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Some elegant approximations and asymptotic formulas of Ramanujan

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Abstract

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Three results from the unorganized pages of Ramanujan's second notebook are proved. The results provide asymptotic expansions and approximations for certain integrals and sums. They evince Ramanujan's extraordinary ability to discern beautiful and unexpected relationships.

Keywords: Ramanujan's notebooks, asymptotic expansion, Watson's lemma, Laplace's theorem, logarithmic integral.

Ramanujan is well known for his asymptotic formulas in number theory. For example, in collaboration with G.H. Hardy, Ramanujan established an asymptotic formula for the partition function $p(n)$. This was a paramount achievement, for no one had ever conjectured that such an asymptotic formula might exist, and moreover, the proof inaugurated the circle method, one of the most important tools in analytic number theory in this century.

On the other hand, until recently, Ramanujan has not been recognized for his many elegant and deep asymptotic formulas in analysis. The reason for this lack of recognition is that almost all of his discoveries had remained buried and unproven in his notebooks [10] until the 1980s. For accounts of much of this work, see [2–4].

In this short note we prove three of Ramanujan's approximations and asymptotic expansions found in the unorganized pages at the end of his second notebook [10]. These three theorems are easier to prove than Ramanujan's deeper discoveries in asymptotic analysis. However, each is evidence of Ramanujan's extraordinary ability to discern beautiful and perhaps surprising relationships.

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Theorem 1 (Ramanujan [10, Volume 2, p.268]). *Let* α, $\beta > 0$ *with* $\alpha\beta = \pi^2$. *Define*

$$I(\alpha) = \alpha^{-1/4}\left(1 + 4\alpha\int_0^\infty \frac{x\,\mathrm{e}^{-\alpha x^2}}{\mathrm{e}^{2\pi x} - 1}\,\mathrm{d}x\right).$$

Then

$$I(\alpha) = I(\beta). \tag{1}$$

Moreover,

$$I(\alpha) = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4}, \quad \text{"nearly".} \tag{2}$$

(Ramanujan often used the words "nearly" or "very nearly" at the end of his approximations or asymptotic formulas.)

We shall not prove (1) here. It appears in Ramanujan's first letter to Hardy [11, p. xxvi] and, in fact, was proved by Ramanujan in two papers [8], [11, pp. 53–58] and [9], [11, pp. 72–77]. This result was also established by Preece [7].

Proof of (2). Recall [6, p.282] the generating function for the Bernoulli numbers B_n, $n \geqslant 0$,

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \quad |x| < 2\pi.$$

Employing Watson's lemma [6, p.71], we find that, as β tends to ∞,

$$I(\beta) \sim \beta^{-1/4} + \sum_{n=0}^{\infty} I_n, \tag{3}$$

where

$$I_n = 4\beta^{3/4}\frac{B_n(2\pi)^{n-1}}{n!}\int_0^\infty \mathrm{e}^{-\beta x^2} x^n\,\mathrm{d}x.$$

Elementary calculations show that

$$I_0 = \alpha^{-1/4}, \qquad I_1 = -\beta^{-1/4}, \qquad I_2 = \tfrac{1}{6}\alpha^{3/4} \quad \text{and} \quad I_4 = -\tfrac{1}{60}\alpha^{7/4}.$$

Using these values in (3), we derive the asymptotic expansion, as β tends to ∞, or as α tends to 0,

$$I(\alpha) = I(\beta) \sim \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \frac{\alpha^{7/4}}{60} + \cdots. \tag{4}$$

On the other hand, by Taylor's theorem, for α sufficiently small and positive (or β sufficiently large),

$$\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4} = \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \left(\frac{1}{24} - \frac{1}{4\pi^2}\right)\alpha^{7/4} + \cdots. \tag{5}$$

Now,

$$\frac{1}{60} = 0.01666\ldots \quad \text{and} \quad \frac{1}{24} - \frac{1}{4\pi^2} = 0.01633\ldots .$$

Thus, there is a remarkable agreement between (4) and (5). This concludes the proof of Theorem 1. □

Theorem 2 (Ramanujan [10, Volume 2, p.318]). *As a tends to ∞,*

$$\int_a^\infty \left(\frac{a}{x}\right)^x \mathrm{d}x \sim \sum_{k=0}^\infty \left(\frac{-k}{a}\right)^k. \tag{6}$$

Note that the integrand equals the summand if one substitutes $-k$ for x. Before giving a rigorous proof of Theorem 2, we shall present a heuristic argument for (6) that Ramanujan likely found.

Immediately before Theorem 2, Ramanujan offers the equality

$$\int_0^\infty \frac{\phi(x)}{x^x} \mathrm{d}x = \sum_{k=-\infty}^\infty \frac{\phi(k)}{k^k}. \tag{7}$$

For the first integral on the right side below, we apply (7). The second integral on the right side actually appears in [10, p.283]. This integral is easily evaluated by expanding the integrand in a power series in $x \log(a/x)$ and integrating termwise. Accordingly, we find that (heuristically)

$$\begin{aligned}\int_a^\infty \left(\frac{a}{x}\right)^x \mathrm{d}x &= \int_0^\infty \left(\frac{a}{x}\right)^x \mathrm{d}x - \int_0^a \left(\frac{a}{x}\right)^x \mathrm{d}x \sim \sum_{k=-\infty}^\infty \frac{a^k}{k^k} - \sum_{k=1}^\infty \frac{a^k}{k^k} \\ &= \sum_{k=-\infty}^0 \frac{a^k}{k^k} = \sum_{k=0}^\infty \left(\frac{-k}{a}\right)^k.\end{aligned}$$

Proof of Theorem 2. First, letting $x = au$, we find that

$$\int_a^\infty \left(\frac{a}{x}\right)^x \mathrm{d}x = a\int_1^\infty \mathrm{e}^{-au \log u} \mathrm{d}u. \tag{8}$$

We shall now use [10, Volume 2, Chapter 3, Section 15, Example 2], [1, p.75], which can be proved by Lagrange inversion. In the notation of Example 2, let $m = 1$, $n = -1$, $p = 1$, $a = t$ and $x = u$. Thus, for $t = u \log u$ and $|t| \leqslant 1/\mathrm{e}$,

$$u - 1 = \sum_{k=1}^\infty \frac{(-1)^{k-1}(k-1)^{k-1}}{k!} t^k. \tag{9}$$

Using (9), we apply Laplace's method [6, pp. 85–86] to obtain an asymptotic expansion as a tends to ∞ for the integral on the right side of (8). To help the reader, we remark that (8) corresponds to [6, (8.04)]. Also, note the parenthetical remark prior to the statement of [6, Theorem 8.1, p.86]. Hence, by a direct application of Laplace's theorem, as a tends to ∞,

$$\int_1^\infty \mathrm{e}^{-au \log u} \mathrm{d}u \sim \sum_{k=0}^\infty \Gamma(k+1)(k+1)\frac{(-1)^k k^k}{(k+1)!a^{k+1}} = \sum_{k=0}^\infty \frac{(-1)^k k^k}{a^{k+1}}.$$

Using this in (8), we complete the proof of (6). □

Another approach to Theorem 2 is via Watson's lemma. Let $t = u \log u$ and let $\phi(t) = 1/(\log u + 1)$. Then by Watson's lemma [6, p.71], as a tends to ∞,

$$\int_a^\infty \left(\frac{a}{x}\right)^x \mathrm{d}x = a\int_1^\infty \mathrm{e}^{-au\log u}\,\mathrm{d}u = a\int_0^\infty \mathrm{e}^{-at}\phi(t)\,\mathrm{d}t \sim \sum_{k=0}^\infty \frac{\phi^{(k)}(0)}{a^k}.$$

Since $\phi(t) = \mathrm{d}u/\mathrm{d}t$, upon differentiating both sides of (9) with respect to t, we find that $\phi(0) = 1$ and

$$\phi^{(k)}(0) = (-k)^k, \quad k \geqslant 1, \tag{10}$$

which completes our second proof of (6). □

Before stating and proving Theorem 3, we introduce some notation and offer a preliminary lemma. Ramanujan defines the logarithmic integral $\mathrm{Li}(x)$ by

$$\mathrm{Li}(x) = \mathrm{PV}\int_0^x \frac{\mathrm{d}t}{\log t}, \quad x > 1.$$

Define the unique positive number μ by

$$\mathrm{Li}(\mu) = 0. \tag{11}$$

Ramanujan [10] and Soldner [5, p.88] numerically calculated μ. We used the computer algebra system MACSYMA to also calculate μ. Table 1 summarizes these calculations.

Lemma. *Let μ be defined by* (11), *and let γ denote Euler's constant. Then, for $x > 1$,*

$$\mathrm{Li}(x) = \gamma + \log\log x + \sum_{k=1}^\infty \frac{\log^k x}{k!k}.$$

Proof. Observe that, for $x > 1$,

$$\begin{aligned}\mathrm{Li}(x) &= \mathrm{PV}\int_0^x \frac{\mathrm{d}t}{\log t} = -\mathrm{PV}\int_{-\log x}^\infty \frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u \\ &= \int_0^1 \frac{1-\mathrm{e}^{-u}}{u}\,\mathrm{d}u - \int_1^\infty \frac{\mathrm{e}^{-u}}{u}\,\mathrm{d}u + \int_1^{\log x} \frac{\mathrm{d}u}{u} + \int_{-\log x}^0 \frac{1-\mathrm{e}^{-u}}{u}\,\mathrm{d}u \\ &= \gamma + \log\log x + \sum_{k=1}^\infty \frac{\log^k x}{k!k},\end{aligned}$$

where we have used a well-known representation for γ [1, p.103]. □

Table 1

Source	μ
Ramanujan	1.45136380
Soldner	1.4513692346
MACSYMA	1.4513692349

For another proof of the lemma, see [5, pp. 3, 11].

We first give Ramanujan's version of Theorem 3: if

$$S := \sum_{k=1}^{n} \frac{1}{k}\left(1+\frac{1}{p}\right)^k = \log p, \tag{12}$$

then $n = (p+\frac{1}{2})\log \mu - \frac{1}{2}$.

We might interpret Ramanujan's statement as giving an estimate for S when $n = [(p+\frac{1}{2})\log \mu - \frac{1}{2}]$. With such an interpretation, the error made in the approximation by $\log p$ is $O(1/p)$. However, if p is chosen so that $n = (p+\frac{1}{2})\log \mu - \frac{1}{2}$ is an integer, then, as stated in Theorem 3, the error term is $O(1/p^2)$. Amazingly, Ramanujan found the precise linear function of p that yields an error term of $O(1/p^2)$. Thus, if the constant $-\frac{1}{2}$ in the definition of n is replaced by any other constant, the error term is $O(1/p)$.

Theorem 3 (Ramanujan [10, Volume 2, p.318]). *Let S be defined by* (12), *and let $n = (p+\frac{1}{2})\log \mu - \frac{1}{2}$ be a positive integer. Then as p tends to ∞,*

$$S = \log p + O(p^{-2}). \tag{13}$$

Proof. Setting $y = \log \mu$, applying the lemma, and using (11), we find that

$$0 = \gamma + \log y + \sum_{k=1}^{\infty} \frac{y^k}{k!k} = \gamma + \log y + \int_0^y \frac{e^t - 1}{t}\, dt. \tag{14}$$

Setting $x = 1 + 1/p$ and using the definition of S, a familiar estimate for a partial sum of the harmonic series [6, p.292], and (14), we deduce that, as n tends to ∞,

$$\begin{aligned} S &= \sum_{k=1}^{n} \frac{x^k - 1}{k} + \sum_{k=1}^{n} \frac{1}{k} \\ &= \sum_{k=1}^{n} \frac{x^k - 1}{k} + \log n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \\ &= \sum_{k=1}^{n} \frac{x^k - 1}{k} + \log\left(\frac{n}{y}\right) + \frac{1}{2n} - \int_0^y \frac{e^t - 1}{t} dt + O\left(\frac{1}{p^2}\right). \end{aligned} \tag{15}$$

Next, applying the Euler–Maclaurin summation formula [6, p.285], we deduce that, as p tends to ∞,

$$\sum_{k=1}^{n} \frac{x^k - 1}{k} = -\tfrac{1}{2}\log x + \frac{x^n - 1}{2n} + \int_0^n \frac{x^t - 1}{t}\, dt + O\left(\frac{1}{p^2}\right). \tag{16}$$

Employing (16) in (15), we find that

$$\begin{aligned} S &= \int_0^n \frac{x^t - 1}{t} dt - \int_0^y \frac{e^t - 1}{t} dt - \tfrac{1}{2}\log x + \log\left(\frac{n}{y}\right) + \frac{x^n}{2n} + O\left(\frac{1}{p^2}\right) \\ &= \int_y^{n \log x} f(t)\, dt - \tfrac{1}{2}\log x + \log\left(p + \tfrac{1}{2} - \frac{1}{2y}\right) + \frac{x^n}{2n} + O\left(\frac{1}{p^2}\right), \end{aligned} \tag{17}$$

where $f(t)=(e^t-1)/t$. Now,

$$-\tfrac{1}{2}\log x=-\frac{1}{2p}+O\left(\frac{1}{p^2}\right),$$

$$\log\left(p+\tfrac{1}{2}-\frac{1}{2y}\right)=\log p+\frac{1}{2p}-\frac{1}{2py}+O\left(\frac{1}{p^2}\right)$$

and

$$\frac{x^n}{2n}=\frac{(1+1/p)^{py+(y-1)/2}}{2py+(y-1)}=\frac{e^y}{2py}+O\left(\frac{1}{p^2}\right),$$

as p tends to ∞. Substituting these estimates in (17), we find that

$$S=\int_y^{n\log x}f(t)\,dt+\frac{f(y)}{2p}+\log p+O\left(\frac{1}{p^2}\right),$$

as p tends to ∞. Thus, it remains to prove that

$$\int_{n\log x}^{y}f(t)\,dt=\frac{f(y)}{2p}+O\left(\frac{1}{p^2}\right). \tag{18}$$

Now,

$$n\log x=y-\frac{1}{2p}+O\left(\frac{1}{p^2}\right). \tag{19}$$

By the first mean value theorem for integrals and (19),

$$\int_{n\log x}^{y}f(t)\,dt=f(u)(y-n\log x)=\frac{f(u)}{2p}+O\left(\frac{1}{p^2}\right), \tag{20}$$

for some value u such that $n\log x<u<y$. By the mean value theorem, there exists a value v such that $u<v<y$ and

$$f(u)=f(y)+(u-y)f'(v)=f(y)+O\left(\frac{1}{p}\right), \tag{21}$$

where the last equality follows from (19), since f' is bounded on (u, y). Using (21) in (20), we complete the proof of (18). □

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