac{1}{a} \sum_{k=1}^{\infty} \frac{(n+k)^{r+k}}{a^{k-1}(k-1)!}, \end{aligned}$$

from which the desired recursion follows.

Entries 13 and 14 are concerned with functions of the form

$$\varphi(x)=\sum_{n=0}^{\infty} f_n(x)\, a^n,$$

where the coefficients are polynomials in x such that

$$f_n(x+y)=\sum_{k=0}^{n} f_k(x)\, f_{n-k}(y), \qquad 0\leqslant n<\infty, \tag{13.1}$$

where x and y are arbitrary real numbers. Thus, φ satisfies the relation

$$\varphi(x+y)=\varphi(x)\,\varphi(y) \tag{13.2}$$

for all values of x and y.

We first prove a general theorem and corollary from which we shall deduce the identities of Entries 13 and 14.

THEOREM. *Let p and q be constants with $q\neq 0$. Let $f_n(x)$, $0\leqslant n<\infty$, be a sequence of polynomials satisfying the difference equations*

$$f_n(x+q)-f_n(x)=qf_{n-1}(x+p), \qquad n\geqslant 1, \tag{13.3}$$

together with the initial conditions

$$f_0(x)\equiv 1 \tag{13.4}$$

and

$$f_n(0)=0, \qquad n\geqslant 1. \tag{13.5}$$

Then $f_n(x)$ satisfies (13.1).

Before commencing the proof, it might be noted that the theorem remains true if the factor q on the right side of (13.3) is replaced by a third arbitrary constant r, $r\neq 0$. For if the solutions of (13.3)–(13.5) are denoted by $f_n(x)$, those solutions under the modified conditions are $(r/q)^n f_n(x)$, $0\leqslant n<\infty$. Thus, for the apparent generalization, both sides of (13.1) are merely multiplied by $(r/q)^n$.

Proof of Theorem. We shall induct on n. By (13.4), relation (13.1) holds for $n=0$. Suppose that (13.1) is valid for all values of x and y when $0\leqslant n\leqslant m-1$.

We shall first show that (13.1) is true for all values of x when $n=m$ and $y=q$. From (13.3) and (13.5),

$$f_n(q)=qf_{n-1}(p), \qquad n\geqslant 1. \tag{13.6}$$

By (13.3), (13.6), and the induction hypothesis,

$$\begin{aligned} f_m(x+q) &= f_m(x) + q \sum_{k=0}^{m-1} f_k(x)\, f_{m-1-k}(p) \\ &= f_m(x) + \sum_{k=0}^{m-1} f_k(x)\, f_{m-k}(q) \\ &= \sum_{k=0}^{m} f_k(x)\, f_{m-k}(q), \end{aligned} \tag{13.7}$$

by (13.4).

Next, we shall show that if (13.1) is valid for all values of x when $n = m$ and y has a particular value y_0, then (13.1) also holds for all values of x when $n = m$ and $y = y_0 + q$. By (13.7),

$$\begin{aligned} f_m(x + y_0 + q) &= \sum_{k=0}^{m} f_k(x + y_0)\, f_{m-k}(q) \\ &= \sum_{k=0}^{m} f_{m-k}(q) \sum_{j=0}^{k} f_j(x)\, f_{k-j}(y_0), \end{aligned}$$

by our assumptions. Now invert the order of summation and put $r = k - j$ to obtain

$$\begin{aligned} f_m(x + y_0 + q) &= \sum_{j=0}^{m} f_j(x) \sum_{r=0}^{m-j} f_r(y_0)\, f_{m-j-r}(q) \\ &= \sum_{j=0}^{m} f_j(x)\, f_{m-j}(y_0 + q), \end{aligned}$$

by the induction hypothesis and by (13.7) when $j = 0$. We have thus shown that (13.1) is valid for all values of x when $n = m$ and y is any positive integral multiple of q. In other words, the polynomial identity (13.1), when $n = m$, is valid for all x and an infinite number of values of y, and so must be valid for all x and all y.

COROLLARY. *Let p and q be constants. Then the polynomials $f_0(x) = 1$, $f_1(x) = x$, and*

$$f_n(x) = \frac{x}{n!} \prod_{k=1}^{n-1} (x + np - kq), \qquad n \geqslant 2,$$

satisfy (13.1).

Proof. For $q \neq 0$, the result is obvious for $n = 0$ and follows from the theorem when $n \geqslant 1$. For $q = 0$, the result follows by continuity in q.

The theorem above and its corollary are not explicitly stated by Ramanujan in his notebooks. The theorem's proof that we have given was supplied to Wilson by U. S. Haslam-Jones. The theorem and corollary are now part of a general theory developed by Rota and Mullin [58, p. 182]. The polynomials in the corollary were first introduced in the literature by Jensen [42] in 1902 and later by Gould [27], and are essentially what are now called the Gould polynomials [57, pp. 733–736; 56, p. 115]. See Gould's papers [27–32] and a paper of Carlitz [14] for several formulae and the context in which these polynomials arise.

ENTRY 13. *Let* $f(n) = nF_{-1}(n)$, *where* F_{-1} *is defined by* (12.1). *Assume that a is real, with* $|a| \geqslant e$. *Then there exists a positive real number x satisfying the relation* $x = a \log x$ *such that for any real number n,* $x^n = f(n)$.

Proof. By the corollary with $p = 1$ and $q = 0$ and by (13.2), $f(m)f(n) = f(m+n)$, where m and n are arbitrary real numbers. Hence, if n is any positive integer, $f(n) = x^n$, where $x = f(1)$. This relation may be extended to negative integers n by using the equality $f(n)\,f(-n) = f(0) = 1$. It can further be extended to all rational numbers r/s upon noting that $\{f(r/s)\}^s = f(r) = x^r$. For $|a| \geqslant e$, $f(n)$ converges uniformly on any compact interval in the variable n. Hence, $f(n)$ is continuous for all n. It follows that $f(n) = x^n$ for all real values of n. Hence, for $|a| \geqslant e$,

$$f'(n) = x^n \log x = \sum_{k=1}^{\infty} \frac{(n+k)^{k-1}}{a^k k!} + \sum_{k=1}^{\infty} \frac{n(k-1)(n+k)^{k-2}}{a^k k!},$$

since both of these series converge uniformly on any compact interval in n. Thus,

$$f'(0) = \log x = \sum_{k=1}^{\infty} \frac{k^{k-2}}{a^k (k-1)!} = \frac{f(1)}{a} = \frac{x}{a}.$$

This completes the proof.

Let a now be complex and consider the relation $x = a \log x$, where x is to be regarded as a function of a. By considering, for example, the graph of $x/\log x$ for real values of x, we see that for $a > e$ there are two branches x_1 and x_2 of the function x that have real values. Thus, $a = e$ is a branch point. One branch, say x_1, decreases from e to 1 as a increases from e to $+\infty$. The other branch x_2 increases from e to $+\infty$ as a increases from e to $+\infty$. Since $f(1)$ tends to 1 as a tends to ∞, it follows that, for $a > e$, $f(1)$ defines a branch of the function $x(a)$ that is real and lies between 1 and e. Entry 13 thus shows that $f(n) = x_1^n$.

COROLLARY. *Let z be an arbitrary complex number, and suppose that w is any complex number such that $|e^{w-1}| \geqslant |w|$. Then*

$$e^z = \sum_{k=0}^{\infty} \frac{z(z+kw)^{k-1} e^{-kw}}{k!}. \tag{13.8}$$

Proof. First suppose that w is real. Apply Entry 13 with $x = e^w$, and so $a = e^w/w$. Then for any real number n,

$$f(n) = e^{nw} = \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1} w^k e^{-kw}}{k!}.$$

Letting $z = nw$, we deduce (13.8) for real z and real w with $|w| \leqslant |e^{w-1}|$. By analytic continuation in each of the variables w and z, we complete the proof.

For brevity, we shall now define $c_0(n) = 1$, $c_1(n) = n$, and

$$c_k(n) = n \prod_{j=1}^{k-1} (n + kp - jq), \qquad k \geqslant 2. \tag{14.1}$$

Define, for complex a,

$$\varphi(n) = \sum_{k=0}^{\infty} \frac{c_k(n)\, a^k}{k!}. \tag{14.2}$$

If $p = q = 0$, $\varphi(n) = e^{an}$. If $p \neq 0$ but $q = 0$, then $\varphi(n) = f(n/p)$, where f is the function defined in Entry 13 but with a replaced by $1/ap$. If $p = 0$ and $q \neq 0$, then $\varphi(n) = (1 + aq)^{n/q}$. If $p = q \neq 0$, then $\varphi(n) = (1 - ap)^{-n/p}$. Thus, in the sequel we may suppose that none of the parameters p, q, and $p - q$ is equal to 0. Furthermore, without loss of generality, we may assume that p and q are positive, for, in a more explicit notation,

$$\begin{aligned}\varphi(n) &= \varphi(n; p, q, a) = \varphi(-n; -p, -q, -a)\\ &= \varphi(n; p-q, -q, a) = \varphi(-n; q-p, q, -a).\end{aligned}$$

Now, by Stirling's formula, as k tends to ∞,

$$\begin{aligned}\left|\frac{c_k(n)}{k!}\right| &= \left| nq^{k-1} \Gamma\left(\frac{n+kp}{q}\right) \Big/ k!\, \Gamma\left(\frac{n+kp}{q} - k + 1\right)\right|\\ &\sim ck^{-3/2} p^{pk/q} |p-q|^{-k(p-q)/q},\end{aligned}$$

where the constant c depends upon p, q, and n but not k. Thus, $\varphi(n)$ converges for

$$|a| \leqslant p^{-p/q}\,|p-q|^{(p-q)/q}. \qquad (14.3)$$

ENTRY 14. *Let φ be defined by* (14.2) *and let p and q be as specified above. If x is a certain root of the equation*

$$aqx^p - x^q + 1 = 0, \qquad (14.4)$$

then $\varphi(n) = x^n$ for every real number n.

Proof. By the same type of argument as that in the proof of Entry 13,

$$\varphi(n) = x^n, \qquad (14.5)$$

where $\varphi(1) = x$ and n is any real number. Next, by a direct calculation,

$$\frac{1}{k!}\,c_k(q) - \frac{q}{(k-1)!}\,c_{k-1}(p) = 0.$$

Hence, since $c_0(n) = 1$,

$$\varphi(q) - aq\varphi(p) = 1.$$

In other words, by (14.5), x satisfies (14.4), and the proof is complete.

Note that, by (14.2), $x = x(a)$ tends to 1 as a tends to 0.

COROLLARY 1. *Let n be real and suppose that $|a| \leqslant 1/4$. Then*

$$(2/(1+\sqrt{1-4a}))^n = 1 + na + n\sum_{k=2}^{\infty} \frac{\Gamma(n+2k)\,a^k}{\Gamma(n+k+1)\,k!}.$$

Proof. In Entry 14, let $p = 2q$. The root of (14.4) which tends to unity as a tends to 0 is given by

$$x^q = (1-\sqrt{1-4aq})/2aq = 2/(1+\sqrt{1-4aq}).$$

Thus, by Entry 14 and (14.1),

$$(2/(1+\sqrt{1-4aq}))^n = \varphi(nq) = 1 + nqa + nq\sum_{k=2}^{\infty}\left\{\prod_{j=k+1}^{2k-1}(nq+jq)\right\}\frac{a^k}{k!},$$

where $|a| < 1/(4q)$, by (14.3). Setting $q = 1$ in the equalities above we complete the proof.

COROLLARY 2. *Let n be real and assume that* $|a| \leqslant 1$. *Then*

$$(a+\sqrt{1+a^2})^n = 1 + na + \sum_{k=2}^{\infty} \frac{b_k(n)\, a^k}{k!},$$

where, for $k \geqslant 2$,

$$\begin{aligned} b_k(n) &= n^2(n^2-2^2)(n^2-4^2)\cdots(n^2-(k-2)^2), && \text{if } k \text{ is even,} \\ &= n(n^2-1^2)(n^2-3^2)\cdots(n^2-(k-2)^2), && \text{if } k \text{ is odd.} \end{aligned}$$

Proof. Let $q = 2p$ in Entry 14. The root of (14.4) which tends to 1 as a approaches 0 is given by $x^p = ap + \sqrt{a^2p^2+1}$. Hence, by Entry 14,

$$(ap+\sqrt{a^2p^2+1})^n = \varphi(np) = 1 + npa + \sum_{k=2}^{\infty} \frac{c_k(np)\, a^k}{k!},$$

where $c_k(np)$ is given by (14.1), and where $|a| \leqslant 1/p$ by (14.3). Now let $p = 1$ and $q = 2$ in the above equalities. Then it is not very hard to see that $c_k(n) = b_k(n)$, and the proof is complete.

Entries 13 and 14 have a long history. Entry 14 was first established by Lambert [45, pp. 38–40] in a paper published in 1758. In 1770, Lagrange [44] published a proof of the celebrated "Lagrange inversion formula." As an application, he derived Entry 14 [44, pp. 53–56]. Entries 13 and 14 appear as problems illustrating the Lagrange inversion formula in the text of Pólya and Szegö [50, pp. 125–126]. In 1779, in a paper stimulated by the work of Lambert, Euler [24] proved both Entries 13 and 14. Rothe [59] rediscovered the special case $q = 1$ of Entry 14 in his dissertation published in 1793. Entries 13 and 14 also follow from Abel's [1; 2, pp. 102–103] generalization of the binomial theorem and are sometimes attributed to him. Entry 13 was rediscovered in 1844 by Eisenstein [21; 22, pp. 122–125] who was apparently unaware of earlier work. Another proof was given by Jensen [42] in 1902. The result is also found in Gould's paper [28, p. 412]. A similar theorem of a more general type has also been established by Gould [30, Theorem 7]. Entry 14 is similar to further results of Gould [27, p. 85; 30, Theorem 1]. Entry 14 has also been generalized in a different direction; solutions of certain algebraic equations can be represented by hypergeometric series. Further references can be found in Birkeland's paper [10]. Hardy [40, p. 194] refers to Ramanujan's work on (14.4). In fact, Ramanujan discusses (14.4) in his quarterly reports [52, p. xvi] which the first author hopes to examine in detail in another paper. The corollary of Entry 13 is essentially a reformulation of an exercise in Bromwich's book [11, p. 160]. (See also [11, p. 195].) An application of this corollary has been given by Rogers [55]. Jackson [41] has found a q-analog of this corollary as well as of Abel's theorems and related results. Finally, we mention that Gould [33] has compiled an extensive bibliography of papers related to Entries 13 and 14, the aforementioned convolution theorem of Abel, and similar results.

ENTRY 15. *Define* u, $0 < u < 1$, *by*

$$u - \log u = 1 + x^2/2, \tag{15.1}$$

where x *is real. Then*

$$\sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} e^{-k(1+x^2/2)} = ue^{1+x^2/2}. \tag{15.2}$$

Furthermore, for sufficiently small positive x,

$$u = \sum_{k=0}^{\infty} b_k x^k,$$

where $b_0 = 1$, $b_1 = -1$, $b_2 = \frac{1}{3}$, $b_3 = -\frac{1}{36}$, $b_4 = -\frac{1}{270}$, *and, in general, the coefficients* b_k *are found successively by substituting into the identity*

$$\frac{1}{2}x^2 = \sum_{j=2}^{\infty} \frac{(1-u)^j}{j}.$$

Proof. In Entry 13, let $x = e^u$ and $a = e^u/u$, so $a = \exp(1 + x^2/2)$ by (15.1). We then find that

$$f(1) = e^u = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} e^{-k(1+x^2/2)},$$

from which (15.2) follows.

Define $F(u) = 2(u - 1 - \log u) = x^2$. Note that F is analytic in a neighborhood N of $u = 1$ and that $F'(1) = 0$ and $F''(1) \neq 0$. For $u \in N$, we may then write $F(u) = G(u)^2 = x^2$, where $G(u)$ is analytic and one-to-one on N, and where, say, $x = G(u)$ and $G'(1) < 0$. Thus, there exists an analytic inverse of $G(u) = x$ in a neighborhood of $x = 0$ of the form

$$u = G^{-1}(x) = \sum_{k=0}^{\infty} b_k x^k.$$

When $0 < u < 1$, the equalities above hold with $x > 0$, since $G'(1) < 0$. We have $b_0 = 1$ and $b_1 < 0$. By (15.1) and Taylor's theorem,

$$\frac{1}{2}x^2 = u - 1 - \log u = \sum_{j=2}^{\infty} \frac{(1-u)^j}{j},$$

and so the coefficients b_k may be calculated as indicated.

EXAMPLE 1. Let m be real and let $0 < n \leqslant 2$. Then

$$2^m = \sum_{k=0}^{\infty} \frac{m\Gamma(m+kn)}{\Gamma(m+kn-k+1)\, 2^{kn}k!}.$$

This example is highly interesting. Ramanujan, in fact, claims the result is true for $0 < n < \infty$. The series does converge for $0 < n < \infty$. However, it converges to a different value for $n > 2$. In the proof below, we shall prove this last fact as well.

Proof. In Entry 14, replace n by m, let $q = 1$, replace p by n, and let $a = 2^{-n}$, where $0 < n < \infty$. We find that if x is a certain root of

$$f(x) \equiv 2^{-n}x^n - x + 1 = 0, \tag{15.3}$$

then

$$x^m = \sum_{k=0}^{\infty} \frac{m\Gamma(m+kn)}{\Gamma(m+kn-k+1)\, 2^{kn}k!}, \tag{15.4}$$

provided the series converges. By (14.3), the series above converges if

$$2^{-n} \leqslant n^{-n}\, |n-1|^{n-1}, \tag{15.5}$$

for $n \neq 1$. By the remarks made prior to Entry 14, the series in (15.4) converges for $n = 1$, in which case (15.5) would be interpreted as $2^{-1} \leqslant 1$. We now show that (15.5) holds for $0 < n < \infty$. Letting $g(x) = (x/2)^x\, |1-x|^{1-x}$, we want to show that $g(x) \leqslant 1$ for $x \geqslant 0$. By elementary calculus, we find that $g(x)$ decreases for $0 < x < \frac{2}{3}$ from the value $g(0) = 1$. On $\frac{2}{3} < x < 2$, g increases to the value $g(2) = 1$. For $x > 2$, g decreases. Thus, $g(x) \leqslant 1$ for $x \geqslant 0$, and (15.5) is valid for $0 < n < \infty$.

Now, obviously, $x = 2$ is a root of (15.3). Since $f'(x) = n2^{-n}x^{n-1} - 1$, we see that there is a unique positive value $x = \xi$ such that $f'(\xi) = 0$. Hence, (15.3) has at most one positive root in addition to the root $x = 2$. If $0 < n \leqslant 1$, $f(0)$ is positive while $f(+\infty)$ is negative. Hence, $x = 2$ is the only positive root of (15.3). Thus, Example 1 is established for $0 \leqslant n < 1$. If $n > 1$, both $f(0)$ and $f(+\infty)$ are positive. Thus, in addition to the root $x = 2$, (15.3) has another (not necessarily distinct) positive root $x = \alpha$, and clearly $\alpha > 1$. Now $\alpha = 2$ if and only if $f'(2) = 0$, which happens only when $n = 2$. Thus, Example 1 is valid for $n = 2$. Observe that $f'(2)$ has the same sign as

$n-2$. Thus, $\alpha > 2$ if $1 < n < 2$, but $\alpha < 2$ if $n > 2$. Also, as n tends to $1+$, α tends to ∞; as n tends to 2, α tends to 2.

To complete the argument, we only need to show that the series in (15.4) converges uniformly in $1 \leqslant n \leqslant 2$ for one particular value of $m \neq 0$; the sum is then a continuous function of n, and so $x = 2$ for $1 < n < 2$. Choose $m = -\frac{1}{2}$. By (15.5) and Stirling's formula, there exists a constant K, independent of k and n, such that

$$|\Gamma(-\tfrac{1}{2} + kn)/\Gamma(kn - k + \tfrac{1}{2})\, 2^{kn} k!| \leqslant K k^{-3/2}.$$

Hence, the series in (15.4) converges uniformly in n on any interval in $[1, \infty)$ when $m = -\frac{1}{2}$. This completes the proof that (15.4) holds for $x = 2$ and any real value of m when $0 < n \leqslant 2$.

Last, we shall show that for $n > 2$, (15.4) is valid for $x = \alpha < 2$. By the argument above, the series in (15.4) converges uniformly for $2 \leqslant n < \infty$, and so it is sufficient to prove the assertion for one particular value of n greater than 2. We shall choose $m = 1$ and $n = 3$. Then from (15.3) we see that $\alpha = \sqrt{5} - 1$. We therefore must prove that

$$\sum_{k=0}^{\infty} \frac{\Gamma(3k+1)}{\Gamma(2k+2)\, 8^k k!} = \sqrt{5} - 1. \tag{15.6}$$

Now since this sum is known to be equal to 2 or $\sqrt{5} - 1$, it suffices to show that this sum is less than 2. Let

$$a_k = \frac{\Gamma(3k+1)}{\Gamma(2k+2)\, 8^k k!}.$$

Then for $k \geqslant 2$,

$$\frac{a_{k+1}}{a_k} = \frac{3(3k+1)(3k+2)}{16(k+1)(2k+3)} = \frac{27}{32} - \frac{81k+69}{32(k+1)(2k+3)} < \frac{27}{32}.$$

Hence,

$$\sum_{k=0}^{\infty} a_k < 1 + \tfrac{1}{8} + \tfrac{3}{64} \sum_{k=0}^{\infty} (\tfrac{27}{32})^k = 1 + \tfrac{1}{8} + \tfrac{3}{10} < 2.$$

This establishes (15.6). Thus, we have shown that the equality in Example 1 is valid for $0 < n \leqslant 2$ and invalid for $n > 2$.

In connection with all of the examples below, it should be kept in mind that in the proofs the relevant root X of the equation $X = A \log X$, $A \geqslant e$, as was emphasized in the remarks made after Entry 13, is that root which lies between 1 and e. If X is a root of $X = -A \log X$, $A \geqslant e$, then there is no ambiguity, as the root, which is between 0 and 1, is unique.

EXAMPLE 2. Let a be positive and suppose that m, n, and p are real and nonzero. Define the positive real number x by the relation $(\log x)^m = ax^n$. Then, for $a \leqslant |m/en|^m$,

$$x^p = p \sum_{k=0}^{\infty} \frac{(mp+nk)^{k-1} a^{k/m}}{m^{k-1} k!}.$$

Proof. Define $y > 0$ by $x^n = y^m$. A short calculation gives

$$\frac{\log y}{y} = \frac{n}{m} a^{1/m}.$$

We now apply Entry 13 with x replaced by y and a replaced by $(m/n) a^{-1/m}$. For $|m/n| a^{-1/m} \geqslant e$, we then have

$$x^p = y^{mp/n} = \sum_{k=0}^{\infty} \frac{(mp/n)(mp/n+k)^{k-1}}{(ma^{-1/m}/n)^k k!},$$

from which the desired result follows.

Examples 3i–3viii and 4i–4iv arise from Entry 13 by suitable changes of variables. Example 3i is essentially the same as Entry 13 except that x is now defined in either of two ways by $x = \pm a \log x$. For Example 3ii, a is replaced by $\pm a/\log a$ in Entry 13. The case $x = 1$ in the first equality of Example 3iii is a problem posed in [49]. This problem can also be deduced from the corollary to Entry 13.

EXAMPLE 3iii. Let a be real with $|a| \leqslant e$ and define the real number x by either of the two equalities $x = ae^{\pm x}$. Then

$$e^x = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1} a^k}{k!} \quad \text{and} \quad e^{-x} = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}(-a)^k}{k!},$$

respectively.

Proof. Let $x = \log y$. Then we have, respectively, $y/\log y = 1/a$ and $y^{-1}/\log(y^{-1}) = -1/a$. Now apply Entry 13 with a replaced by $1/a$ and $-1/a$, respectively.

Example 3iv follows from Entry 13 upon replacing a by $\pm 1/\log a$.

EXAMPLE 3v. Let a be positive with $|\log a| \leqslant e$. Define a positive real number x by either of the two relations $x^{\pm x} = a$. Then

$$\frac{1}{x} = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}(\mp \log a)^k}{k!}.$$

Proof. Let $x = 1/y$. Then it follows that $\mp(\log y)/y = \log a$. Now apply Entry 13 with x replaced by y and a replaced by $\mp 1/\log a$.

Example 3vi is identical to Example 3iii except that the relation $x = ae^{\pm x}$ has been replaced by $x = ae^{\mp x}$ in the second notebook.

EXAMPLE 3vii. Let a be real and define the real number x by either of the relations $e^x \pm x = a$. Then, respectively, if $a \leqslant -1$,

$$e^{-e^x} = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}(-1)^k e^{ak}}{k!},$$

and if $a \geqslant 1$,

$$e^{e^x} = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1} e^{-ak}}{k!}.$$

Proof. Let $x = \log\log y$. Then $e^a = y(\log y)^{\pm 1}$. In the former case $\log(1/y)/(1/y) = -e^a$, and in the latter case $y/\log y = e^a$. Now apply Entry 13. In the former case x is replaced by $1/y$ and a is replaced by $-e^{-a}$; in the latter case x is replaced by y and a by e^a.

Example 3viii is the same as Example 3vii, except that x has been replaced by $\log x$.

EXAMPLE 4i. Let $x > 0$ and define v by $v = x^v$. Then for $|\log x| \leqslant 1/e$,

$$v = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}(\log x)^k}{k!}.$$

Proof. In Entry 13, replace a by $1/\log x$. Then x is replaced by v.

For v Ramanujan writes

$$x^{x^{x^{\&c}}}.$$

It is curious that Eisenstein [21] used this same notation. In Examples 4ii–4iv, similar notations are employed by Ramanujan.

Example 4ii is the same as Example 3vii but with x and a replaced by v and $\pm x$, respectively. Example 4iii is simply a reformulation of Entry 13 with x replaced by e^v and a replaced by x. Example 4iv is another version of Example 4ii, but with x replaced by $-x$ in the former equality and v replaced by $-v$ in the latter equality; in other words, x and v satisfy either of the relations $v = \pm\log(x + v)$.

Ramanujan next attempts to generalize Entry 13 by considering the functions $\varphi_r(n)$ defined by the equation

$$x^n \varphi_r(n) = F_r(n), \tag{16.1}$$

where $x = a \log x$, $F_r(n)$ is defined in (12.1), and where a is specified prior to Entry 12. Thus, $\varphi_{-1}(n) = 1/n$.

ENTRY 16a. *If r is an integer and n is any complex number, then*

$$n\varphi_r(n) = \varphi_{r+1}(n) - \log x \varphi_{r+1}(n+1). \tag{16.2}$$

Proof. Using (16.1) and the relation $x/a = \log x$ in Entry 12, we readily deduce (16.2).

Putting $r = -2$ in (16.2), we find that

$$\varphi_{-2}(n) = \frac{1}{n^2} - \frac{\log x}{n(n+1)} = \frac{1 - \log x}{n(n+1)} + \frac{1}{n^2(n+1)}.$$

Letting $r = -3$ in (16.2), we get

$$\begin{aligned}\varphi_{-3}(n) &= \frac{1-\log x}{n^2(n+1)} + \frac{1}{n^3(n+1)} - \frac{\log x}{n}\left\{\frac{1-\log x}{(n+1)(n+2)} + \frac{1}{(n+1)^2(n+2)}\right\} \\ &= \frac{(1-\log x)^2}{n(n+1)(n+2)} + \frac{(3n+2)(1-\log x)}{n^2(n+1)^2(n+2)} + \frac{3n+2}{n^3(n+1)^2(n+2)}.\end{aligned}$$

Both of these formulas for $\varphi_{-2}(n)$ and $\varphi_{-3}(n)$ were given by Ramanujan. It is clear from the recursion formula (16.2) that $\varphi_{-k}(n)$, $k \geqslant 1$, is a polynomial in $\log x$ of degree $k-1$ with the coefficients being rational functions of n.

We now turn to the calculation of $\varphi_r(n)$ when r is nonnegative. Let $x = e^u$, and so $a = e^u/u$. Putting $\varphi_r(n) = g_r(n, u) = g_r(u)$, we find from (16.1) that

$$\begin{aligned}g_r(u) &= \sum_{k=0}^{\infty} \frac{(n+k)^{r+k} e^{-u(n+k)} u^k}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+k)^{r+k+j}(-1)^j u^{k+j}}{k!\,j!}.\end{aligned}$$

We first calculate $g_0(u)$. Using the latter representation above, we find that

$$g_0^{(m)}(u) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(n+k)^{k+j}(-1)^j (k+j)!\, u^{k+j-m}}{k!\,j!\,(k+j-m)!}, \qquad m \geqslant 0,$$

and so $g_0^{(m)}(0) = S(m, m)$, where

$$S(m, m) = \sum_{k=0}^{m} (-1)^{k+m} \binom{m}{k} (n+k)^m.$$

It is well known [35, p. 134] that $S(m, m) = m!$ Thus,

$$g_0(u) = \sum_{m=0}^{\infty} u^m = \frac{1}{1-u}. \tag{16.3}$$

Next, we shall show, by induction on r, that

$$g_r(u) = \sum_{k=1}^{r+1} \frac{\psi_k(r, n)}{(1-u)^{r+k}}, \qquad r \geqslant 0, \tag{16.4}$$

where $\psi_k(r, n)$, $1 \leqslant k \leqslant r+1$, is independent of u. By (16.3), (16.4) is valid for $r = 0$. A direct calculation shows that, for $r \geqslant 1$,

$$\begin{aligned}(1-u)\, g_r(u) &= n g_{r-1}(u) + u g'_{r-1}(u) \\ &= n g_{r-1}(u) - (1-u)\, g'_{r-1}(u) + g'_{r-1}(u).\end{aligned}$$

Using the induction hypothesis, we thus find from (16.4) that

$$\begin{aligned}(1-u)\, g_r(u) = n \sum_{k=1}^{r} \frac{\psi_k(r-1, n)}{(1-u)^{r+k-1}} - \sum_{k=1}^{r} \frac{(r+k-1)\, \psi_k(r-1, n)}{(1-u)^{r+k-1}} \\ + \sum_{k=1}^{r} \frac{(r+k-1)\, \psi_k(r-1, n)}{(1-u)^{r+k}},\end{aligned}$$

or

$$g_r(u) = \sum_{k=1}^{r+1} \frac{(n-r-k+1)\, \psi_k(r-1, n) + (r+k-2)\, \psi_{k-1}(r-1, n)}{(1-u)^{r+k}},$$

where we define $\psi_0(r-1, n) = 0 = \psi_{r+1}(r-1, n)$. Hence, (16.4) is established, and moreover we have proven the recursion formula

$$\psi_k(r, n) = (n-r-k+1)\, \psi_k(r-1, n) + (r+k-2)\, \psi_{k-1}(r-1, n), \tag{16.5}$$

where $1 \leqslant k \leqslant r+1$.

ENTRY 16b. *Let r and t be integers such that $1 \leqslant t \leqslant r+2$. Then*

$$\psi_t(r+1, n) = (n-1)\, \psi_t(r, n-1) + \psi_{t-1}(r+1, n) - \psi_{t-1}(r+1, n-1),$$

where $\psi_t(r, n) = 0$ if $t \notin \{k\colon 1 \leqslant k \leqslant r+1\}$.

Proof. Employing (16.4) in (16.2), we obtain the identity

$$n \sum_{k=1}^{r+1} \frac{\psi_k(r,n)}{(1-\log x)^{r+k}}$$
$$= \sum_{k=1}^{r+2} \frac{\psi_k(r+1,n)}{(1-\log x)^{r+k+1}} + \{(1-\log x)-1\} \sum_{k=1}^{r+2} \frac{\psi_k(r+1,n+1)}{(1-\log x)^{r+k+1}}.$$

Equating coefficients of $(1-\log x)^{-r-t}$, $1 \leqslant t \leqslant r+2$, on both sides and replacing n by $n-1$, we deduce the desired recursion formula.

Using either of the recursion formulas given in (16.5) and Entry 16b, together with the value $\psi_1(0,n)=1$ from (16.3), we can successively calculate the coefficients of $\varphi_1(n)$, $\varphi_2(n)$, $\varphi_3(n)$,.... We thus find that

$$\varphi_1(n) = \frac{n-1}{(1-\log x)^2} + \frac{1}{(1-\log x)^3},$$

$$\varphi_2(n) = \frac{(n-1)(n-2)}{(1-\log x)^3} + \left((n-1)(n-2)\left\{\frac{1}{n-2}+\frac{2}{n-1}\right\}\Big/(1-\log x)^4\right)$$
$$+ \frac{3}{(1-\log x)^5},$$

$$\varphi_3(n) = \frac{(n-1)(n-2)(n-3)}{(1-\log x)^4}$$
$$+ \left((n-1)(n-2)(n-3)\left\{\frac{1}{n-3}+\frac{2}{n-2}+\frac{3}{n-1}\right\}\Big/(1-\log x)^5\right)$$
$$+ \frac{15n-35}{(1-\log x)^6} + \frac{15}{(1-\log x)^7},$$

all of which are given by Ramanujan.

COROLLARY 1. *Let x be complex. If $-\rho < n < 1$, where ρ is the unique real root of $ye^{y+1}=1$, then*

$$\frac{e^x}{1-n} = \sum_{k=0}^{\infty} \frac{(x+kn)^k}{k!\,e^{kn}}. \tag{16.6}$$

If $n > 1$, then

$$\frac{e^{mx/n}}{1-m} = \sum_{k=0}^{\infty} \frac{(x+kn)^k}{k!\,e^{kn}}, \tag{16.7}$$

where $0 < m < 1$ and $e^m/m = e^n/n$.

Proof. By (12.1), (16.1), and (16.3), for any y and for $|a| > e$,

$$F_0(y) = \sum_{k=0}^{\infty} \frac{(y+k)^k}{a^k k!} = \frac{t^y}{1-\log t}, \tag{16.8}$$

where $t = a \log t$. Recall that if $a > e$, then t is that root of this equation which lies between 1 and e; if $a < -e$, then t denotes the unique real root of this equation.

When $n = 0$, (16.6) reduces to the Maclaurin series for e^x. Let $a = e^n/n$, where $-\rho < n < 1$, $n \neq 0$. In this case, $|a| > e$, and the appropriate root t is equal to e^n. Putting $y = x/n$, we find that (16.8) reduces to (16.6).

If $n = 1$, the series in (16.6) diverges, but if $n > 1$ it converges. In the latter case, let $a = e^m/m$, where m is defined in the hypotheses. Then by (16.6),

$$\begin{aligned}\frac{e^{my}}{1-m} &= \sum_{k=0}^{\infty} \frac{(y+k)^k}{(e^m/m)^k\, k!} = \sum_{k=0}^{\infty} \frac{(y+k)^k}{(e^n/n)^k\, k!} \\ &= \sum_{k=0}^{\infty} \frac{(yn+kn)^k}{k!\, e^{kn}}.\end{aligned}$$

Thus, (16.7) readily follows.

As indicated in the proof, this very interesting corollary is a generalization of the familiar Maclaurin series for e^x. Characteristically, Ramanujan states no conditions on n for (16.6) to hold. By Newton's method, it may readily be shown that $\rho = 0.27846454\ldots$. The second part of Corollary 1, namely (16.7), is not given by Ramanujan. Equality (16.6) was apparently first established by Jensen [42]. Other proofs have been given by Dupart *et al.* [20] and by Gould [29]. Carlitz [13] has employed this corollary in establishing the orthogonality of a certain set of polynomials.

COROLLARY 2. *For each nonnegative integer r,*

$$n^r = \lim_{a \to \infty} \varphi_r(n) = \sum_{k=1}^{r+1} \psi_k(r, n).$$

Proof. The first equality above follows immediately from the definitions of $F_r(n)$ and $\varphi_r(n)$ given in (12.1) and (16.1), respectively. The second equality follows from the definition of $\psi_k(r, n)$ in (16.4).

Next, fix $a > 1/e$. For real h, define $x > 0$ by the relation

$$x^x = a^a e^h. \tag{17.1}$$

Then $x\log x = a\log a + h$ and $(1+\log x)\,dx/dh = 1$, i.e.,

$$(x + a\log a + h)\frac{dx}{dh} = x. \tag{17.2}$$

At $h=0$, $x=a$ and

$$\frac{dx}{dh} = \frac{1}{1+\log a} \equiv n. \tag{17.3}$$

Since $h = h(x)$ extends to a one-to-one analytic function in a neighborhood of a, there is an analytic inverse $x = x(h)$ in a neighborhood of the origin. Thus we have an expansion of the form

$$\frac{x-a}{a} = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{A_k}{k!}\left(\frac{h}{a}\right)^k, \tag{17.4}$$

where $|h|$ is sufficiently small.

ENTRY 17. *For $r \geqslant 2$, we have*

$$A_r = n(r-2)A_{r-1} + n\sum_{k=1}^{r-1}\binom{r-1}{k}A_kA_{r-k}. \tag{17.5}$$

Proof. Substituting (17.4) in the differential equation (17.2), we obtain the identity

$$\left(\log a + \frac{h}{a}\right)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}$$
$$= \left\{1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{k!}\left(\frac{h}{a}\right)^k\right\}\left\{1 - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}\right\},$$

or, by (17.3),

$$\left(1 + n\frac{h}{a}\right)\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}$$
$$= n + n\sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{k!}\left(\frac{h}{a}\right)^k\left\{1 - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{A_k}{(k-1)!}\left(\frac{h}{a}\right)^{k-1}\right\}.$$

Equating coefficients of $(h/a)^{r-1}$, $r \geqslant 2$, we obtain

$$\frac{A_r}{(r-1)!} - n\frac{A_{r-1}}{(r-2)!} = n\left\{-\frac{A_{r-1}}{(r-1)!} + \sum_{k=1}^{r-1}\frac{A_k}{k!}\frac{A_{r-k}}{(r-1-k)!}\right\}.$$

The recurrence relation (17.5) now easily follows.

In Ramanujan's version of (17.5) the terms with $k=j$ and $k=r-j$, $1 \leqslant j \leqslant r/2$, are combined. However, he has incorrectly written the last term. On p. 36, line 4 (in the pagination of [53, Vol. II]), multiply $A_{(r-1)/2}$ by $A_{(r+1)/2}$ and square $A_{r/2}$.

We have seen that $dx/dh = 1/(1+\log x) \equiv N$ and

$$A_1 = \left.\frac{dx}{dh}\right|_{h=0} = N|_{x=a} = n. \tag{17.6}$$

In general, it follows from (17.4) that for $r \geqslant 1$,

$$A_r = (-a)^{r-1} \left.\frac{d^r x}{dh^r}\right|_{h=0}. \tag{17.7}$$

Inducting on r, we find that there are numbers $a(r,k)$ for which

$$\sum_{k=0}^{r-2} a(r,k)\, N^{2r-k-1} = (-x)^{r-1} \frac{d^r x}{dh^r}, \qquad r \geqslant 2. \tag{17.8}$$

Differentiating both sides of (17.8) with respect to h and comparing coefficients of N^{2r-k+1}, we obtain the following recursion formula given by Ramanujan:

$$a(r+1,k) = (r-1)\, a(r,k-1) + (2r-k-1)\, a(r,k), \tag{17.9}$$

where $r \geqslant 2$, $0 \leqslant k \leqslant r-1$, and $a(r,k)$ is defined to be 0 when $k<0$ or $k > r-2$. Setting $h=0$ in (17.8) and using (17.7), we have

$$A_r = \sum_{k=0}^{r-2} a(r,k)\, n^{2r-k-1}, \qquad r \geqslant 2. \tag{17.10}$$

From (17.6), (17.9), and (17.10), Ramanujan has calculated A_r $(1 \leqslant r \leqslant 7)$ as follows:

$$A_1 = n$$

$$A_2 = n^3$$

$$A_3 = 3n^5 + n^4$$

$$A_4 = 15n^7 + 10n^6 + 2n^5$$

$$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$$

$$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$$

$$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8.$$

EXAMPLE 1. For $n=1$ and $r \geqslant 2$,

$$\sum_{k=0}^{r-2} a(r, k) = A_r = (r-1)^{r-1}.$$

Proof. The first equality follows from setting $n=1$ in (17.6). In (17.1), let $a=1$ and $x=1/y$. We then easily find that $y/\log y = -1/h$. Now apply Entry 13 with $n=-1$, x replaced by y, and $a=-1/h$. Accordingly, we find that, for $|he| \leqslant 1$,

$$x = 1/y = -\sum_{k=0}^{\infty} \frac{(k-1)^{k-1}(-h)^k}{k!}.$$

On the other hand, setting $a=1$ in (17.4) yields

$$x = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} A_k h^k}{k!}.$$

A comparison of these two series yields the desired result.

EXAMPLE 2. Fix a, $0 < a < e$. For real h, define $x > 0$ by the relation $x^{1/x} = a^{1/a} e^h$. Then for sufficiently small $|h|$,

$$\frac{a}{x} = 1 - \sum_{k=1}^{\infty} \frac{A_k (ah)^k}{k!}.$$

Proof. Setting $x = 1/y$, we find that $y^y = (1/a)^{1/a} e^{-h}$. Now use (17.4) with a replaced by $1/a$, h replaced by $-h$, and x replaced by y. The desired equality now easily follows.

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