Mh. Math. 84, 265—273 (1977)

Monatshefte für Mathematik

Dedekind Sums and Class Numbers

By

Bruce C. Berndt, Urbana, Ill., and **Ronald J. Evans**, La Jolla, Cal.

(Received 6 December 1976)

Dedicated to L. Carlitz on his 70^{th} Birthday

Abstract

Formulas for the class number of an imaginary quadratic number field are proved. Some of these formulas were previously established by Berndt and by Goldstein and Razar with the use of analytic methods. The proofs given here use Dirichlet's classical class number formula, but otherwise the proofs are completely elementary. A key ingredient in the proofs is the reciprocity theorem for Dedekind—Rademacher sums.

Throughout the sequel, M denotes a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integral entries a, b, c, and d such that $ad - bc = 1$; k denotes a natural number; and χ denotes a character $(\bmod k)$. Write

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Let $\Gamma(k)$ be the group of matrices M such that $M \equiv \pm I \pmod k$. Let $G(k)$ denote the group of matrices generated by S_k and T. Write $A(k) = G(k) \cap \Gamma(k)$ and set $A(k)T = \{MT : M \in A(k)\}$.

Given an imaginary quadratic number field of discriminant $-k$, let $h(-k)$ denote its class number and let w_k denote the number of roots of unity in this field. As usual, let

$$((x)) = \begin{cases} x - [x] - 1/2, & \text{if } x \text{ is not an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

In 1976, GOLDSTEIN and RAZAR [5, p. 358] derived the following formula for $h(-k)^2$:

$$h(-k)^2 = \frac{w_k^2}{4} \sum_{m \pmod{ck}} \sum_{n \pmod{k}} \chi(mn) \left(\left(\frac{m}{ck}\right)\right) \left(\left(\frac{am+cn}{ck}\right)\right), \quad (1)$$

when $M \in A(k)T$, $c > 0$, and χ is odd, real, and primitive. Their proof depends upon two different formulations of the transformation formulae for

$$\sum_{m,n=1}^{\infty} \frac{\chi(m)\chi(n)}{n} e^{2\pi i m n z/k}, \quad (2)$$

where $\mathrm{Im}(z) > 0$. One of these formulations is a special case of a general theorem proved in 1973 by BERNDT [2, Theorems 2, 3]. See also [3, Theorem 1]. Moreover, using the transformation formulae for (2), but in a manner less complicated than that in [5], BERNDT [2, Theorem 4] essentially established the following class number formula:

$$\begin{aligned} \frac{4h(-k)^2}{w_k^2} = & \sum_{m \pmod{ck}} \sum_{n \pmod{k}} \chi(mn) \left(\left(\frac{m}{ck}\right)\right) \left(\left(\frac{am+cn}{ck}\right)\right) + \\ & + \sum_{m \pmod{ak}} \sum_{n \pmod{k}} \chi(mn) \left(\left(\frac{m}{ak}\right)\right) \left(\left(\frac{cm+an}{ak}\right)\right), \end{aligned} \quad (3)$$

where χ is odd, real, and primitive, and where a and c are positive integers such that either $k \mid a$ or $k \mid c$. Thus, (3) was established under less restrictions than (1). The formulation of (3) in [2] is in terms of Dedekind character sums and the generalized Bernoulli number $B_1(\chi)$. However, by the use of Dirichlet's class number formula [6, p. 405]

$$h(-k) = -\frac{w_k}{2k} \sum_{n=1}^{k-1} \chi(n)\, n, \quad (4)$$

it follows at once that $B_1(\chi) = -2h(-k)/w_k$, and so the formulation (3) is easily seen to be equivalent to that in [2, Theorem 4].

In this paper, we prove by elementary methods that the second double sum on the right side of (3) vanishes whenever $M \in A(k)T$ and χ is odd, real, and primitive (see Theorem 7 and Lemma 2).

We also give completely elementary proofs of (1) and (3) (see Theorems 4 and 8 and Lemma 2).

If h and k are integers with $k > 0$ and if x and y are real numbers, recall that the Dedekind—Rademacher sum $s(h,k;x,y)$ is defined by

$$s(h,k;x,y) = \sum_{n \pmod{k}} \left(\left(\frac{h(n+y)}{k} + x\right)\right) \left(\left(\frac{n+y}{k}\right)\right).$$

In particular, if x and y are both integers, $s(h,k;x,y)$ is the ordinary Dedekind sum $s(h,k)$. It is easily seen that $s(h,k;x,y) = s(th,tk;x,y)$ for each positive integer t.

Let $\mathscr{B}_2(x) = B_2(x-[x])$, where $B_2(x)$ denotes the second Bernoulli polynomial. Let

$$\delta(z) = \begin{cases} 1, & \text{if } z \text{ is an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

A principal ingredient in our proofs is the following reciprocity theorem for Dedekind—Rademacher sums [4], [7]. If h and k are coprime, positive integers, then

$$s(h,k;x,y) + s(k,h;y,x) = -\tfrac{1}{4}\delta(x)\delta(y) + ((x))((y)) + \frac{h}{2k}\mathscr{B}_2(y) + \frac{k}{2h}\mathscr{B}_2(x) + \frac{1}{2hk}\mathscr{B}_2(hy+kx). \tag{5}$$

For real x and integral m, define

$$F(m,x;\chi) = \sum_{n \pmod{k}} \chi(n) \left(\left(\frac{m(n+x)}{k}\right)\right).$$

Lemma 1. *Suppose that x is real, χ is primitive, and $(m,k) > 1$. Then $F(m,x;\chi) = 0$.*

Proof. Write $m_1 = m/(m,k)$ and $k_1 = k/(m,k)$. Then

$$F(m,x;\chi) = \sum_{n \pmod{k}} \chi(n) \left(\left(\frac{m_1(n+x)}{k_1}\right)\right) = \sum_{b \pmod{k_1}} \sum_{\substack{n \pmod{k} \\ n \equiv b \pmod{k_1}}} \chi(n) \left(\left(\frac{m_1(b+x)}{k_1}\right)\right).$$

For each positive integer r, let H_r denote the group of reduced residue classes $(\bmod r)$. It remains to show that for each $b \in H_{k_1}$,

$$\sum_{\substack{n (\bmod k) \\ n \equiv b (\bmod k_1)}} \chi(n) = 0. \tag{6}$$

Now, reduction $(\bmod k_1)$ is a homomorphism ϱ from H_k onto H_{k_1}. Choose $c \in H_k$ such that $\varrho(c) = b$. Then the sum in (6) may be written as

$$\sum_{\substack{n (\bmod k) \\ n \equiv 1 (\bmod k_1)}} \chi(cn) = \chi(c) \sum_{n \in \ker \varrho} \chi(n) = 0,$$

since by the definition of primitivity in [1, p. 168], $\chi(n) \neq 1$ for some $n \in \ker \varrho$. Q. e. d.

Let a and c be integers with $a > 0$. Define

$$s(c, a; \chi) = \sum_{n (\bmod k)} \chi(n) s(an + c, ak). \tag{7}$$

Lemma 2. *Let χ be primitive. Then*

$$s(c, a; \chi) = \sum_{m (\bmod ak)} \sum_{n (\bmod k)} \bar{\chi}(m) \chi(n) \left(\left(\frac{m}{ak}\right)\right) \left(\left(\frac{an + cm}{ak}\right)\right).$$

Proof. We have

$$\begin{aligned} s(c, a; \chi) &= \sum_{m (\bmod ak)} \sum_{n (\bmod k)} \chi(n) \left(\left(\frac{m}{ak}\right)\right) \left(\left(\frac{m(an + c)}{ak}\right)\right) = \\ &= \sum_{m (\bmod ak)} \left(\left(\frac{m}{ak}\right)\right) F(m, c/a; \chi). \end{aligned} \tag{8}$$

By Lemma 1, $F(m, c/a; \chi) = 0$ when $(m, k) > 1$. Thus, we may restrict the sum on m in (8) by the condition $(m, k) = 1$. Replacing n by $nm^{-1} (\bmod k)$ in (8), we obtain the desired result. Q. e. d.

Theorem 3. *Let a and c be positive integers with $k \mid a$. Let χ be odd. Then*

$$s(c, a; \chi) + s(a, c; \bar{\chi}) = \sum_{n (\bmod k)} \chi(n) s(n, k).$$

Proof. Since $s(a, c; \chi)$ is unchanged when both a and c are divided by (a, c), we may assume that $(a, c) = 1$.

Setting $m = k\mu + \nu$, $0 \leqslant \mu \leqslant a-1$, $0 \leqslant \nu \leqslant k-1$, we find that

$$s(c,a;\chi) = \sum_{n(\mathrm{mod}\,k)} \chi(n) \sum_{m(\mathrm{mod}\,ak)} \left(\left(\frac{m}{ak}\right)\right)\left(\left(\frac{m(an+c)}{ak}\right)\right) =$$

$$= \sum_{n(\mathrm{mod}\,k)} \chi(n) \sum_{\mu(\mathrm{mod}\,a)} \sum_{\nu(\mathrm{mod}\,k)} \left(\left(\frac{\mu+\nu/k}{a}\right)\right)\left(\left(\frac{c(\mu+\nu/k)}{a} + \frac{\nu n}{k}\right)\right) =$$

$$= \sum_{n(\mathrm{mod}\,k)} \chi(n) \sum_{\nu(\mathrm{mod}\,k)} s(c,a;\nu n/k, \nu/k).$$

Applying the reciprocity theorem (5), we get

$$s(c,a;\chi) = \sum_{n(\mathrm{mod}\,k)} \chi(n) \sum_{\nu(\mathrm{mod}\,k)} \left\{ -s(a,c;\nu/k,\nu n/k) - \tfrac{1}{4}\,\delta\left(\frac{\nu}{k}\right)\delta\left(\frac{\nu n}{k}\right) + \right.$$
$$+ \left(\left(\frac{\nu}{k}\right)\right)\left(\left(\frac{\nu n}{k}\right)\right) + \frac{c}{2a}\mathscr{B}_2\left(\frac{\nu}{k}\right) + \frac{a}{2c}\mathscr{B}_2\left(\frac{\nu n}{k}\right) + \tag{9}$$
$$\left. + \frac{1}{2ac}\mathscr{B}_2\left(\frac{c\nu}{k} + \frac{a\nu n}{k}\right)\right\}.$$

Since $k \mid a$, $\mathscr{B}_2(\nu(c+an)/k) = \mathscr{B}_2(\nu c/k)$. Also, $\mathscr{B}_2(\nu n/k)$ is an even function of n. Thus, by summing on n, we see that the contributions of the second, fourth, fifth, and sixth expressions in curly brackets above yield 0. Thus, (9) becomes

$$s(c,a;\chi) + \sum_{n(\mathrm{mod}\,k)} \chi(n) \sum_{\nu(\mathrm{mod}\,k)} s(a,c;\nu/k,\nu n/k) = \sum_{n(\mathrm{mod}\,k)} \chi(n)\, s(n,k). \tag{10}$$

It remains to show that the double sum in (10) is $s(a,c;\bar{\chi})$. Replacing n by $n^{-1} (\mathrm{mod}\,k)$ and then replacing ν by νn, we find that

$$\sum_{n(\mathrm{mod}\,k)} \chi(n) \sum_{\nu(\mathrm{mod}\,k)} s(a,c;\nu/k,\nu n/k) =$$

$$= \sum_{n(\mathrm{mod}\,k)} \bar{\chi}(n) \sum_{\nu(\mathrm{mod}\,k)} \sum_{\mu(\mathrm{mod}\,c)} \left(\left(\frac{\mu+\nu/k}{c}\right)\right)\left(\left(\frac{a(\mu+\nu/k)}{c} + \frac{\nu n}{k}\right)\right) =$$

$$= \sum_{n(\mathrm{mod}\,k)} \bar{\chi}(n) \sum_{m(\mathrm{mod}\,ck)} \left(\left(\frac{m}{ck}\right)\right)\left(\left(\frac{am+cmn}{ck}\right)\right) =$$

$$= \sum_{n(\mathrm{mod}\,k)} \bar{\chi}(n)\, s(cn+a, ck) = s(a,c;\bar{\chi}).$$

Q. e. d.

Theorem 4. *Let a and c be positive integers with $k \mid a$. Let χ be odd, real, and primitive. Then*

$$\frac{4h(-k)^2}{w_k^2} = s(c, a; \chi) + s(a, c; \chi).$$

Proof. In view of Theorem 3, it suffices to prove that

$$\frac{4h(-k)^2}{w_k^2} = \sum_{n \pmod k} \chi(n)\, s(n, k). \tag{11}$$

Applying Lemma 2 with $a = 1$ and $c = 0$, we find that

$$\sum_{n \pmod k} \chi(n)\, s(n, k) = \left\{ \sum_{m \pmod k} \chi(m) \left(\left(\frac{m}{k}\right)\right) \right\}^2 = \left\{ \frac{1}{k} \sum_{m=1}^{k-1} \chi(m)\, m \right\}^2,$$

and (11) now follows upon the use of (4). Q. e. d.

By symmetry, the condition $k \mid a$ in Theorem 4 may be replaced by $k \mid c$. With the aid of Lemma 2, the class number formula of Theorem 4 is seen to be equivalent to that in (3). For still another (but less elementary) proof of Theorem 4, see [4, Corollary 7.5].

Theorem 5. *We have $A(k) = \langle S_k, T S_k T, \pm I \rangle$.*

Proof. Since

$$T S_k T = \begin{pmatrix} -1 & 0 \\ k & -1 \end{pmatrix},$$

clearly $\langle S_k, T S_k T, \pm I \rangle \subseteq A(k)$. Moreover, since $(T S_1 T) S_1 (T S_1 T) = -T$, the reverse inclusion follows when $k = 1$. Suppose that $k > 1$. Let $M \in A(k)$. Since also $M \in G(k)$, M can be written as

$$M = \pm (S_k^{n_0} T)(S_k^{n_1} T) \ldots (S_k^{n_{r-1}} T) S_k^{n_r},$$

where $r \geqslant 0$ and n_j is an integer with $n_j \neq 0$ for $0 < j < r$. To show that $A(k) \subseteq \langle S_k, T S_k T, \pm I \rangle$, we must show that $2 \mid r$. Suppose that $2 \nmid r$. Then since M and all powers of S_k and $T S_k T$ are in $\Gamma(k)$, it follows that $S_k^{n_0} T \in \Gamma(k)$, which is false. Q. e. d.

Theorem 6. *Let a and c be integers such that $a > 0$, $k \mid a$, and $c^2 \equiv 1 \pmod{ak}$. Let χ be odd. Then $s(c, a; \chi) = 0$.*

Proof. Since $\chi(-1) = -1$, it suffices to show that $s(an + c, ak) = s(-an + c, ak)$ for each integer n. Since $k \mid a$, it is easy to see that $(ak, c-an)=1$. Thus, replacing m by $m(c-an)$ below, we find that

$$s(an+c, ak) = \sum_{m (\mathrm{mod}\, ak)} \left(\left(\frac{m}{ak}\right)\right) \left(\left(\frac{m(an+c)}{ak}\right)\right) =$$

$$= \sum_{m (\mathrm{mod}\, ak)} \left(\left(\frac{m(c-an)}{ak}\right)\right) \left(\left(\frac{m(c^2-a^2n^2)}{ak}\right)\right) =$$

$$= s(c-an, ak),$$

since $c^2-a^2n^2 \equiv 1 (\mathrm{mod}\, ak)$. Q. e. d.

Theorem 7. *Let χ be odd. Then $s(c,a;\chi)=0$ for each $M \in A(k)T$ with $a>0$.*

Proof. We make the convention that $s(en+c, ek)=0$ when $e=0$. Thus, $s(c,e;\chi)=0$ when $e=0$. In particular, $s(c,a;\chi)=0$ when $M=T$.

We induct on the length of the word M in $A(k)T$. Assume that

$$s(c,a;\chi)=0 \text{ for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A(k)T \text{ with } a \geqslant 0.$$

Consider the four matrices

$$-TS_k^{\pm 1}TM = \begin{pmatrix} a & * \\ c \mp ak & * \end{pmatrix}, \quad S_k M = \begin{pmatrix} a+ck & * \\ c & * \end{pmatrix},$$

$$\text{and} \quad S_k^{-1}M = \begin{pmatrix} a-ck & * \\ c & * \end{pmatrix}.$$

By Theorem 5 and induction, it suffices to show that

$$s(c \mp ak, a; \chi) = 0$$

and that

$$s(c, |a \pm ck|; \chi) = s(-c, |a \pm ck|; \chi) = 0. \tag{12}$$

By the definition of $s(c,a;\chi)$ and the induction hypothesis, it follows that

$$s(c \mp ak, a; \chi) = s(c, a; \chi) = 0.$$

It remains to prove (12).

First, suppose that $a=0$. Since $(a,c)=1$, we have $c=\pm 1$. Trivially, $k \mid (a \pm ck)$. Thus, we can apply Theorem 6 to conclude that (12) holds.

Now suppose that $a > 0$. Replacing n by $-n$ in (7), we see that $s(c,a;\chi) = s(-c,a;\chi)$, and so

$$s(e,a;\chi) = 0\,, \tag{13}$$

where $e = |c|$. Similarly, $s(c,f;\chi) = s(-c,f;\chi)$, where $f = |a \pm ck|$. It thus suffices to show that $s(e,f;\chi) = 0$ when $f > 0$. Note that $e > 0$, since $c \equiv \pm 1 \pmod{k}$ and $k > 1$. Hence, we may apply Theorem 3 to deduce that

$$\begin{aligned} s(e,f;\chi) - \sum_{n \,(\mathrm{mod}\, k)} \chi(n)\, s(n,k) &= \\ = -\sum_{n \,(\mathrm{mod}\, k)} \bar{\chi}(n) &\sum_{m \,(\mathrm{mod}\, ek)} \left(\left(\frac{m}{ek}\right)\right) \left(\left(\frac{m(en+f)}{ek}\right)\right) \end{aligned} \tag{14}$$

and

$$\begin{aligned} s(e,a;\chi) - \sum_{n \,(\mathrm{mod}\, k)} \chi(n)\, s(n,k) &= \\ = -\sum_{n \,(\mathrm{mod}\, k)} \bar{\chi}(n) &\sum_{m \,(\mathrm{mod}\, ek)} \left(\left(\frac{m}{ek}\right)\right) \left(\left(\frac{m(en+a)}{ek}\right)\right). \end{aligned} \tag{15}$$

Replacing n by $-n$ in the right side of (15), we observe that this expression remains unchanged when a is replaced by $-a$. It follows that the right side of (15) also remains invariant when a is replaced by f. Thus, (13), (14), and (15) show that

$$s(e,f;\chi) = s(e,a;\chi) = 0\,. \quad \text{Q. e. d.}$$

Theorem 8. *Let $M \in A(k)T$ with $c > 0$. Let χ be odd, real, and primitive. Then*

$$h(-k)^2 = \frac{w_k^2}{4} \sum_{n \,(\mathrm{mod}\, k)} \chi(n)\, s(cn + a, ck)\,. \tag{16}$$

Proof. First, suppose that $a \neq 0$. By Theorem 4,

$$\frac{4h(-k)^2}{w_k^2} = s(c,|a|;\chi) + s(|a|,c;\chi)\,. \tag{17}$$

Since $s(c,|a|;\chi) = s(-c,|a|;\chi)$, it follows from Theorem 7 that $s(c,|a|;\chi) = 0$. Hence, since $s(a,c;\chi) = s(-a,c;\chi)$, (16) follows from (17).

Finally, suppose that $a = 0$. Since $(a,c) = 1$ and $c > 0$, it follows that $c = 1$. By Lemma 2,

$$s(0,1;\chi) = \sum_{m (\operatorname{mod} k)} \sum_{n (\operatorname{mod} k)} \chi(mn) \left(\left(\frac{m}{k}\right)\right) \left(\left(\frac{n}{k}\right)\right) = \left\{\frac{1}{k} \sum_{n=1}^{k-1} \chi(n) n\right\}^2.$$

Thus, upon the use of (4), (16) follows. Q. e. d.

With the aid of Lemma 2, (16) may be converted into the form (1) given by Goldstein and Razar [5].

Corollary 9. *Let χ be real, odd, and primitive. Then*

$$h(-k)^2 = \frac{w_k^2}{4} \sum_{n (\operatorname{mod} k)} \chi(n) s(n,k).$$

Proof. Choose $M = T$ in Theorem 8. Q. e. d.

References

[1] Apostol, T. M.: Introduction to Analytic Number Theory. New York: Springer. 1976.

[2] Berndt, B. C.: Character transformation formulae similar to those for the Dedekind eta-function. Proc. Symp. Pure Math., Vol. 24, p. 9—30. Providence: American Mathematical Society. 1973.

[3] Berndt, B. C.: On Eisenstein series with characters and the values of Dirichlet L-functions. Acta Arith. **28**, 299—320 (1975).

[4] Berndt, B. C.: Reciprocity theorems for Dedekind sums and generalizations. Advances Math. **23**, 285—316 (1976).

[5] Goldstein, L. J., and M. Razar: A generalization of Dirichlet's class number formula. Duke Math. J. **43**, 349—358 (1976).

[6] Hasse, H.: Vorlesungen über Zahlentheorie, 2. Aufl. Berlin—Heidelberg—New York: Springer. 1964.

[7] Rademacher, H.: Some remarks on certain generalized Dedekind sums. Acta Arith. **9**, 97—105 (1964).

B. C. Berndt
Department of Mathematics
University of Illinois
Urbana, IL 61801, U.S.A.

R. J. Evans
Department of Mathematics
University of California at San Diego
La Jolla, CA 92093, U.S.A.