# DISCRETE FREE PRODUCTS OF TWO COMPLEX CYCLIC MATRIX GROUPS

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**1. Introduction.** All 2-by-2 matrices in this paper are to be viewed as linear fractional transformations on the extended complex plane $\mathbb{C}^*$. Let $L^+$ and $L^-$ be the open half-planes to the right and left, respectively, of the extended imaginary axis $L$. Let $\Lambda$ be the set of complex 2-by-2 matrices $A$ with real trace and determinant $\pm 1$ such that $A(L^+) \subset L^-$. Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, where

$$\Omega_1 = \{A \in \Lambda : \det A = 1, |\text{tr } A| \geq 2\},$$

$$\Omega_2 = \{A \in \Lambda : \det A = 1, |\text{tr } A| = 2\cos(\pi/q) \text{ for some integer } q > 2\},$$

$$\Omega_3 = \{A \in \Lambda : \text{tr } A = 0\},$$

and

$$\Omega_4 = \{A \in \Lambda : \det A = -1, A^2 \in \Lambda\}.$$

Observe that the elements of $\Omega_1 \cup \Omega_4$ have infinite order, while those of $\Omega_2 \cup \Omega_3$ have finite order.

We will prove that whenever $A \in \Omega$ and $B \in \Omega$ do not both have a fixed point on $L$, then the group $\langle A, B^t \rangle$ is the discrete free product $\langle A \rangle * \langle B^t \rangle$, where $B^t$ denotes the transpose of $B$. The case where both $A$ and $B$ have a fixed point on $L$ is also discussed. We show in addition that if $A \in \Omega$ and $B \in \Omega$ are real, then $\langle A, B^t \rangle$ is the discrete free product $\langle A \rangle * \langle B^t \rangle$ if and only if for every real $u$,

$$\{A, B^t\} \not\subset \left\{ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 0 & -u \\ 1/u & 0 \end{vmatrix} \right\}.$$

The significance of these results is discussed in §3. In particular, we show there that the latter result implies that the free products in [2, Theorem 1] are all discrete. We thereby fill a gap in [2, §4], wherein the discreteness was proved only in a special case.

Discrete free products of two cyclic matrix groups have been extensively studied, along different lines. For example, Purzitsky [5] has given necessary and sufficient conditions for any group $\langle A, B \rangle$ generated by *real* linear fractional transformations $A$ and $B$ of determinant 1 to be the discrete free product $\langle A \rangle * \langle B \rangle$. (See also [6].)

**2. Definitions and notation.** Let $P$ denote the set of matrices $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ of determinant $\pm 1$ such that either $a, b \geq 0 \geq c, d$ or $a, b \leq 0 \leq c, d$. We reserve the notation $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for a matrix (or transformation) in $\Omega \cap P$. In fact, the symbol $M$ can be used to denote an arbitrary real matrix in $\Omega$, in view of Lemma 5. Given a transformation $M$, we stipulate

without loss of generality that $a, b \geq 0 \geq c, d$. If, moreover, $c(a + d) \geq 0$, $M$ is said to be *plussed*.

Let $\omega_M$ and $\omega'_M$ be the fixed points of $M$, with $|\omega_M| \leq |\omega'_M|$ if the fixed points are on the extended real line $\mathbb{R}^*$, and with Im $\omega_M > 0$ otherwise. Note that $\omega_M$ is finite, because otherwise $\omega_M = \omega'_M = \infty$ and $M$ would have the form $\begin{vmatrix} 1 & * \\ 0 & 1 \end{vmatrix}$, which contradicts the fact that $M \in P$. Note also that Re $\omega_M \leq 0$, since $M(L^+) \subset L^-$. Let $L_M$ denote the extended vertical line through $\omega_M$. Let $L_M^+$ and $L_M^-$ denote the open half-planes to the right and left of $L_M$, respectively.

If $E$ is a nonsingular 2-by-2 matrix and $\mathscr{A}$ is a set of 2-by-2 matrices, write $\mathscr{A}^E = \{A^E : A \in \mathscr{A}\}$, where $A^E = EAE^{-1}$. A 2-by-2 matrix $S$ is said to be an *L-map* if $S(L^+) = L^+$. Note that if $S$ and $S_1$ are $L$-maps, then so are $S^{-1}$ and $S \circ S_1$. In addition, $S(L) = L$, $S(L^-) = L^-$, $\Lambda^S = \Lambda$, and $\Omega^S = \Omega$.

Let $I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $T_u = \begin{vmatrix} 0 & -u \\ 1/u & 0 \end{vmatrix}$, and $W_r = \begin{vmatrix} 1 & r \\ 0 & -1 \end{vmatrix}$. Write $T = T_1$ and $W = W_0$.

Given $U \subset \mathbb{C}^*$, let cl $U$ denote the closure of $U$ in $\mathbb{C}^*$ and let ccl $U$ denote the complement of the closure of $U$ in $\mathbb{C}^*$.

**3. Main results.** We now present the main theorems. The proofs are postponed until §6.

THEOREM 1. *Let $A, B \in \Omega$ and $C = B^W$. Suppose that $A$ and $C$ do not both have a fixed point on $L$. Then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$.*

THEOREM 2. *Let $A, B \in \Omega$ and $C = B^W$. If $A$ and $B$ are real, then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ if and only if for every real $u$, $\{A, C\} \not\subset \{W, T_u\}$.*

Consider the Hecke group $\langle S_\lambda, T \rangle$ generated by $S_\lambda = \begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix}$ and $T = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$. It is well-known that when $\lambda = 2\cos(\pi/q)$ for an integer $q \geq 3$, then $\langle S_\lambda, T \rangle$ is the discrete free product $\langle TS_\lambda \rangle * \langle T \rangle$, and that when $\lambda$ is a *complex* number of modulus $\geq 2$, then $\langle S_\lambda, T \rangle$ is the discrete free product $\langle S_\lambda \rangle * \langle T \rangle$. Theorem 1 is sufficiently general to imply these results. For, if $\lambda = 2\cos(\pi/q)$ for an integer $q \geq 3$, apply Theorem 1 with $A = TS_\lambda$, $B = T$. If $\lambda \in \mathbb{C}$, $|\lambda| \geq 2$, apply Theorem 1 with $A = S_\lambda^G$, $B = T^G$, where $G = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$.

If, in Theorem 1, $C$ is defined to be $B^t$ instead of $B^W$, the resulting statement is equivalent. To see this, we need only show that $\Omega^W = \Omega^t$. Assume that for some $\tau_1 \in L^+$ and some $A \in \Lambda$, $A^{-1}(\tau_1) = \tau_2 \in \text{cl}(L^+)$. Then $A(\tau_2) \in L^+$, so for some $\tau_3 \in L^+$ close to $\tau_2$, $A(\tau_3) \in L^+$. This contradicts the fact that $A(L^+) \subset L^-$. Thus $A^{-1}(L^+) \subset L^-$ for all $A \in \Lambda$. This proves that $\Lambda = \Lambda^{-1}$. Thus $\Omega = \Omega^{-1}$; so $\Omega^t = T\Omega^{-1}T = \Omega^T$. Since $TW = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ is an $L$-map, $\Omega^{TW} = \Omega$. Therefore, $\Omega^t = \Omega^T = \Omega^W$.

Similarly, if $C$ is defined to be $B^t$ instead of $B^W$ in Theorem 2, the resulting statement is equivalent. Another equivalent formulation of Theorem 2 is as follows. "Let

$A, B \in \Omega^V$, where $V = 2^{-1/2} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$, and let $C = B^t$. If $A$ and $B$ are real, then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ if and only if, for every real pair $r, s$ satisfying $s^2 - r^2 = 1$, we have $\{A, C\} \not\subset \left\{ \begin{vmatrix} r & s \\ -s & -r \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right\}$". In the notation of [2], it follows from [2, Lemma 9] and Lemma 5 below that $J \subset \Lambda^V$; hence $K \subset \Omega^V$, so Theorem 2 now immediately implies that the free products in [2, Theorem 1] are all discrete.

Theorem 2 is, in fact, an extension of a theorem of Newman [4, Theorem 15, p. 162]. For if in Theorem 2 the determinants of $A$ and $B$ are restricted to be 1, then Theorem 2 becomes a restatement of Newman's theorem (see Lemma 5 below).

The following simple example shows that the requirement in Theorem 2 that $A$ and $B$ are real cannot be dropped. Let $A = \begin{vmatrix} 1 & i \\ 0 & -1 \end{vmatrix}$ and $C = \begin{vmatrix} 1 & 0 \\ i & -1 \end{vmatrix}$. Then $A \in \Omega$, $C \in \Omega^W$, but $\langle A, C \rangle$ is not the free product $\langle A \rangle * \langle C \rangle$, since $(AC)^3 = I$.

**4. Real conjugates of complex matrices.** Theorem 4 (below) will enable us to focus attention on those matrices in $\Omega$ which are real. First a lemma is proved.

LEMMA 3. *If $N \in \Omega$ fixes $\infty$, then $N = W_\rho$ for some $\rho$. If $N \in \Omega$ fixes $0$, then $N = W_\rho^t$ for some $\rho$.*

*Proof.* Suppose that $N(\infty) = \infty$, so $N = \begin{vmatrix} \alpha & \beta \\ 0 & \delta \end{vmatrix}$. For each $\tau \in L^+$, $N\tau = (\alpha/\delta)\tau + \beta/\delta \in L^-$. Let $\psi = \arg(\alpha/\delta)$ with $-\pi < \psi \le \pi$. Suppose that $\psi \in (-\pi, 0]$. If $\tau \in L^+$ has sufficiently large modulus and $\arg \tau$ is sufficiently close to $\pi/2$, then $N\tau \in L^+$, a contradiction. Suppose then that $\psi \in (0, \pi)$. If $\tau \in L^+$ has sufficiently large modulus and $\arg \tau$ is sufficiently close to $-\pi/2$, then $N\tau \in L^+$, a contradiction. Thus $\psi = \pi$, i.e., $\alpha/\delta < 0$.

Suppose that $\alpha \notin \mathbb{R}$. Since $\operatorname{tr} N$ is real, $\delta = \bar\alpha$. Since $\alpha/\bar\alpha$ is a negative number of modulus 1, $\alpha = -\bar\alpha$. Thus $N = \begin{vmatrix} \alpha & \beta \\ 0 & -\alpha \end{vmatrix}$, i.e., $N = W_\rho$ for some $\rho$.

Suppose that $\alpha \in \mathbb{R}$. Since $\alpha/\delta < 0$, $\det N < 0$. Thus $N \in \Omega_3$ or $N \in \Omega_4$. For all $\tau \in L^+$, $N^2\tau = (\alpha^2/\delta^2)\tau + \beta(\alpha + \delta)/\delta^2$. If $\tau \in \mathbb{R}$ is sufficiently large, then $N^2\tau \in L^+$. Thus $N^2 \notin \Lambda$, so $N \notin \Omega_4$. Therefore, $N \in \Omega_3$. It follows that $N = \begin{vmatrix} \alpha & \beta \\ 0 & -\alpha \end{vmatrix}$, i.e., $N = W_\rho$ for some $\rho$. This proves the first assertion.

Suppose that $N \in \Omega$ fixes $0$. Then, since $TW = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ is an $L$-map, $N^{TW} \in \Omega$ and $N^{TW}$ fixes $\infty$. By the first assertion, $N^{TW} = W_\rho$ for some $\rho$, so $N = W_{-\rho}^t$.

THEOREM 4. *Let $N = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \in \Omega$. Then there is an $L$-map $S$ such that $SNS^{-1}$ is real.*

*Proof.* Note that the fixed points $\omega$ and $\omega'$ of $N$ lie in $\operatorname{cl}(L^-)$, since $N(L^+) \subset L^-$. Define $\omega_1 = \operatorname{Re} \omega$, $\omega_2 = \operatorname{Im} \omega$, $\omega_1' = \operatorname{Re} \omega'$, and $\omega_2' = \operatorname{Im} \omega'$.

*Case* 1: $\omega, \omega' \in L$. Assume that $\omega = \omega'$. If $\omega \neq \infty$, conjugate $N$ by the $L$-map $\begin{vmatrix} 0 & 1 \\ 1 & -\omega \end{vmatrix}$. Thus we may assume that both fixed points of $N$ are $\infty$. Then $N = \begin{vmatrix} 1 & \beta \\ 0 & 1 \end{vmatrix}$ for some $\beta$, which contradicts the fact that $N \in \Lambda$. Hence $\omega \neq \omega'$. If neither $\omega$ nor $\omega'$ is $\infty$, conjugate $N$ by the $L$-map $\begin{vmatrix} 1 & -\omega' \\ \omega - \omega' & -\omega(\omega - \omega') \end{vmatrix}$; if one of $\omega$ or $\omega'$ is $\infty$, say $\omega = \infty$, then conjugate $N$ by the $L$-map $\begin{vmatrix} 1 & -\omega' \\ 0 & 1 \end{vmatrix}$ (which fixes $\omega = \infty$). Thus we may assume that $\omega' = 0$ and $\omega = \infty$. By Lemma 3, $N = W_\rho$ and $N = W'_\nu$ for some pair $\rho, \nu$. Hence $N = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$.

*Case* 2: $\omega' \in L$, $\omega \in L^-$. If $\omega' \neq \infty$, conjugate $N$ by the $L$-map $\begin{vmatrix} 0 & 1 \\ 1 & -\omega' \end{vmatrix}$. Then we may assume that $\omega' = \infty$. We may moreover assume that $\omega = -1$, for otherwise conjugate $N$ by the $L$-map $\begin{vmatrix} i & \omega_2 \\ 0 & -i\omega_1 \end{vmatrix}$ (which fixes $\omega' = \infty$). Thus $N = \begin{vmatrix} \alpha & \alpha - \delta \\ 0 & \delta \end{vmatrix}$. By Lemma 3, $N = \begin{vmatrix} 1 & \rho \\ 0 & -1 \end{vmatrix}$ for some $\rho$. Therefore, $N = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$.

*Case* 3: $\omega, \omega' \in L^-$ and $\omega = \omega'$. We may assume that $\omega = -1$, for otherwise conjugate $N$ by the $L$-map $\begin{vmatrix} i & \omega_2 \\ 0 & -i\omega_1 \end{vmatrix}$. Thus $N = \begin{vmatrix} 1 - \gamma & -\gamma \\ \gamma & 1 + \gamma \end{vmatrix}$. If $\gamma \in \mathbb{R}$, then $N$ is real, so suppose that $\gamma \notin \mathbb{R}$. Let $\eta = \arg \gamma$. Let $S = \begin{vmatrix} 1 & k \\ k & 1 \end{vmatrix}$, where $k = i \tan(\eta/2)$. Then $S$ is an $L$-map and

$$SNS^{-1} = \begin{vmatrix} 1 - \gamma_1 & -\gamma_1 \\ \gamma_1 & 1 + \gamma_1 \end{vmatrix}, \quad \text{where } \gamma_1 = |\gamma| \in \mathbb{R}.$$

*Case* 4: $\omega, \omega' \in L^-$, $\omega \neq \omega'$. We may assume that $\omega' = -1$, otherwise conjugate $N$ by the $L$-map $\begin{vmatrix} i & \omega'_2 \\ 0 & -i\omega'_1 \end{vmatrix}$. Assume that $\omega_2 \neq 0$. Let $S' = \begin{vmatrix} xi & 1 \\ 1 & xi \end{vmatrix}$, where $x$ is a solution of the equation

$$\omega_2 x^2 + x(|\omega|^2 - 1) - \omega_2 = 0.$$

Since the discriminant of this quadratic equation is

$$(\omega_1^2 - 1)^2 + \omega_2^4 + 2\omega_1^2 \omega_2^2 + 2\omega_2^2,$$

it follows that $x \in \mathbb{R}$. Therefore $S'$ is an $L$-map. Since $S'(-1) = -1$ and $S'(\omega) \in \mathbb{R}$ by definition of $x$, we see that $S'N(S')^{-1}$ fixes the two distinct real points $-1$ and $S'(\omega)$. It may thus be assumed without loss of generality that $\omega_2 = 0$, i.e., that $N$ fixes $-1$ and $\omega$, where $\omega < 0$, $\omega \neq -1$. Write $N = \begin{vmatrix} \sigma/2 - y & \beta \\ \gamma & \sigma/2 + y \end{vmatrix}$, where $\sigma = \operatorname{tr} N$. The equalities $N(-1) = -1$ and $N(\omega) = \omega$ imply that $\beta = \gamma\omega$ and $y = (1 - \omega)\gamma/2$. Since

$$\det N = \sigma^2/4 - \gamma^2(1 + \omega)^2/4,$$

we have

$$\gamma^2 = 4(\sigma^2/4 - \det N)/(1+\omega)^2.$$

Since $\sigma$ and $\det N$ are real by the hypothesis $N \in \Omega$, $\gamma$ is real or purely imaginary. If $\gamma \in \mathbb{R}$, then $N$ is real, so suppose that $\gamma$ is purely imaginary. Let $S = \begin{vmatrix} i/\sqrt{-\omega} & 1 \\ 1 & i\sqrt{-\omega} \end{vmatrix}$. Then $S$ is an $L$-map and $SNS^{-1} = \begin{vmatrix} \sigma/2 - z & z(1-\omega)/2\omega \\ z(1-\omega)/2 & \sigma/2 + z \end{vmatrix}$, where $z = i\gamma\sqrt{-\omega}$. Since this matrix is real, the proof is complete.

## 5. Lemmas on real matrices in $\Omega$.

LEMMA 5. *Let* $N = \begin{vmatrix} w & x \\ y & z \end{vmatrix}$ *have real entries. Then* $N \in \Lambda$ *if and only if* $N \in P$.

*Proof.* Suppose that $N \in \Lambda$. It was shown in §3 that $\Lambda = \Lambda^{-1}$. Hence $N^{-1}(\infty) \in \mathrm{cl}(L^-)$, so that either $y, z \geq 0$ or $y, z \leq 0$. Further, for all $t > 0$,

$$\frac{wt + x}{yt + z} < 0.$$

It follows that $N \in P$.

Conversely, suppose that $N \in P$. Then $N^{-1}(\infty) \notin L^+$, so for all $\tau \in L^+$,

$$\mathrm{sgn}\, \mathrm{Re}(N(\tau)) = \mathrm{sgn}\{wy|\tau|^2 + xz + (xy + wz)\mathrm{Re}\,\tau\} < 0.$$

It follows that $N \in \Lambda$. This completes the proof.

The next lemma characterizes those matrices $M \in \Omega$ which map $L_M$ onto a straight line. (Recall that $M$ always denotes a matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \in \Omega$ with $a, b \geq 0$, and $c, d \leq 0$.)

LEMMA 6. *The following are equivalent:*
  (i) $\infty \in M(L_M)$;
  (ii) $M = W_b$, *or both* $\mathrm{tr}\, M = 0$ *and* $\det M = 1$;
  (iii) $M(L_M) = L_M$.

*Proof.* We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) (note that (iii) $\Rightarrow$ (i) is obvious).

*Case* 1: $\det M = 1$. Suppose that $\infty \in M(L_M)$. If $M(\infty) = \infty$, then $M = \begin{vmatrix} a & b \\ 0 & d \end{vmatrix}$, where $a > 0$ and $d < 0$. This contradicts the fact that $\det M = 1$. Hence $M(\infty) \neq \infty$, i.e., $c < 0$. Then $-d/c \in L_M$, so $-d/c = \mathrm{Re}\,\omega_M$. By definition of $\omega_M$,

$$-d/c = \mathrm{Re}\,\omega_M = \mathrm{Re}\{(2c)^{-1}(a - d - \sqrt{(a+d)^2 - 4})\}. \tag{1}$$

It is readily seen that (1) holds if and only if $a + d = 0$. This proves (ii).

To prove that (ii) $\Rightarrow$ (iii), suppose that $\mathrm{tr}\, M = 0$. Then $c \neq 0$ and by the second equality

in (1),

$$M(\infty) = a/c = \mathrm{Re}\ \omega_M = \mathrm{Re}\ \omega'_M.$$

Thus $M$ maps each of $\omega_M$, $\omega'_M$ and $\infty$ into $L_M$, so $M(L_M) = L_M$.

*Case* 2: det $M = -1$. In this case, $\omega_M, \omega'_M \in \mathbb{R}^*$. Suppose that (i) holds. Assume that $c < 0$. Then $-d/c \in L_M$, so $-d/c = \mathrm{Re}\ \omega_M = \omega_M$. Then

$$-d/c = \omega_M = (a - d - \sqrt{(a+d)^2 + 4})/2c,$$

which is impossible. Hence $c = 0$, i.e., $M(\infty) = \infty$. By Lemma 3, $M = W_b$. This proves (ii).

To prove that (ii) $\Rightarrow$ (iii), note that if $M = W_b$, then $M$ fixes the line $\{z : \mathrm{Re}\ z = -b/2\} = L_M$. This completes the proof.

Define $\Pi$ to be the set of (real) $M \in \Omega$ which satisfy (ii) of Lemma 6. Define $D_M$ as follows. If $M \in \Pi$, let $D_M = L_M^-$. If $M \in \Omega_4$, let $D_M$ be the interior of the circle $M^2(L_M)$. For all other $M \in \Omega$, let $D_M$ be the interior of the circle $M(L_M)$. If $M \in \Omega_4$, define $D'_M$ to be the interior of the circle $M(L_M)$. (See Figures 1 through 5.) Note that since $M$ is real and conformal, the circles $M(L_M)$ and $M^2(L_M)$ are orthogonal to the real axis.
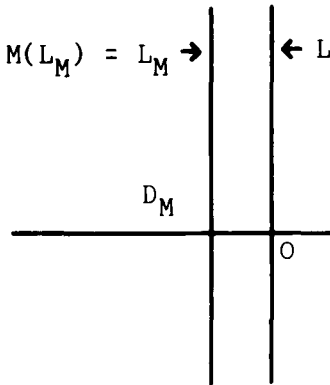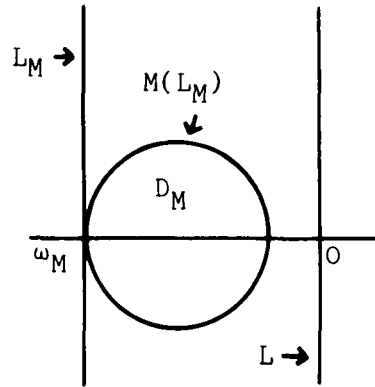


Figure 1. $M \in \Pi$                         Figure 2. $M \in \Omega_1$, $M$ plussed

Lemmas 7 through 11 (below) show that if $M$ is plussed (recall that this means $c(a+d) \geq 0$), then $M(L_M^+) = D_M$ when $M \notin \Omega_4$, and $M(L_M^+) = D'_M$ and $M^2(L_M^+) = D_M$ when $M \in \Omega_4$. Lemmas 8 through 11 also verify that $D_M$ and $D'_M$ are positioned as suggested by Figures 2 through 5. We note that if $\omega_M \in L$, then $M \in \Omega_3$. For if $\omega_M = ir$, then $M = \begin{vmatrix} 0 & -r \\ 1/r & 0 \end{vmatrix}$ when $r > 0$, and $M$ is as given in Lemma 3 when $r = 0$.

LEMMA 7. *If* $M \in \Pi$, *then* $M(L_M^+) = D_M$.

*Proof.* First suppose that $M = W_b$. Then $M$ maps the points $-b/2$, $-b/2 + i$, $\infty$ to $-b/2$, $-b/2 - i$, $\infty$, respectively. Thus $M$ fixes $L_M$ but reverses its orientation. Therefore $M(L_M^+) = L_M^- = D_M$.
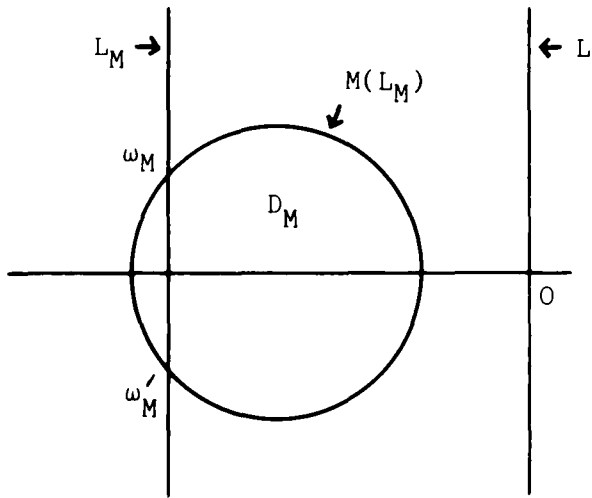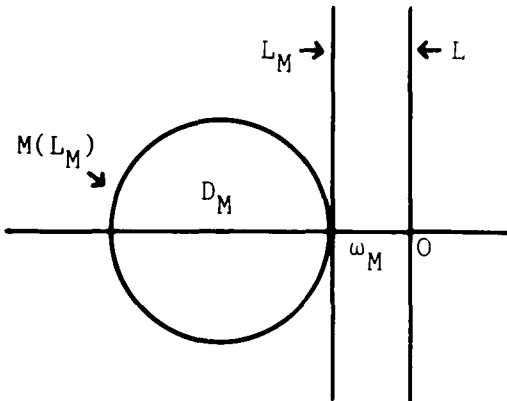
Figure 3. $M \in \Omega_2$, $M$ plussed
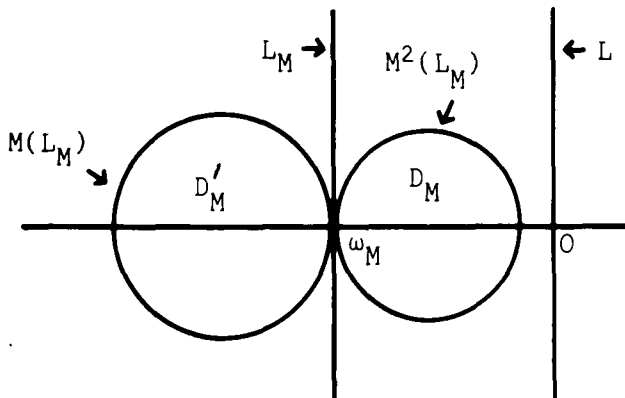


Figure 4. $M \in \Omega_3 \backslash \Pi$



Figure 5. $M \in \Omega_4$, $M$ plussed

Now suppose that $M = \begin{vmatrix} a & b \\ c & -a \end{vmatrix}$ with det $M = 1$. If $c = 0$, then det $M < 0$; hence $c \neq 0$. Therefore, $M$ maps the points $(a - i)/c$, $a/c$, $(a + i)/c$ to $(a - i)/c$, $\infty$, $(a + i)/c$, respectively. Consequently, $M$ fixes $L_M$ but reverses its orientation, so $M(L_M^+) = L_M^- = D_M$. This completes the proof.

LEMMA 8. *If $M \in \Omega_1$ is plussed, then $D_M \subset L^- \cap L_M^+$ and $M(L_M^+) = D_M$.*

*Proof.* It suffices to show that $\omega_M < M(\infty) \leq 0$. For then, clearly $D_M \subset L^- \cap L_M^+$. Also, since the upper half-plane is invariant under $M$, the sequence $M(\omega_M)$, $M(\omega_M + i)$, $M(\infty)$ will determine a clockwise orientation on $M(L_M)$, so $M(L_M^+) = D_M$.

If $c = 0$, then by Lemma 3, $M = W_\rho$ for some $\rho$. Then det $M = -1$, which contradicts the fact that $M \in \Omega_1$. Hence $c < 0$. As $M$ is plussed, $\sigma = a + d < 0$. Hence $\sqrt{\sigma^2 - 4} < -\sigma$. Dividing by $-2c$ and then adding $(a - d)/2c$, we have

$$\omega_M = (a - d - \sqrt{\sigma^2 - 4})/2c < a/c = M(\infty).$$

Finally, $a/c \leq 0$ because $M \in P$. This completes the proof.

LEMMA 9. *If $M \in \Omega_2$ is plussed, then $D_M \subset L^-$ and $M(L_M^+) = D_M$.*

*Proof.* It suffices to show that Re $\omega_M < M(\infty) \leq 0$. As shown in the proof of Lemma 8, $c < 0$, $a/c \leq 0$, and $a + d < 0$. Thus,

$$\text{Re } \omega_M = (a - d)/2c < a/c = M(\infty) \leq 0.$$

LEMMA 10. *If $M \in \Omega_3 \backslash \Pi$, then $D_M \subset L_M^-$ and $M(L_M^+) = D_M$.*

*Proof.* It suffices to show that $M(\infty) < \omega_M$. For then, clearly $D_M \subset L_M^-$. Also, since $M$ maps the upper half-plane to the lower half-plane, the sequence $M(\omega_M)$, $M(\omega_M + i)$, $M(\infty)$ will determine a clockwise orientation on $M(L_M)$, so $M(L_M^+) = D_M$.

If $c = 0$, then $M = W_\rho$ for some $\rho$, which contradicts the fact that $M \notin \Pi$. Hence $c < 0$ and

$$M(\infty) = a/c < (a - 1)/c = \omega_M.$$

This completes the proof.

LEMMA 11. *If $M \in \Omega_4$ is plussed, then $D'_M \subset L_M^-$ $D_M \subset L^- \cap L_M^+$, $M(L_M^+) = D'_M$, and $M(D'_M) = D_M$.*

*Proof.* As in the proof of Lemma 10, it suffices to show that

$$M(\infty) < \omega_M < M^2(\infty) \leq 0.$$

Write $M^2 = \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix}$. Since $M$ is plussed, $c_2 = c(a + d) \geq 0$. Also, $a_2 + d_2 = (a + d)^2 + 2 \geq 2$. This proves that $M^2$ is plussed and that $M^2 \in \Omega_1$. Thus, the inequality $\omega_M < M^2(\infty) \leq 0$ follows exactly as in the proof of Lemma 8.

If $c = 0$, then by Lemma 3, $M = W_\rho$, which contradicts the fact that $M \in \Omega_4$. Hence

$c < 0$. Clearly,

$$-(d + a) < \sqrt{(a + d)^2 + 4}.$$

Dividing by $-2c$ and then adding $(a - d)/2c$, we have

$$M(\infty) = a/c < (a - d - \sqrt{(a + d)^2 + 4})/2c = \omega_M.$$

This completes the proof.

LEMMA 12. *Define* $\mathcal{O}_M = L_M^- \cup D_M$. *If* $M$ *is plussed, then* $M^n(\mathrm{ccl}\,\mathcal{O}_M) \subset \mathcal{O}_M$ *for all* $n$ *such that* $M^n \neq I$.

*Proof.* Case 1: $M \in \Omega_1$. We have $M(L_M^+) = D_M \subset L_M^+$ by Lemma 8, and hence

$$M^n(L_M^+) \subset D_M \subset \mathcal{O}_M \qquad (n > 0). \tag{2}$$

Also, $M^{-1}(\mathrm{ccl}\,D_M) = L_M^- \subset \mathrm{ccl}\,D_M$, and hence

$$M^{-n}(\mathrm{ccl}\,D_M) \subset L_M^- \subset \mathcal{O}_M \qquad (n > 0). \tag{3}$$

Since $\mathrm{ccl}\,\mathcal{O}_M = L_M^+ \cap \mathrm{ccl}\,D_M$, the desired result follows from (2) and (3).

*Case 2:* $M \in \Omega_2$. Let $q$ be the order of $M$. Fix $\tau \in \mathrm{ccl}\,\mathcal{O}_M$. Let $K_\tau = \{z \in \mathbb{C}^* : |t_1(z)| = |t_1(\tau)|\}$ where $t_1 = \begin{vmatrix} 1 & -\omega_M \\ 1 & -\omega_M' \end{vmatrix}$. If $\tau \in \mathbb{R}$, then $K_\tau = \mathbb{R}^*$. If $\tau \notin \mathbb{R}$, then $K_\tau$ is a circle through $\tau$ such that $M(K_\tau) = K_\tau$. (This can be shown using the formula $t(W(\tau)) = \rho^{-2}t(\tau)$ which occurs in [1, p. 112].) If $\mathrm{Im}\,\tau > 0$, then $K_\tau \subset \{z : \mathrm{Im}\,z > 0\}$ and $\omega_M$ is inside $K_\tau$; if $\mathrm{Im}\,\tau < 0$, then $K_\tau \subset \{z : \mathrm{Im}\,z < 0\}$ and $\omega_M'$ is inside $K_\tau$. Thus $K_\tau \cap \mathcal{O}_M$ is an arc or a ray. (See Figure 6.) By Lemma 9, $M(L_M^+) = D_M$ and hence also $M^{-1}(\mathrm{ccl}\,D_M) = L_M^-$. Consequently, $M(\tau) \in \mathcal{O}_M$ and $M^{-1}(\tau) \in \mathcal{O}_M$. We claim that the points $\tau$, $M(\tau), \ldots, M^{q-1}(\tau)$ occur in that cyclic order on $K_\tau$. To see this, choose a complex linear fractional transformation $Y$ such that $Y(\infty)$ and $Y(0)$ are the fixed points of $M$. Then $X = Y^{-1}MY = \begin{vmatrix} u & 0 \\ 0 & 1/u \end{vmatrix}$ for some complex $u \neq 0$. We have $\alpha = X^q(\alpha) = u^{2q}\alpha$ for all $\alpha$, so $u^{2q} = 1$. Since
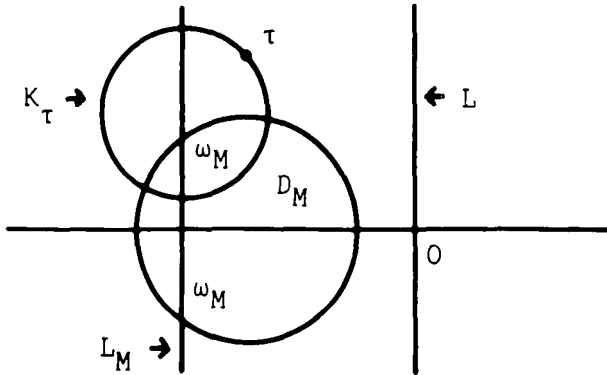


Figure 6

$M \in \Omega_2$, $u^2 = \exp(\pm 2\pi i/q)$. Let $\alpha = Y^{-1}(\tau)$. The points $\alpha, X(\alpha), X^2(\alpha), \ldots, X^{q-1}(\alpha)$ occur in that cyclic order on $Y^{-1}K_\tau$. This proves the claim. It follows that the points $M(\tau), \ldots, M^{q-1}(\tau)$ all lie on $K_\tau \cap \mathcal{O}_M$. Thus $M^n(\tau) \in \mathcal{O}_M$ for all $n$ such that $M^n \neq I$.

*Case* 3: $M \in \Omega_3$. By Lemmas 7 and 10,

$$M(\text{ccl } \mathcal{O}_M) = M(L_M^+) = D_M \subset L_M^- = \mathcal{O}_M.$$

Since $M$ is an involution, this is the desired result.

*Case* 4: $M \in \Omega_4$. By Lemma 11, $M^2(L_M^+) = D_M \subset L_M^+$, and hence

$$M^{2n}(L_M^+) \subset D_M \subset \mathcal{O}_M \qquad (n > 0). \tag{4}$$

Also, $M^{-2}(\text{ccl } D_M) = L_M^- \subset \text{ccl } D_M$, and hence

$$M^{-2n}(\text{ccl } D_M) \subset L_M^- \subset \mathcal{O}_M \qquad (n > 0). \tag{5}$$

For all $n > 0$, we have, by (5) and Lemma 11,

$$M^{-2n+1}(L_M^+) = M^{-2n}(D_M') \subset M^{-2n}(\text{ccl } D_M) \subset \mathcal{O}_M. \tag{6}$$

For all $n > 0$, we have, by (4) and Lemma 11,

$$M^{2n-1}(L_M^+) \subset M^{-1}(D_M) = D_M' \subset L_M^- \subset \mathcal{O}_M. \tag{7}$$

The desired result now follows from (4), (5), (6) and (7). This completes the proof.

## 6. Proofs of Theorems 1 and 2.

*Proof of Theorem* 1. By Theorem 4, there exist $L$-maps $Q_1$ and $Q_2$ such that $X = Q_1^{-1}AQ_1$ and $Y = Q_2^{-1}BQ_2$ are real. Since $A \in \Omega$ and $Q_1$ is an $L$-map, $X \in \Omega$. Similarly, $Y \in \Omega$. We may assume without loss of generality that $X$ is plussed, otherwise replace $A$ by $A^{-1}$. Similarly, assume that $Y$ is plussed. Let $Z = Y^W$ (recall that $W = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$). Note that $Z = Q_3^{-1}CQ_3$, where $Q_3 = Q_2^W$. Define $\mathcal{O}_X$ and $\mathcal{O}_Y$ as in Lemma 12. Note that $\mathcal{O}_X \subset L^-$ and $W(\mathcal{O}_Y) \subset L^+$ (see Figures 1 through 5). Since $Q_1$ and $Q_3$ are $L$-maps, $\mathcal{O}_1 = Q_1(\mathcal{O}_X) \subset L^-$ and $\mathcal{O}_3 = Q_3(W(\mathcal{O}_Y)) \subset L^+$. Define $F = \text{ccl}(\mathcal{O}_1 \cup \mathcal{O}_3)$. Since, by hypothesis, at least one of $A$ and $C$ fixes no point of $L$, we have $F \neq \varnothing$. To see this, suppose, for example, that $A$ fixes no point of $L$. Then $X$ fixes no point of $L$, so $L^- \cap \text{ccl}(\mathcal{O}_X) \neq \varnothing$ (see Figures 1 through 5). Thus $L^- \cap \text{ccl}(\mathcal{O}_1) \neq \varnothing$, so $F \neq \varnothing$.

To show that $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$, we will show that for every nontrivial reduced word $U$ in $\langle A, C \rangle$, $U(F) \cap F = \varnothing$. To show this, it suffices to prove that for all integers $n$ such that $A^n \neq I$,

$$A^n(F \cup \mathcal{O}_3) \subset \mathcal{O}_1, \tag{8}$$

and that for all integers $n$ such that $C^n \neq I$,

$$C^n(F \cup \mathcal{O}_1) \subset \mathcal{O}_3. \tag{9}$$

We give an example to illustrate why it suffices to prove (8) and (9). Let $\tau \in F$ and let $U$

be the reduced word $A^5C^{-3}A^4$. By (8), $\tau_1 = A^4\tau \in \mathcal{O}_1$. By (9), $\tau_2 = C^{-3}\tau_1 \in \mathcal{O}_3$. By (8), $U(\tau) = A^5\tau_2 \in \mathcal{O}_1$, so $U(\tau) \notin F$.

Putting $M = X$ in Lemma 12, we have $X^n(\text{ccl } \mathcal{O}_X) \subset \mathcal{O}_X$ for all $n$ such that $X^n \neq I$. Since

$$Q_1(\text{ccl } \mathcal{O}_X) = \text{ccl}(Q_1(\mathcal{O}_X)) = \text{ccl}(\mathcal{O}_1),$$

it follows that $A^n(\text{ccl } \mathcal{O}_1) \subset \mathcal{O}_1$, for all $n$ such that $A^n \neq I$. This yields (8). Putting $M = Y$ in Lemma 12, we have $WZ^nW(\text{ccl } \mathcal{O}_Y) \subset \mathcal{O}_Y$ for all $n$ such that $Z^n \neq I$. Hence $C^n(\text{ccl } \mathcal{O}_3) \subset \mathcal{O}_3$ for all $n$ such that $C^n \neq I$. This yields (9) and completes the proof.

Before proving Theorem 2, we prove the following lemma.

LEMMA 13. *Let $A$ and $C$ be linear fractional transformations of order 2 such that $A \neq C$ and $AC$ is not elliptic. Then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$.*

*Proof.* The reduced words in $\langle A, C \rangle$ are the words alternating in the symbols $A$ and $C$, e.g., $ACACACA$. Since $A \neq C$, $AC \neq I$. Since $AC$ is not elliptic, $AC$ has infinite order. Hence $\langle A, C \rangle = \langle A \rangle * \langle C \rangle$. Moreover, by [3, Theorem 1E, p. 87], $\langle AC \rangle$ is discrete. Since $[\langle A, C \rangle : \langle AC \rangle] = 2$, it follows that $\langle A, C \rangle$ is discrete. This completes the proof.

*Proof of Theorem 2.* If $\{A, C\} \subset \{W, T_u\}$ for some $u$, clearly $\langle A, C \rangle$ is not the free product of $\langle A \rangle$ and $\langle C \rangle$. Conversely, assume that for every $u$, $\{A, C\} \not\subset \{W, T_u\}$. It may be assumed that both $A$ and $C$ have a fixed point on $L$, otherwise $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ by Theorem 1. As $W(L) = L$, $B$ also has a fixed point on $L$. Let $M \in \{A, B\}$ (where we write $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ with $a, b \geq 0$ and $c, d \leq 0$). If $M$ fixes 0 or $\infty$, then $M = W_b$ (with $b \geq 0$) or $M = W_c^t$ (with $c \leq 0$) by Lemma 3. If $M$ fixes a point $ui$ $(0 < u < \infty)$, then an easy calculation shows that $M = \begin{vmatrix} a & -cu^2 \\ c & a \end{vmatrix}$. But since the diagonal elements of $M$ cannot both be positive or negative, $a = 0$. Hence $M = T_u$. Since for every $u$, $\{A, C\} \not\subset \{W, T_u\}$, $\langle A, C \rangle$ is one of the following groups or their transposes:

$$G_1 = \langle W_r, W_s \rangle \qquad (r \leq 0, s > 0), \tag{10}$$

$$G_2 = \langle W_r^t, W_s \rangle \qquad (r > 0, s > 0), \tag{11}$$

$$G_3 = \langle W_s, T_u \rangle \qquad (s \neq 0), \tag{12}$$

or

$$G_4 = \langle T_u, T_v \rangle \qquad (u > v > 0). \tag{13}$$

Let $U_i$ and $V_i$ be the first and second given generator of $G_i$, respectively ($i = 1, 2, 3, 4$). It remains to show that each $G_i$ is the discrete free product $\langle U_i \rangle * \langle V_i \rangle$. It is easily checked that $U_i$ and $V_i$ are involutions and that $U_i V_i$ is not elliptic ($i = 1, 2, 3, 4$). (Note, for example, that $U_3 V_3$ is loxodromic.) Hence the result follows from Lemma 13 and the proof is complete.

**7. Transformations with fixed points on $L$.**   We discuss the structure of $\langle A, C \rangle$ where $A \in \Omega$, $C \in \Omega^W$, and both $A$ and $C$ have a fixed point on $L$. The case where these fixed points are equal is treated in Theorem 14; the remaining case is treated in Theorem 15.

THEOREM 14. *Let $A \in \Omega$ and $C \in \Omega^W$. Suppose that $A$ and $C$ both fix $\omega \in L$. Then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ unless $A = C$.*

*Proof.* If $\omega \neq \infty$, conjugate $A$ and $C$ by the $L$-map $\begin{vmatrix} 0 & 1 \\ 1 & -\omega \end{vmatrix}$. Thus we may assume without loss of generality that $\omega = \infty$. By Lemma 3, $A = W_\rho$ and $C = W_\nu^W = W_{-\nu}$ for some pair $\rho$, $\nu$. Suppose that $A \neq C$. Then $AC = \begin{vmatrix} 1 & u \\ 0 & 1 \end{vmatrix}$ for some $u \neq 0$. The result thus follows from Lemma 13.

THEOREM 15. *Let $A \in \Omega$ and $C \in \Omega^W$. Suppose that $A$ fixes $\omega_A \in L$ and $C$ fixes $\omega_C \in L$, where $\omega_A \neq \omega_C$. Then there exists an $L$-map $S$ such that $A^S = W_\rho$ and $C^S = W_\nu^t$ for some pair $\rho$, $\nu$. Also $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ if and only if $A \neq C$ and $AC$ is not elliptic.*

*Proof.* We begin by proving the first assertion. If $\omega_A \neq \infty$, conjugate $A$ and $C$ by the $L$-map $\begin{vmatrix} 0 & 1 \\ 1 & -\omega_A \end{vmatrix}$. Thus it may be assumed without loss of generality that $\omega_A = \infty$. Let $S$ be the $L$-map $\begin{vmatrix} 1 & -\omega_C \\ 0 & 1 \end{vmatrix}$ (note that $\omega_C \neq \omega_A = \infty$). Then $A^S(\infty) = \infty$ and $C^S(0) = 0$. Since $(C^S)^W \in \Omega$ and $(C^S)^W(0) = 0$, it follows from Lemma 3 that $A^S = W_\rho$ for some $\rho$ and $C^{WS} = W_{-\nu}^t$ for some $\nu$. Hence $C^S = W_\nu^t$.

To prove the second assertion, assume that $AC$ is elliptic. If $AC$ has finite order, then $\langle A, C \rangle$ is not the free product $\langle A \rangle * \langle C \rangle$. If $AC$ has infinite order, then $\langle AC \rangle$, and hence $\langle A, C \rangle$, is not discrete. Conversely, assume that $A \neq C$ and that $AC$ is not elliptic. By the first assertion of this theorem, $A$ and $C$ are involutions. Thus $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ by Lemma 13.

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